Various aspects of unipotent group actions in algebraic geometry

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§1. Unipotent group actions on complete varieties

1.1. Let \( k \) be an algebraically closed field of characteristic zero. Let \( G \) be a connected algebraic group defined over \( k \). Assume that \( G \) acts non-trivially on an algebraic variety \( V \),

\[
\sigma : G \times V \longrightarrow V.
\]

Then we have the canonical Lie algebra homomorphism

\[
\sigma_* : \mathfrak{g} := \text{Lie}(G) \longrightarrow \mathcal{D}^0(V, \mathcal{O}_V),
\]

where \( \mathcal{O}_V := (\Omega^1_{V/k})^* \). If \( V \) is smooth over \( k \), \( \mathcal{O}_V \) is a locally free \( O_V \)-Module associated with the tangent bundle \( T_V \). For every element \( \tau \) of \( \mathfrak{g} \), \( \sigma_*(\tau) \) is thus a holomorphic (tangent) vector field of \( V \).

Now assume that \( V \) is a nonsingular projective variety defined over \( k \). Let \( X \) be a holomorphic vector field on \( V \) such that \( X \neq 0 \). A point \( P \) of \( V \) is said to be a zero of \( X \) if \( X(P) = 0 \); the set of all zeros of \( X \) is denoted by
Zero(X), which is a closed subvariety of X. Let P ∈ Zero(X).
Then we can consider the Lie derivative L_X:

\[ L_X : T_{V,P} \to T_{V,P} ; \quad L_X(Y) = [X,Y]. \]

X is said to be generic at P (or X has a simple zero at P) if L_X is nondegenerate on \( T_{V,P} \). X is said to be generic (or X has only simple zeros) if L_X is nondegenerate for every zero P of X. If X has a simple zero at P, we can consider the eigenvalues \( \theta_1(P), \ldots, \theta_n(P) \) of L_X, where \( n = \dim V \). The existence of holomorphic vector fields (or actions of algebraic groups) on V imposes some restrictions on the topology and the numerical characters of V. We shall quote some of the known results.

1.2. Let V be a nonsingular projective variety defined over k and let X be a holomorphic vector field on V such that X ≠ 0. Let Z := Zero(X). Define the contraction operator \( i_X \) as follows:

\[ i_X : \Omega_X^P \to \Omega_X^{P-1} \]

\[ i_X(fdx_1 \wedge \ldots \wedge dx_p) = f(\sum_{i=1}^P (-1)^{i-1} X(x_i)dx_1 \wedge \ldots \wedge dx_{i} \wedge \ldots \wedge dx_p). \]

The definition is well-defined, and if \( \omega^P \) is an element of \( H^0(V, \Omega^P_V) \) then \( i_X(\omega^P) \in H^0(V, \Omega^{P-1}_V) \). Let

\[ \mathcal{Y}^1 := \{ X \in H^0(V, \Theta_V) \mid i_X : H^0(V, \Omega^1_V) \to H^0(V, \Theta_V) \}. \]

is the zero map

Then \( \mathcal{Y}^1 \) is a Lie subalgebra of \( H^0(V, \Theta_V) \).
1.2.1. THEOREM. With the above notations, we have:

(1) (Kobayashi [7]). If 0 ≤ dim Z < n := dim V, then \( P_m(V) = 0 \) for every \( m > 0 \). Hence \( \kappa(V) = -\infty \).

(2) (Carrell–Lieberman [1]). Assume that \( Z \neq \emptyset \). Then

\[ h^p,q = \dim_k H^q(V, \Omega^p_V) = 0 \quad \text{whenever} \quad |p-q| > \dim_k Z. \]

(3) (Carrell–Lieberman [1]). Every element \( X \) of \( \mathfrak{g}_j^{1} \) has zeros. Hence, if \( h^{1,0}(V) = \dim H^0(V, \Omega^1_V) = 0 \) then every holomorphic vector field has zero. Hence, if \( V \) has a holomorphic vector field without zero, \( h^{1,0}(V) > 0 \).

1.2.2. COROLLARY. Assume that \( \dim V = 2 \) and \( V \) has a holomorphic vector field \( X \) with \( \dim \operatorname{Zero}(X) = 0 \). Then \( V \) is rational.

