Various aspects of unipotent group actions in algebraic geometry

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§ 1. Unipotent group actions on complete varieties

1.1. Let $k$ be an algebraically closed field of characteristic zero. Let $G$ be a connected algebraic group defined over $k$. Assume that $G$ acts non-trivially on an algebraic variety $V$, 

$$\sigma : G \times V \to V.$$  

Then we have the canonical Lie algebra homomorphism 

$$\sigma_* : \mathfrak{g} := \text{Lie}(G) \to H^0(V, \mathcal{O}_V),$$

where $\mathcal{O}_V := (\Omega^1_{V/k})^*$. If $V$ is smooth over $k$, $\mathcal{O}_V$ is a locally free $O_V$-Module associated with the tangent bundle $T_V$. For every element $\tau$ of $\mathfrak{g}$, $\sigma_*(\tau)$ is thus a holomorphic (tangent) vector field of $V$.

Now assume that $V$ is a nonsingular projective variety defined over $k$. Let $X$ be a holomorphic vector field on $V$ such that $X \neq 0$. A point $P$ of $V$ is said to be a zero of $X$ if $X(P) = 0$; the set of all zeros of $X$ is denoted by
Zero(X), which is a closed subvariety of X. Let P ∈ Zero(X).
Then we can consider the Lie derivative $L_X$:

$$L_X : T_{V,P} \rightarrow T_{V,P} \ ; \ L_X(Y) = [X,Y].$$

X is said to be generic at P (or X has a simple zero at P) if $L_X$ is nondegenerate on $T_{V,P}$. X is said to be generic (or X has only simple zeros) if $L_X$ is nondegenerate for every zero P of X. If X has a simple zero at P, we can consider the eigenvalues $\theta_1(P), \ldots, \theta_n(P)$ of $L_X$, where $n = \dim V$. The existence of holomorphic vector fields (or actions of algebraic groups) on V imposes some restrictions on the topology and the numerical characters of V. We shall quote some of the known results.

1.2. Let V be a nonsingular projective variety defined over k and let X be a holomorphic vector field on V such that $X \neq 0$. Let $Z := \text{Zero}(X)$. Define the contraction operator $i_X$ as follows:

$$i_X : \Omega^P_X \rightarrow \Omega^{P-1}_X$$

$$i_X(fdx_1 \wedge \ldots \wedge dx_P) = f(\sum_{i=1}^{P} (-1)^{i-1}X(x_i)dx_1 \wedge \ldots \wedge dx_i \wedge \ldots \wedge dx_P).$$

The definition is well-defined, and if $\omega^P$ is an element of $H^0(V, \Omega^P_V)$ then $i_X(\omega^P) \in H^0(V, \Omega^{P-1}_V)$. Let

$$\mathfrak{f}^1_V = \{ X \in H^0(V, \Theta_V) \mid i_X : H^0(V, \Omega^1_V) \rightarrow H^0(V, \Omega_V) \}.$$  

is the zero map

Then $\mathfrak{f}^1_V$ is a Lie subalgebra of $H^0(V, \Theta_V)$. 

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1.2.1. **Theorem.** With the above notations, we have:

1. (Kobayashi [7]). If \(0 \leq \dim Z < n := \dim V\), then \(P_m(V) = 0\) for every \(m > 0\). Hence \(\kappa(V) = -\infty\).

2. (Carrell-Lieberman [1]). Assume that \(Z \neq \emptyset\). Then
\[
h^p,q = \dim_k H^q(V, \Omega^p_V) = 0 \quad \text{whenever} \quad |p-q| > \dim_k Z.
\]

3. (Carrell-Lieberman [1]). Every element \(X\) of \(L^1_J\) has zeros. Hence, if \(h^{1,0}(V) = \dim H^0(V, \Omega^1_V) = 0\) then every holomorphic vector field has zero. Hence, if \(V\) has a holomorphic vector field without zero, \(h^{1,0}(V) > 0\).

1.2.2. **Corollary.** Assume that \(\dim V = 2\) and \(V\) has a holomorphic vector field \(X\) with \(\dim \text{Zero}(X) = 0\). Then \(V\) is rational.

**Proof.** The assumption \(\dim \text{Zero}(X) = 0\) implies \(h^{1,0}(V) = 0\). Since \(X \neq 0\), we have \(P_m(V) = 0\) for every \(m > 0\). Hence \(V\) is rational by Castelnuovo's criterion of rationality.

1.2.3. **Theorem.** Let \(k = \mathbb{C}\). Assume that \(V\) has a holomorphic vector field \(X\) possessing only simple zeros. For a point \(P\) of \(\text{Zero}(X)\), let \(\theta_1(P), \ldots, \theta_n(P)\) be the eigenvalues of \(L_X\). Assume that \(\text{Re} \theta_i(P) \neq 0\) for \(1 \leq i \leq n\) and every point \(P \in \text{Zero}(X)\). Then the Betti numbers of \(V\) are given as follows:

\[
b_{2p}(V) = \#\{P \in \text{Zero}(X) | \#\{j | \text{Re} \theta_j(P) > 0, 1 \leq j \leq n\} = p\}
\]

\[
b_{2p+1}(V) = 0, \quad \text{(cf. Carrell-Lieberman [1]).}
\]

1.3. **Examples.**

1. Let \(G\) be a semi-simple algebraic group, let \(P\) be
a parabolic subgroup of $G$, let $T$ be a maximal torus with $T \subseteq P$ and let $V := G/P$. Let $t$ be a regular element of infinite order in $T$ such that there exists a one-dimensional subtorus $S$ of $T$ passing through $t$. Let $S$ act on $V$ via left translations of $G$. Let $X$ be a holomorphic vector field on $V$ defined by the canonical Lie algebra homomorphism

$$\alpha_* : \mathfrak{j} := \text{Lie}(S) \longrightarrow H^0(V, \mathcal{O}_V).$$

Then $\text{Zero}(X)$ is a finite set and $X$ has only simple zeros.

**Proof.** We claim that:

$(gP)$ is a fixed point of $S \mapsto g^{-1}tg \in P \mapsto g \in N(T)P$.

Indeed, $S$ is the closure of $\{t^m \mid m \in \mathbb{Z}\}$, and hence $(gP)$ is a fixed point if and only if $g^{-1}tg \in P$. Then $t \in gPg^{-1}$.

Since $t$ is a regular element, $T \subseteq gPg^{-1}$. Hence $g^{-1}Tg \subseteq P$.

Therefore $g^{-1}Tg = p^{-1}Tp$ for some element $p \in P$. Hence $gp^{-1} \in N(T)$. Since $\#(N(T)P/P) < +\infty$, there are only finitely many fixed points of $S$ on $V$. Let $(gP)$ be a fixed point of $S$. Let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of $G$ and $P$, respectively.

Now, $T_V(gP)$ is identified with $\mathfrak{g}/\mathfrak{p}$ via $\mathfrak{g}_{\mathfrak{p}} : \mathfrak{g}/\mathfrak{p} \longrightarrow T_V(gP)$.

Then the Lie derivative $L_X$ on $T_V(gP)$ is identified with

$$Y \pmod{\mathfrak{p}} \mapsto \text{Ad}(g^{-1}tg)(Y) \pmod{\mathfrak{p}}.$$

Noting that $g^{-1}tg \in P$, we know that $L_X$ is non-degenerate at $(gP)$.