Proof. The assumption \( \dim \operatorname{Zero}(X) = 0 \) implies \( h^{1,0}(V) = 0 \). Since \( X \neq 0 \), we have \( P_m(V) = 0 \) for every \( m > 0 \). Hence \( V \) is rational by Castelnuovo's criterion of rationality.

1.2.3. THEOREM. Let \( k = \mathbb{C} \). Assume that \( V \) has a holomorphic vector field \( X \) possessing only simple zeros. For a point \( P \) of \( \operatorname{Zero}(X) \), let \( \theta_1(P), \ldots, \theta_n(P) \) be the eigenvalues of \( L_X \).

Assume that \( \Re \theta_i(P) \neq 0 \) for \( 1 \leq i \leq n \) and every point \( P \in \operatorname{Zero}(X) \). Then the Betti numbers of \( V \) are given as follows:

\[ b_{2p}(V) = \# \{ P \in \operatorname{Zero}(X) | \# \{ j | \Re \theta_j(P) > 0, 1 \leq j \leq n \} = p \} \]

\[ b_{2p+1}(V) = 0, \quad \text{(cf. Carrell–Lieberman [1]).} \]

1.3. Examples.

(1) Let \( G \) be a semi-simple algebraic group, let \( P \) be
a parabolic subgroup of $G$, let $T$ be a maximal torus with $T \subseteq P$ and let $V := G/P$. Let $t$ be a regular element of infinite order in $T$ such that there exists a one-dimensional subtorus $S$ of $T$ passing through $t$. Let $S$ act on $V$ via left translations of $G$. Let $X$ be a holomorphic vector field on $V$ defined by the canonical Lie algebra homomorphism

$$\alpha_* : \mathfrak{g} := \text{Lie}(S) \longrightarrow H^0(V, \mathcal{O}_V).$$

Then \textbf{Zero}(X) is a finite set and $X$ has only simple zeros.

\textbf{Proof.} We claim that:

$(gP)$ is a fixed point of $S \mapsto g^{-1}tg \in P \mapsto g \in N(T)P$.

Indeed, $S$ is the closure of $\{tm \mid m \in \mathbb{Z}\}$, and hence $(gP)$ is a fixed point if and only if $g^{-1}tg \in P$. Then $t \in gp^{-1}$.

Therefore $g^{-1}tg = p^{-1}tp$ for some element $p \in P$. Hence $gp^{-1} \in N(T)$. Since $\#(N(T)P/P) < +\infty$, there are only finitely many fixed points of $S$ on $V$. Let $(gP)$ be a fixed point of $S$. Let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of $G$ and $P$, respectively.

Now, $T_V, (gP)$ is identified with $\mathfrak{g}/\mathfrak{p}$ via $g_{*,*} : \mathfrak{g}/\mathfrak{p} \longrightarrow T_V, (gP)$.

Then the Lie derivative $L_X$ on $T_V, (gP)$ is identified with

$$Y \pmod{\mathfrak{p}} \mapsto \text{Ad}(g^{-1}tg)(Y) \pmod{\mathfrak{p}}.$$

Noting that $g^{-1}tg \in P$, we know that $L_X$ is non-degenerate at $(gP)$.

(2) Let $V = \mathbb{P}^n_k$ with homogeneous coordinates $(x_0, x_1, \ldots, x_n)$. Let $\alpha_0, \ldots, \alpha_n$ be pairwise prime integers such that $\alpha_0 + \ldots + \alpha_n$
= 0. Let \( G_m \) act on \( V \) via
\[
t(x_0, x_1, \ldots, x_n) = (t^{\alpha_0} x_0, t^{\alpha_1} x_1, \ldots, t^{\alpha_n} x_n).
\]
Then the fixed points of \( G_m \) on \( \mathbb{P}^n \) are \( O_i \)'s, where \( O_i = (0, \ldots, 0, 1, 0, \ldots, 0) \). Let \( u_j = x_j/x_i \) and \( \xi_j = \frac{\partial}{\partial u_j} \) for \( 0 \leq j \leq n \) and \( j \neq i \). Then we have
\[
T_{\mathbb{P}^n, O_i} = \sum_{j=0}^{n} k \xi_j \quad \text{and} \quad L_X(\xi_j) = (\alpha_j - \alpha_i) \xi_j.
\]