(2) Let $V = \mathbb{P}^n_k$ with homogeneous coordinates $(x_0, x_1, \ldots, x_n)$. Let $\alpha_0, \ldots, \alpha_n$ be pairwise prime integers such that $\alpha_0 + \cdots + \alpha_n$.
= 0. Let \( G_m \) act on \( V \) via
\[
t(x_0, x_1, \ldots, x_n) = (t^{a_0}x_0, t^{a_1}x_1, \ldots, t^{a_n}x_n).
\]
Then the fixed points of \( G_m \) on \( \mathbb{P}^n \) are \( O_i \)'s, where \( O_i = (0, \ldots, 0, 1, 0, \ldots, 0) \). Let \( u_j = x_j/x_i \) and \( \xi_j = \frac{\partial}{\partial u_j} \) for \( 0 \leq j \leq n \) and \( j \neq i \). Then we have
\[
T_{\mathbb{P}^n, O_i} = \sum_{j=0}^{n} k\xi_j \quad \text{and} \quad L_X(\xi_j) = (\alpha_j - \alpha_i)\xi_j.
\]
Instead, consider the following action of \( G_a \) on \( \mathbb{P}^n_k \),
\[
G_a = \{ \exp(tA) \mid t \in k, A = \begin{pmatrix} 0 & 1 & 0 \\ & & \\ 0 & \ddots & 1 \\ & & 0 \end{pmatrix} \in M_{n+1}(k) \}
\]
\[
t \left( \begin{array}{c} x_0 \\ x_1 \\ \vdots \\ x_n \end{array} \right) = \exp(tA) \left( \begin{array}{c} x_0 \\ x_1 \\ \vdots \\ x_n \end{array} \right).
\]
Then \( O := (1,0,\ldots,0) \) is the unique fixed point of \( G_a \). The holomorphic vector field \( X \) on \( \mathbb{P}^n \) defined by this action has the following Lie derivative \( L_X \) on \( T_{\mathbb{P}^n, O} \);
\[
u_j = x_j/x_0, \quad \xi_j = \frac{\partial}{\partial u_j} \quad (1 \leq j \leq n),
\]
\[
T_{\mathbb{P}^n, O} = \sum_{j=1}^{n} k\xi_j,
\]
\[
L_X(\xi_j) = 0 \quad \text{if} \quad j = 1; = -\xi_{j-1} \quad \text{if} \quad j > 1.
\]
Hence the zero of \( X \) at \( O \) is not simple.

1.4. Now, we shall be mainly interested in the unipotent group actions on complete algebraic varieties. A main problem is the Carrell conjecture, which we shall state below.
Let $G$ be a unipotent algebraic group defined over $k$. We shall summarize some of the known results on unipotent group actions.

1.4.1. THEOREM. (1) [Borel fixed point theorem] (cf. Fogarty [2], Horrocks [6]). If a connected solvable affine algebraic group $G$ acts on a complete algebraic variety $V$ then the fixed point locus $V^G$ is nonempty. If $G$ is unipotent, $V^G$ is connected if and only if $V$ is connected.

(2) Let $G$ be a connected affine algebraic group. Then $G$ is unipotent if and only if, for any connected complete variety $V$ on which $G$ acts, $V^G$ is connected (cf. Fogarty [2]).

(3) Let $G$ and $V$ be the same as in (2) above. Then the canonical inclusion $\iota : V^G \hookrightarrow V$ induces an equivalence between the categories of etale coverings $\text{Et}(V)$ and $\text{Et}(V^G)$. In particular, the inclusion $\iota$ yields an isomorphism of algebraic fundamental groups,

$$\pi_* : \pi_1(V^G)_{\text{alg}} \xrightarrow{\sim} \pi_1(V)_{\text{alg}},$$

(cf. Horrocks [6]).

1.4.2. We also recall the following result:

THEOREM (Matsumura [8]). Assume that $V$ is a nonsingular complete variety. Then the group of all birational automorphisms of $V$, $\text{Bir}(V)$, contains an affine algebraic group of positive dimension if and only if $V$ is birationally equivalent to $\mathbb{P}^1 \times W$ ($V \sim \mathbb{P}^1 \times W$ as notation), where $W$ is a complete variety. Thus, if the Kodaira dimension $\kappa(V) \geq 0$, $\text{Bir}(V)$ cannot contain any affine algebraic group.
1.4.3. Now, we consider the following:

CARRELL CONJECTURE. Assume that a connected unipotent group $G$ acts on a nonsingular projective variety $V$ in such a way that the fixed point locus $V^G$ consists of a single point. Then $V$ is rational.

1.4.4. A partial solution of the above conjecture is this:

THEOREM. Let $G$ and $V$ be the same as in the Carrell conjecture. Then we have:

(1) If $\dim V \leq 2$, the Carrell conjecture is affirmative.

(2) If $\dim V = 3$, $V$ is one of the following:

(i) $V$ is rational,

(ii) $V \sim \mathbb{P}^1 \times W$, where $W$ is a nonsingular projective surface with $\kappa(W) \geq 1$ and $p_g = q = 0$. Moreover, $W$ is simply connected.

Proof. Without loss of generality, we may assume that the action of $G$ is effective, i.e., the canonical homomorphism $G \to \text{Aut} \ (V)$ is injective.