Instead, consider the following action of \( G_a \) on \( \mathbb{P}_k^n \),
\[
G_a = \{ \exp(tA) \mid t \in k, A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & 0 \end{pmatrix} \in M_{n+1}(k) \}
\]
\[
t(x_0, x_1, \ldots, x_n) = \exp(tA) \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}.
\]
Then \( O := (1, 0, \ldots, 0) \) is the unique fixed point of \( G_a \). The holomorphic vector field \( X \) on \( \mathbb{P}^n \) defined by this action has the following Lie derivative \( L_X \) on \( T_{n, O} \mathbb{P}^n \):
\[
u_j = x_j/x_0, \quad \xi_j = \frac{\partial}{\partial u_j} \quad (1 \leq j \leq n),
\]
\[
T_{\mathbb{P}^n, O} = \sum_{j=1}^{n} k \xi_j,
\]
\[
L_X(\xi_j) = 0 \quad \text{if} \quad j = 1; = -\xi_{j-1} \quad \text{if} \quad j > 1.
\]
Hence the zero of \( X \) at \( O \) is not simple.

1.4. Now, we shall be mainly interested in the unipotent group actions on complete algebraic varieties. A main problem is the Carrell conjecture, which we shall state below.
Let $G$ be a unipotent algebraic group defined over $k$. We shall summarize some of the known results on unipotent group actions.

1.4.1. THEOREM. (1) [Borel fixed point theorem](cf. Fogarty [2], Horrocks [6]). If a connected solvable affine algebraic group $G$ acts on a complete algebraic variety $V$ then the fixed point locus $V^G$ is nonempty. If $G$ is unipotent, $V^G$ is connected if and only if $V$ is connected.

(2) Let $G$ be a connected affine algebraic group. Then $G$ is unipotent if and only if, for any connected complete variety $V$ on which $G$ acts, $V^G$ is connected (cf. Fogarty [2]).

(3) Let $G$ and $V$ be the same as in (2) above. Then the canonical inclusion $\iota : V^G \hookrightarrow V$ induces an equivalence between the categories of etale coverings $\text{Et}(V)$ and $\text{Et}(V^G)$. In particular, the inclusion $\iota$ yields an isomorphism of algebraic fundamental groups,

$$i_* : \pi_1(V^G)_{\text{alg}} \xrightarrow{\sim} \pi_1(V)_{\text{alg}},$$

(cf. Horrocks [6]).

1.4.2. We also recall the following result:

THEOREM (Matsumura [8]). Assume that $V$ is a nonsingular complete variety. Then the group of all birational automorphisms of $V$, $\text{Bir}(V)$, contains an affine algebraic group of positive dimension if and only if $V$ is birationally equivalent to $\mathbb{P}^1 \times W$ ($V \cong \mathbb{P}^1 \times W$ as notation), where $W$ is a complete variety. Thus, if the Kodaira dimension $\kappa(V) \geq 0$, $\text{Bir}(V)$ cannot contain any affine algebraic group.
1.4.3. Now, we consider the following:

**CARRELL CONJECTURE.** Assume that a connected unipotent group $G$ acts on a nonsingular projective variety $V$ in such a way that the fixed point locus $V^G$ consists of a single point. Then $V$ is rational.

1.4.4. A partial solution of the above conjecture is this:

**THEOREM.** Let $G$ and $V$ be the same as in the Carrell conjecture. Then we have:

1. If $\dim V \leq 2$, the Carrell conjecture is affirmative.
2. If $\dim V = 3$, $V$ is one of the following:
   1. $V$ is rational,
   2. $V \sim \mathbb{P}^1 \times W$, where $W$ is a nonsingular projective surface with $\kappa(W) \geq 1$ and $p_g = q = 0$. Moreover, $W$ is simply connected.

**Proof.** Without loss of generality, we may assume that the action of $G$ is effective, i.e., the canonical homomorphism $G \longrightarrow \text{Aut} (V)$ is injective.

1. The case $\dim V = 1$ is obvious by virtue of Matsumura's theorem. Suppose that $\dim V = 2$. If $\dim G = 1$, the action of $G$ on $V$ gives rise to a holomorphic vector field $X$ on $V$ such that $\text{Zero}(X) = V^G$, which consists of a single point. Then, by virtue of Corollary 1.2.2, $V$ is rational. Assume that $\dim G = 2$. Then $G$ is commutative, i.e., $G \cong G_1 \times G_2$ with $G_1 \cong G_2 \cong G_a$. By virtue of Matsumura's theorem, $V \sim \mathbb{P}^1 \times C$, where $C$ is a complete nonsingular model of $k(V^G)$.

Then $G_2$ acts on $C$ effectively. Hence $C \sim \mathbb{P}^1$, and $V \sim \mathbb{P}^1 \times \mathbb{P}^1$. 

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Namely, $V$ is rational.

(2) Assume that $\dim V = 3$. Consider first the case where $\dim G = 3$. Then $G$ has a central series of subgroups

$$G \supset G_1 \supset G_2 \supset (1),$$

such that $G/G_1 \cong G_1/G_2 \cong G_2/G_0$. By Matsumura's theorem, $V \cong \mathbb{P}^1 \times W$, where $W$ is a nonsingular projective surface such that $W$ is a complete nonsingular model of $k(V^G)$ and the unipotent group $G/G_2$ acts effectively on $W$. By virtue of the above case where $\dim V = \dim G = 2$, we conclude that $W \cong \mathbb{P}^1 \times \mathbb{P}^1$; note that we did not use in the proof the assumption $V^G = \{\text{single point}\}$. Hence $V \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Suppose next that $\dim G = 2$. By a similar reasoning as above, we know that $V \cong \mathbb{P}^1 \times \mathbb{P}^1 \times C$, where $C$ is a nonsingular complete curve.

Since $V^G$ consists of a single point, we know that $\pi_1(V)_{\text{alg}} = (0)$ (cf. Theorem 1.4.1, (3)). Since $\pi_1(V)_{\text{alg}} \cong \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \times C)_{\text{alg}}$, we know that $\pi_1(C)_{\text{alg}} = (0)$, i.e., $C$ is simply connected. This implies that $C \cong \mathbb{P}^1$. Hence $V$ is rational. Suppose finally that $\dim G = 1$. By virtue of Matsumura's theorem, $V \cong \mathbb{P}^1 \times W$, where $W$ is a complete nonsingular model of $k(V^G)$. We may assume that $W$ is relatively minimal. Let $p_1$ and $p_2$ be the canonical projections from $\mathbb{P}^1 \times W$ to $\mathbb{P}^1$ and $W$, respectively. Then we have,

$$\Omega^1_{\mathbb{P}^1 \times W} \cong p_1^* \Omega^1_{\mathbb{P}^1} + p_2^* \Omega^1_W$$

and

$$\Omega^2_{\mathbb{P}^1 \times W} \cong p_1^* \Omega^1_{\mathbb{P}^1} \wedge p_2^* \Omega^1_W + p_2^* \Omega^2_W.$$

By virtue of Theorem 1.2.1, (1), we have $h^{1,0}(V) = 0$ for
i = 1, 2, because the action of $G$ yields a holomorphic vector field $X$ on $V$ with $\text{Zero}(X) = V^G = \{\text{single point}\}$. Since $h^i,0(V)$ is a birational invariant (cf. Griffiths-Harris [16; p. 494]), we know that $h^1,0(W) = h^2,0(W) = 0$. Hence $p_g = q = 0$ for $W$. Moreover, since $\pi_1(V)_{\text{alg}} = (0)$, we know that $\pi_1(W)_{\text{alg}} = (0)$. Namely, $W$ is simply connected. If $W$ is rational, $V$ is rational. Suppose that $W$ is not rational. If $\kappa(W) = 0$ then $p_g = q = 0$ implies that $W$ is an Enriques surface, which is, however, not simply connected. Hence $\kappa(W) \geq 1$.

Q.E.D.