(1) The case $\dim V = 1$ is obvious by virtue of Matsumura's theorem. Suppose that $\dim V = 2$. If $\dim G = 1$, the action of $G$ on $V$ gives rise to a holomorphic vector field $X$ on $V$ such that $\text{Zero}(X) = V^G$, which consists of a single point. Then, by virtue of Corollary 1.2.2, $V$ is rational. Assume that $\dim G = 2$. Then $G$ is commutative, i.e., $G \cong G_1 \times G_2$ with $G_1 \cong G_2 \cong G_a$. By virtue of Matsumura's theorem, $V \sim \mathbb{P}^1 \times C$, where $C$ is a complete nonsingular model of $k(V^G)$. Then $G_2$ acts on $C$ effectively. Hence $C \sim \mathbb{P}^1$, and $V \sim \mathbb{P}^1 \times \mathbb{P}^1$. 

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Namely, $V$ is rational.

(2) Assume that $\dim V = 3$. Consider first the case where $\dim G = 3$. Then $G$ has a central series of subgroups

$$G \supseteq G_1 \supseteq G_2 \supseteq 1,$$

such that $G/G_1 \cong G_1/G_2 \cong G_2 \cong G_a$. By Matsumura's theorem, $V \sim \mathbb{P}^1 \times W$, where $W$ is a nonsingular projective surface such that $W$ is a complete nonsingular model of $k(V^G)$ and the unipotent group $G/G_2$ acts effectively on $W$. By virtue of the above case where $\dim V = \dim G = 2$, we conclude that $W \sim \mathbb{P}^1 \times \mathbb{P}^1$; note that we did not use in the proof the assumption $\mathcal{V}^G = \{\text{single point}\}$. Hence $V \sim \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Suppose next that $\dim G = 2$. By a similar reasoning as above, we know that $V \sim \mathbb{P}^1 \times \mathbb{P}^1 \times C$, where $C$ is a nonsingular complete curve.

Since $\mathcal{V}^G$ consists of a single point, we know that $\pi_1(V)_{\text{alg}} = (0)$ (cf. Theorem 1.4.1, (3)). Since $\pi_1(V)_{\text{alg}} \cong \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \times C)_{\text{alg}}$, we know that $\pi_1(C)_{\text{alg}} = (0)$, i.e., $C$ is simply connected.

This implies that $C \cong \mathbb{P}^1$. Hence $V$ is rational. Suppose finally that $\dim G = 1$. By virtue of Matsumura's theorem, $V \sim \mathbb{P}^1 \times W$, where $W$ is a complete nonsingular model of $k(V^G)$.

We may assume that $W$ is relatively minimal. Let $p_1$ and $p_2$ be the canonical projections from $\mathbb{P}^1 \times W$ to $\mathbb{P}^1$ and $W$, respectively. Then we have,

$$\mathcal{O}_{\mathbb{P}^1 \times W} \cong p_1^* \mathcal{O}_{\mathbb{P}^1} + p_2^* \mathcal{O}_W,$$

$$\mathcal{O}_{\mathbb{P}^1 \times W} \cong p_1^* \mathcal{O}_{\mathbb{P}^1} \wedge p_2^* \mathcal{O}_W + p_2^* \mathcal{O}_W^2.$$

By virtue of Theorem 1.2.1, (1), we have $h^{1,0}(V) = 0$ for
i = 1, 2, because the action of \( G \) yields a holomorphic vector field \( X \) on \( V \) with \( \text{Zero}(X) = V^G = \{ \text{single point} \} \). Since \( h^{1,0}(V) \) is a birational invariant (cf. Griffiths-Harris [16; p.494]), we know that \( h^{1,0}(W) = h^{2,0}(W) = 0 \). Hence \( p_g = q = 0 \) for \( W \). Moreover, since \( \pi_1(V)_{\text{alg}} = (0) \), we know that \( \pi_1(W)_{\text{alg}} = (0) \). Namely, \( W \) is simply connected. If \( W \) is rational, \( V \) is rational. Suppose that \( W \) is not rational. If \( \kappa(W) = 0 \) then \( p_g = q = 0 \) implies that \( W \) is an Enriques surface, which is, however, not simply connected. Hence \( \kappa(W) \geq 1 \).

Q.E.D.

1.5. We shall give the following result on the existence of a \( G_a \)-action.

LEMMA (cf. [9; p. 35]). Let \( W \) be a variety defined over \( k \) and let \( \pi : V \rightarrow W \) be a \( \mathbb{P}^1 \)-bundle over \( W \). If there exists a nontrivial \( G_a \)-action on \( V \) whose orbits are contained in fibers of the projection \( \pi \), then the fixed point locus \( V^G \) contains a cross-section \( S \) of \( \pi \). Then there exists a locally free \( O_W \)-module \( E \) of rank 2 such that \( E \) is an extension of \( O_W \) by an invertible sheaf \( L \) on \( W \), \( V \cong \mathbb{P}(E) \), and \( S \) is the cross-section corresponding to \( L \). Moreover, we have \( H^0(W,L^{-1}) \neq 0 \). Conversely, if \( H^0(W,L^{-1}) \neq 0 \), there exists a \( G_a \)-action on \( V \) along fibers of \( \pi \).

Proof. Let \( V_1^G \) be the union of irreducible components of \( V^G \) of codimension 1, and consider \( V_1^G \) as a reduced effective divisor on \( V \). Since \( G_a \) acts on \( V \) along fibers of \( \pi \), each general fiber contains one and only one fixed point. Hence
(V^G, \mathfrak{L}) = 1$, where $\mathfrak{L}$ is a general fiber of $\pi$. This implies that $V^G$ contains only one irreducible component $S$, which is a cross-section of $\pi$. Let $L = O_S(S)$ and let $E = \pi_* O_V(S)$. Then we have an exact sequence,

$$0 \rightarrow O_L \rightarrow E \rightarrow L \rightarrow 0.$$ 

By construction, $S$ is the cross-section corresponding to $L$. The remaining part is proved in [9; p. 35]. Q.E.D.

§ 2. Unipotent group actions on affine varieties

2.1. Let $k$ be an algebraically closed field of characteristic $\geq 0$. Recall the following very well-known results:

2.1.1. THEOREM (Nagata [13], Haboush [5]). Let $R$ be a finitely generated $k$-algebra and let $G$ be a connected reductive algebraic group. Assume that $G$ acts on $R$ as $k$-automorphisms of $R$ in such a way that:

For every $f \in R$, a $k$-submodule $\sum_{g \in G} f^g k$ of $R$ is a finite $k$-module; then we say that $G$ acts rationally on $R$.

Let $R^G$ be the subring consisting of $G$-invariant elements in $R$. Then $R^G$ is finitely generated over $k$.

2.1.2. THEOREM (Nagata [14]). There exists a unipotent algebraic group $G$ acting rationally on a polynomial ring $R := k[x_1, \ldots, x_n]$ such that $R^G$ is not finitely generated over $k$.

The writer believes that there should exist a rational action of the additive group $\mathbb{G}_a$ on a polynomial ring $R = k[x_1, \ldots, x_n]$ such that $R^{\mathbb{G}_a}$ is not finitely generated over $k$. If there is
such an action, we must have \( n \geq 4 \) by virtue of Zariski's theorem (cf. Nagata [14]), and the action is not linear by virtue of the following result of Weitzenböck:

2.1.3. **Theorem** (cf. Seshadri [15]). Let there be given a linear action of \( G_a \) on a polynomial ring \( R = k[x_1, \ldots, x_n] \), where \( \text{char}(k) = 0 \). Then \( R^a \) is finitely generated over \( k \).

2.2. For the sake of simplicity, we assume that \( \text{char}(k) = 0 \).

**Theorem.** Assume that \( G_a \) acts non-trivially on a polynomial ring \( R = k[x_1, \ldots, x_n] \), where \( n \leq 3 \). Let \( A \) be the \( G_a \)-invariant subring of \( R \). Then we have:

1. \( A \) is a finitely generated over \( k \), and \( A \) is a unique factorization domain.

2. If either \( n \leq 2 \) or \( A \) is regular then \( A \) is a polynomial ring over \( k \).

**Proof.** For the proof of the assertion (1) and the case \( n \leq 2 \) in the assertion (2), see Miyanishi [9; §§ 1, 3 of Chap.I]. We shall prove the assertion (2) in the case where \( n = 3 \) and \( A \) is regular.