1.5. We shall give the following result on the existence of a $G_a$-action.

**LEMMA** (cf. [9; p. 35]). Let $W$ be a variety defined over $k$ and let $\pi : V \rightarrow W$ be a $\mathbb{P}^1$-bundle over $W$. If there exists a nontrivial $G_a$-action on $V$ whose orbits are contained in fibers of the projection $\pi$, then the fixed point locus $V^G$ contains a cross-section $S$ of $\pi$. Then there exists a locally free $O_W$-module $E$ of rank 2 such that $E$ is an extension of $O_W$ by an invertible sheaf $L$ on $W$, $V \cong \mathbb{P}(E)$, and $S$ is the cross-section corresponding to $L$. Moreover, we have $H^0(W,L^{-1}) \neq 0$. Conversely, if $H^0(W,L^{-1}) \neq 0$, there exists a $G_a$-action on $V$ along fibers of $\pi$.

**Proof.** Let $V^G_1$ be the union of irreducible components of $V^G$ of codimension 1, and consider $V^G_1$ as a reduced effective divisor on $V$. Since $G_a$ acts on $V$ along fibers of $\pi$, each general fiber contains one and only one fixed point. Hence
$(V_1^G \cdot \ell) = 1$, where $\ell$ is a general fiber of $\pi$. This implies that $V_1^G$ contains only one irreducible component $S$, which is a cross-section of $\pi$. Let $L = O_S(S)$ and let $E = \pi_* O_Y(S)$. Then we have an exact sequence,

$$0 \rightarrow O_W \rightarrow E \rightarrow L \rightarrow 0.$$

By construction, $S$ is the cross-section corresponding to $L$. The remaining part is proved in [9; p. 35]. Q.E.D.

§ 2. **Unipotent group actions on affine varieties**

2.1. Let $k$ be an algebraically closed field of characteristic $\geq 0$. Recall the following very well-known results:

2.1.1. **THEOREM** (Nagata [13], Haboush [5]). Let $R$ be a finitely generated $k$-algebra and let $G$ be a connected reductive algebraic group. Assume that $G$ acts on $R$ as $k$-automorphisms of $R$ in such a way that:

For every $f \in R$, a $k$-submodule $\sum_{g \in G} f^g k$ of $R$ is a finite $k$-module; then we say that $G$ acts rationally on $R$.

Let $R^G$ be the subring consisting of $G$-invariant elements in $R$. Then $R^G$ is finitely generated over $k$.

2.1.2. **THEOREM** (Nagata [14]). There exists a unipotent algebraic group $G$ acting rationally on a polynomial ring $R := k[x_1, \ldots, x_n]$ such that $R^G$ is not finitely generated over $k$.

The writer believes that there should exist a rational action of the additive group $G_a$ on a polynomial ring $R = k[x_1, \ldots, x_n]$ such that $R^{G_a}$ is not finitely generated over $k$. If there is
such an action, we must have \( n \geq 4 \) by virtue of Zariski's theorem (cf. Nagata [14]), and the action is not linear by virtue of the following result of Weitzenböck:

2.1.3. THEOREM (cf. Seshadri [15]). Let there be given a linear action of \( G_a \) on a polynomial ring \( R = k[x_1, \ldots, x_n] \), where \( \text{char}(k) = 0 \). Then \( R^a \) is finitely generated over \( k \).

2.2. For the sake of simplicity, we assume that \( \text{char}(k) = 0 \).

THEOREM. Assume that \( G_a \) acts non-trivially on a polynomial ring \( R = k[x_1, \ldots, x_n] \), where \( n \leq 3 \). Let \( A \) be the \( G_a \)-invariant subring of \( R \). Then we have:

1. \( A \) is a finitely generated over \( k \), and \( A \) is a unique factorization domain.

2. If either \( n \leq 2 \) or \( A \) is regular then \( A \) is a polynomial ring over \( k \).

Proof. For the proof of the assertion (1) and the case \( n \leq 2 \) in the assertion (2), see Miyanishi [9; §§ 1, 3 of Chap.1]. We shall prove the assertion (2) in the case where \( n = 3 \) and \( A \) is regular.

(i) By virtue of Zariski's theorem [14; p. 52], \( A \) is finitely generated over \( k \). Moreover, \( A \) is a UFD and the set \( A^* \) of all invertible elements of \( A \) is \( k^* := k-(0) \). Let \( Y := \text{Spec}(R) \), let \( X := \text{Spec}(A) \) and let \( \pi : Y \rightarrow X \) be the dominant morphism induced by the injection \( A \hookrightarrow R \). We shall prove that the logarithmic Kodaira dimension of \( X \) has value \( \kappa(X) = -\infty \).

Then we can apply the following characterization of the affine plane (cf. Miyanishi-Sugie [12] and Fujita [3] as well as the
papers of Iitaka's given in the references of these papers):

Let \( X = \text{Spec}(A) \) be a nonsingular affine surface. Then \( X \cong \mathbb{A}^2_k \) if and only if \( A \) is a UFD, \( A^* = k^* \) and \( \bar{k}(X) = \infty \).

(ii) We claim that \( \pi : Y \to X \) is a faithfully flat, equi-dimensional morphism of dimension 1.

We shall first show that \( \pi \) is surjective. Suppose \( \pi \) is not surjective. Then there exists a maximal ideal \( \mathfrak{m} \) of \( A \) such that \( \mathfrak{m} \mathfrak{R} = \mathfrak{R} \). Let \( (0, t_0) \) be a discrete valuation ring of the quotient field \( K \) of \( A \) such that \( L \) dominates \( A_{\mathfrak{m}} \).

Let \( \mathfrak{R}' = \mathfrak{R} \otimes L \), which is identified with a subring of the field \( L = k(x_1, x_2, x_3) \). Let \( \Delta \) be a locally nilpotent derivation on \( \mathfrak{R} \) associated with the given \( G_a \)-action on \( Y \) (cf. [9; § 1, Chap. I]). Then \( \Delta \) extends naturally to a locally nilpotent \( \mathfrak{R} \)-derivation in \( \mathfrak{R}' \), and \( \mathfrak{R} \) is the ring of \( \Delta \)-invariants in \( \mathfrak{R}' \), i.e., \( \mathfrak{R} = \{ r \in \mathfrak{R}'; \Delta(r) = 0 \} \). By assumption, we have \( t\mathfrak{R}' = \mathfrak{R}' \), where \( t \) is a uniformisant of \( \mathfrak{R} \). Hence \( t\mathfrak{R} = 1 \) for some element \( r \in \mathfrak{R}' \). Then \( t\Delta(r) = 0 \), whence \( r \in \mathfrak{R} \). This is a contradiction. Thus \( \pi \) is surjective.

Secondly, we shall show that every irreducible component of a fiber of \( \pi \) has dimension 1. Note that general fibers of \( \pi \) are isomorphic to \( \mathbb{A}^1_k \) (cf. [9; § 1, Chap. I]). Hence each irreducible component of a fiber has dimension \( \geq 1 \). Suppose that an irreducible component \( T \) of a fiber \( \pi^*(P) \) (with \( P \in X \)) has dimension 2. Since \( \mathfrak{R} \) is a UFD, there exists an irreducible element \( a \in \mathfrak{R} \) such that \( T = \text{Spec}(\mathfrak{R}/a\mathfrak{R}) \). Since \( T \) is \( G_a \)-stable, \( a \) is \( G_a \)-invariant, i.e., \( a \in A \). Let \( C = \ldots \)
Spec(A/aA). Since A is a UFD, C is an irreducible curve on X and $\pi^{-1}(C) = T \subset \pi^{-1}(P)$. This is a contradiction because $\pi$ is surjective. Thus $\pi$ is an equi-dimensional morphism of dimension 1.

Finally, we shall show that $R$ is flat over $A$. Let $q$ be a prime ideal of $R$ and let $p = q \cap A$. Then $R_q$ dominates $A_p$. Since $A_p$ is regular and $R_q$ is Cohen-Macaulay, $R_q$ is flat over $A_p$ (cf. EGA [4; IV,15.4.2]). Hence $\pi$ is faithfully flat.

(iii) Let $U := \{P \in X; \pi^*(P) \text{ is irreducible and reduced} \}$. Then, by virtue of [9; Th.4.1.1,p.46], $W := \pi^{-1}(U)$ is an $A^1$-bundle over $U$. We claim that $\kappa(X) = -\infty$.

Let $H$ be a hyperplane in $Y = A^3_k$ such that $H \cap W \neq \emptyset$. Suppose $\kappa(X) \geq 0$. Let C be an irreducible curve on $H$. Consider a morphism:

$$\varphi : C \times A^1_k \hookrightarrow H \times A^1_k = Y \xrightarrow{\pi} X,$$

and assume that $\varphi$ is a dominant morphism. Since $\dim(C \times A^1_k) = \dim V = 2$, we have

$$-\infty = \kappa(C \times A^1_k) \geq \kappa(X) \geq 0,$$

which is a contradiction. Hence $\varphi$ is not a dominant morphism.