(i) By virtue of Zariski's theorem [14; p. 52], \( A \) is finitely generated over \( k \). Moreover, \( A \) is a UFD and the set \( A^* \) of all invertible elements of \( A \) is \( k^* = k-(0) \). Let \( Y := \text{Spec}(R) \), let \( X := \text{Spec}(A) \) and let \( \pi : Y \rightarrow X \) be the dominant morphism induced by the injection \( A \hookrightarrow R \). We shall prove that the logarithmic Kodaira dimension of \( X \) has value \( \kappa(X) = -\infty \). Then we can apply the following characterization of the affine plane (cf. Miyanishi-Sugie [12] and Fujita [3] as well as the
papers of Iitaka's given in the references of these papers):  

Let \( X = \text{Spec}(A) \) be a nonsingular affine surface. Then \( X \cong \mathbb{A}^2_k \) if and only if \( A \) is a UFD, \( A^* = k^* \) and \( \overline{\kappa}(X) = -\infty \).

(ii) We claim that \( \pi : Y \to X \) is a faithfully flat, equi-dimensional morphism of dimension 1.

We shall first show that \( \pi \) is surjective. Suppose \( \pi \) is not surjective. Then there exists a maximal ideal \( m \) of \( A \) such that \( mR = R \). Let \( (O, tO) \) be a discrete valuation ring of the quotient field \( K \) of \( A \) such that \( O \) dominates \( A_m \). Let \( R' = R \otimes O \), which is identified with a subring of the field \( L := k(x_1, x_2, x_3) \). Let \( \Delta \) be a locally nilpotent derivation on \( R \) associated with the given \( G_a \)-action on \( Y \) (cf. [9; § 1, Chap. I]). Then \( \Delta \) extends naturally to a locally nilpotent \( O \)-derivation in \( R' \), and \( O \) is the ring of \( \Delta \)-invariants in \( R' \), i.e., \( O = \{ r \in R'; \Delta(r) = 0 \} \). By assumption, we have \( tR' = R' \), where \( t \) is a uniformising element of \( O \). Hence \( t \Delta(r) = 0 \) for some element \( r \in R' \). Then \( t \Delta(r) = 0 \), whence \( r \in O \). This is a contradiction. Thus \( \pi \) is surjective.

Secondly, we shall show that every irreducible component of a fiber of \( \pi \) has dimension 1. Note that general fibers of \( \pi \) are isomorphic to \( \mathbb{A}^1_k \) (cf. [9; § 1, Chap. I]). Hence each irreducible component of a fiber has dimension \( \geq 1 \). Suppose that an irreducible component \( T \) of a fiber \( \pi^*(P) \) (with \( P \in X \)) has dimension 2. Since \( R \) is a UFD, there exists an irreducible element \( a \in R \) such that \( T = \text{Spec}(R/aR) \). Since \( T \) is \( G_a \)-stable, \( a \) is \( G_a \)-invariant, i.e., \( a \in A \). Let \( C := \ldots \)}
Spec(A/aA). Since A is a UFD, C is an irreducible curve on X and \( \pi^{-1}(C) = T \subset \pi^{-1}(P) \). This is a contradiction because \( \pi \) is surjective. Thus \( \pi \) is an equi-dimensional morphism of dimension 1.

Finally, we shall show that \( R \) is flat over \( A \). Let \( q \) be a prime ideal of \( R \) and let \( p = q \cap A \). Then \( R_q \) dominates \( A_p \).

Since \( A_p \) is regular and \( R_q \) is Cohen-Macaulay, \( R_q \) is flat over \( A_p \) (cf. EGA [4, IV, 15.4.2]). Hence \( \pi \) is faithfully flat.

(iii) Let \( U := \{ P \in X; \pi^*(P) \) is irreducible and reduced\}. Then, by virtue of [9, Th.4.1.1,p.46], \( W := \pi^{-1}(U) \) is an \( \mathbb{A}^1 \)-bundle over \( U \). We claim that \( \overline{k}(X) = -\infty \).

Let \( H \) be a hyperplane in \( Y = \mathbb{A}^3_k \) such that \( H \cap W \neq \emptyset \).

Suppose \( \overline{k}(X) \geq 0 \). Let \( C \) be an irreducible curve on \( H \).

Consider a morphism:

\[
\varphi : C \times \mathbb{A}^1_k \rightarrow H \times \mathbb{A}^1_k = Y \xrightarrow{\pi} X,
\]

and assume that \( \varphi \) is a dominant morphism. Since \( \dim(C \times \mathbb{A}^1_k) = \dim V = 2 \), we have

\[
-\infty = \overline{k}(C \times \mathbb{A}^1_k) \geq \overline{k}(X) \geq 0,
\]

which is a contradiction. Hence \( \varphi \) is not a dominant morphism.

Let \( D \) be the closure of \( \varphi(C \times \mathbb{A}^1_k) \) in \( X \). Then \( C \times \mathbb{A}^1_k \subset \pi^{-1}(D) \). Suppose \( C \cap W \neq \emptyset \). Then the general fibers of \( \pi : \pi^{-1}(D) \rightarrow D \) are isomorphic to \( \mathbb{A}^1_k \). This implies that \( \pi^{-1}(D) \) is irreducible and reduced. Since \( \dim(C \times \mathbb{A}^1_k) = \dim \pi^{-1}(D) = 2 \), we have \( C \times \mathbb{A}^1_k = \pi^{-1}(D) \).

Let \( Q \) be a point on \( H \), and let \( C_1, \ldots, C_r \) be irreducible
curves on \( H \) such that \( C_1 \cap \ldots \cap C_r = \{ Q \} \) and that \( C_i \cap W \neq \emptyset \) for \( 1 \leq i \leq r \). For any point \( Q \) on \( H \), we can find such a set of irreducible curves. Indeed, \( H \) is the affine plane \( \mathbb{A}_k^2 \) and \( H \cap (Y-W) \) has dimension \( \leq 1 \); thus we have only to take a set of suitably chosen lines on \( H \) passing through \( Q \). Let \( D_i \) be the irreducible curve which is the closure of \( \pi(C_i \times \mathbb{A}_k^1) \) on \( X \) for \( 1 \leq i \leq r \). Then \( C_i \times \mathbb{A}_k^1 = \pi^{-1}(D_i) \) for \( 1 \leq i \leq r \).

Since we have

\[
(Q) \times \mathbb{A}_k^1 = (C_1 \cap \ldots \cap C_r) \times \mathbb{A}_k^1 = (C_1 \times \mathbb{A}_k^1) \cap \ldots \cap (C_r \times \mathbb{A}_k^1) = \pi^{-1}(D_1) \cap \ldots \cap \pi^{-1}(D_r) = \pi^{-1}(D_1 \cap \ldots \cap D_r),
\]

we know that \( D_1 \cap \ldots \cap D_r = \{ P \} \), \( P \) being a point on \( X \). The correspondence \( Q \mapsto P \) defines a morphism \( \psi : H \rightarrow X \) such that \( (Q) \times \mathbb{A}_k^1 = \pi^{-1}(P) \). If \( \psi \) is a dominant morphism, we have

\[
-\infty = \overline{k}(H) \geq \overline{k}(X) \geq 0,
\]
which is a contradiction. Hence \( \psi \) is not a dominant morphism.

Let \( F \) be the closure of \( \psi(H) \) in \( X \). Then, for every point \( P \) of \( F \), we have \( \dim \psi^{-1}(P) \geq 1 \), and \( \pi(\psi^{-1}(P) \times \mathbb{A}_k^1) = \psi(\psi^{-1}(P)) = P \). This contradicts the assertion proved in the step (ii). Therefore \( \overline{k}(X) = -\infty \).

Natural as it is, the situation of \( G_a \)-actions on a polynomial ring \( R = k[x_1, \ldots, x_n] \) becomes complicated and worse as \( n \) increases. If \( n \leq 2 \), \( R \) is a polynomial ring in one variable over the subring \( A \) of \( G_a \)-invariants (cf. [9; §1, Chap.I]). However, this does not hold in the case where \( n = 3 \). Still, the property that \( A \) be a polynomial ring seems to hold without the assumption that \( A \) is regular. When \( n = 3 \),
another criterion for $A$ to be a polynomial ring is that $A$
contains one of coordinates $x_1, x_2, x_3$. Thus, if $G_a$ acts
linearly on $R = k[x_1, x_2, x_3]$, then $A$ is a polynomial ring.
Perhaps, $A$ no longer is a polynomial ring for a general $G_a$-
action on $R$ if $n \geq 4$.

2.3. Finally, we shall state the following result without proof:

THEOREM (cf. [11]). Assume that char($k$) = 0. Let $X =$
Spec$(A)$ be a normal affine surface defined over $k$, possessing
a non-trivial action of the additive group $G_a$. Then every
singular point of $X$ is a cyclic quotient singularity.

The result no longer holds if $X$ is not affine.

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