Let $D$ be the closure of $\varphi(C \times A^1_k)$ in $X$. Then $C \times A^1_k \subset \pi^{-1}(D)$. Suppose $C \cap W \neq \emptyset$. Then the general fibers of $\pi : \pi^{-1}(D) \rightarrow D$ are isomorphic to $A^1_k$. This implies that $\pi^{-1}(D)$ is irreducible and reduced. Since $\dim(C \times A^1_k) = \dim \pi^{-1}(D) = 2$, we have $C \times A^1_k = \pi^{-1}(D)$.

Let $Q$ be a point on $H$, and let $C_1, \ldots, C_r$ be irreducible
curves on $H$ such that $C_1 \cap \ldots \cap C_r = \{Q\}$ and that $C_i \cap W \neq \emptyset$ for $1 \leq i \leq r$. For any point $Q$ on $H$, we can find such a set of irreducible curves. Indeed, $H$ is the affine plane $\mathbb{A}^2_k$ and $H \cap (Y-W)$ has dimension $\leq 1$; thus we have only to take a set of suitably chosen lines on $H$ passing through $Q$. Let $D_i$ be the irreducible curve which is the closure of $\pi(C_i \times \mathbb{A}^1_k)$ on $X$ for $1 \leq i \leq r$. Then $C_i \times \mathbb{A}^1_k = \pi^{-1}(D_i)$ for $1 \leq i \leq r$. Since we have

$$(Q) \times \mathbb{A}^1_k = (C_1 \cap \ldots \cap C_r) \times \mathbb{A}^1_k = (C_1 \times \mathbb{A}^1_k) \cap \ldots \cap (C_r \times \mathbb{A}^1_k) = \pi^{-1}(D_1) \cap \ldots \cap \pi^{-1}(D_r) = \pi^{-1}(D_1 \cap \ldots \cap D_r),$$

we know that $D_1 \cap \ldots \cap D_r = \{P\}$, $P$ being a point on $X$. The correspondence $Q \mapsto P$ defines a morphism $\psi : H \rightarrow X$ such that $(Q) \times \mathbb{A}^1_k = \pi^{-1}(P)$. If $\psi$ is a dominant morphism, we have

$$-\infty = \bar{k}(H) \geq \bar{k}(X) \geq 0,$$

which is a contradiction. Hence $\psi$ is not a dominant morphism. Let $F$ be the closure of $\psi(H)$ in $X$. Then, for every point $P$ of $F$, we have $\dim \psi^{-1}(P) \geq 1$, and $\pi(\psi^{-1}(P) \times \mathbb{A}^1_k) = \psi(\psi^{-1}(P)) = P$. This contradicts the assertion proved in the step (ii). Therefore $\bar{k}(X) = -\infty$. Q.E.D.

Natural as it is, the situation of $G_a$-actions on a polynomial ring $R = k[x_1, \ldots, x_n]$ becomes complicated and worse as $n$ increases. If $n \leq 2$, $R$ is a polynomial ring in one variable over the subring $A$ of $G_a$-invariants (cf. [9; §1, Chap.I]). However, this does not hold in the case where $n = 3$. Still, the property that $A$ be a polynomial ring seems to hold without the assumption that $A$ is regular. When $n = 3$, - 4 -
another criterion for $A$ to be a polynomial ring is that $A$
contains one of coordinates $x_1, x_2, x_3$. Thus, if $G_a$
acts linearly on $R = k[x_1, x_2, x_3]$, then $A$ is a polynomial ring.
Perhaps, $A$ no longer is a polynomial ring for a general $G_a$-
action on $R$ if $n \geq 4$.

2.3. Finally, we shall state the following result without proof:

**THEOREM (cf. [11]).** Assume that $\text{char}(k) = 0$. Let $X =$
$\text{Spec}(A)$ be a normal affine surface defined over $k$, possessing
a non-trivial action of the additive group $G_a$. Then every
singular point of $X$ is a cyclic quotient singularity.

The result no longer holds if $X$ is not affine.

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