<table>
<thead>
<tr>
<th>Title</th>
<th>Various Aspects of unipotent Group Actions in Algebraic Geometry (Lie Algebras, Algebraic Groups and Related Topics)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>MIYANISHI, MASAYOSHI</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1980), 394: 229-244</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1980-08</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/104981">http://hdl.handle.net/2433/104981</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Various aspects of unipotent group actions in algebraic geometry

Masayoshi Miyanishi (Osaka University)

§ 1. Unipotent group actions on complete varieties

1.1. Let $k$ be an algebraically closed field of characteristic zero. Let $G$ be a connected algebraic group defined over $k$. Assume that $G$ acts non-trivially on an algebraic variety $V$,

$$\sigma : G \times V \to V.$$  

Then we have the canonical Lie algebra homomorphism

$$\sigma_* : \mathfrak{g} := \text{Lie}(G) \to H^0(V, \mathcal{O}_V),$$

where $\mathcal{O}_V := (\Omega^1_{V/k})^*$. If $V$ is smooth over $k$, $\mathcal{O}_V$ is a locally free $O_V$-module associated with the tangent bundle $T_V$. For every element $\tau$ of $\mathfrak{g}$, $\sigma_*(\tau)$ is thus a holomorphic (tangent) vector field of $V$.

Now assume that $V$ is a nonsingular projective variety defined over $k$. Let $X$ be a holomorphic vector field on $V$ such that $X \neq 0$. A point $P$ of $V$ is said to be a zero of $X$ if $X(P) = 0$; the set of all zeros of $X$ is denoted by

---
Zero(X), which is a closed subvariety of X. Let P ∈ Zero(X). Then we can consider the Lie derivative L_X:

\[ L_X : T_{V,P} \rightarrow T_{V,P} ; \quad L_X(Y) = [X,Y]. \]

X is said to be generic at P (or X has a simple zero at P) if \( L_X \) is nondegenerate on \( T_{V,P} \); X is said to be generic (or X has only simple zeros) if \( L_X \) is nondegenerate for every zero \( P \) of X. If X has a simple zero at P, we can consider the eigenvalues \( \theta_1(P), \ldots, \theta_n(P) \) of \( L_X \), where \( n = \dim V \). The existence of holomorphic vector fields (or actions of algebraic groups) on V imposes some restrictions on the topology and the numerical characters of V. We shall quote some of the known results.

1.2. Let V be a nonsingular projective variety defined over k and let X be a holomorphic vector field on V such that \( X \neq 0 \). Let \( Z := \text{Zero}(X) \). Define the contraction operator \( i_X \) as follows:

\[ i_X : \Omega^P_X \rightarrow \Omega^{P-1}_X \]

\[ i_X(\sum_{i=1}^{P} (-1)^{i-1} X(x_i) dx_1 \wedge \ldots \wedge dx_i \wedge \ldots \wedge dx_P) = f(\sum_{i=1}^{P} (-1)^{i-1} X(x_i) dx_1 \wedge \ldots \wedge dx_i \wedge \ldots \wedge dx_P). \]

The definition is well-defined, and if \( \omega^P \) is an element of \( H^0(V, \Omega^P_V) \) then \( i_X(\omega^P) \in H^0(V, \Omega^{P-1}_V) \). Let

\[ \mathcal{F}_1 := \{ X \in H^0(V, \Theta_V) \mid i_X : H^0(V, \Omega^1_V) \rightarrow H^0(V, \mathcal{O}_V) \}. \]

is the zero map. Then \( \mathcal{F}_1 \) is a Lie subalgebra of \( H^0(V, \Theta_V) \).
1.2.1. **THEOREM.** With the above notations, we have:

(1) (Kobayashi [7]). If \(0 < \dim Z < n:= \dim V\), then \(\mathcal{P}_m(V) = 0\) for every \(m > 0\). Hence \(\kappa(V) = -\infty\).

(2) (Carrell–Lieberman [1]). Assume that \(Z \neq \emptyset\). Then

\[
\mathcal{H}^p_q = \dim_k \mathcal{H}^q(V, \Omega^p_V) = 0 \text{ whenever } |p-q| > \dim_k Z.
\]

(3) (Carrell–Lieberman [1]). Every element \(X\) of \(\mathcal{F}_f^1\) has zeros. Hence, if \(h^{1,0}(V) = \dim \mathcal{H}^0(V, \Omega^1_V) = 0\) then every holomorphic vector field has zero. Hence, if \(V\) has a holomorphic vector field without zero, \(h^{1,0}(V) > 0\).

1.2.2. **COROLLARY.** Assume that \(\dim V = 2\) and \(V\) has a holomorphic vector field \(X\) with \(\dim \text{Zero}(X) = 0\). Then \(V\) is rational.

**Proof.** The assumption \(\dim \text{Zero}(X) = 0\) implies \(h^{1,0}(V) = 0\). Since \(X \neq 0\), we have \(\mathcal{P}_m(V) = 0\) for every \(m > 0\). Hence \(V\) is rational by Castelnuovo's criterion of rationality.

1.2.3. **THEOREM.** Let \(k = \mathbb{C}\). Assume that \(V\) has a holomorphic vector field \(X\) possessing only simple zeros. For a point \(P\) of \(\text{Zero}(X)\), let \(\theta_1(P), \ldots, \theta_n(P)\) be the eigenvalues of \(L_X\).

Assume that \(\Re \theta_i(P) \neq 0\) for \(1 \leq i \leq n\) and every point \(P \in \text{Zero}(X)\). Then the Betti numbers of \(V\) are given as follows:

\[
\begin{align*}
\mathcal{b}_2(P)(V) &= \#\{P \in \text{Zero}(X)\} \quad \#\{j \mid \Re \theta_j(P) > 0, 1 \leq j \leq n\} = p \\
\mathcal{b}_{2p+1}(V) &= 0, \text{ (cf. Carrell–Lieberman [1])}.
\end{align*}
\]

1.3. **Examples.**

(1) Let \(G\) be a semi-simple algebraic group, let \(P\) be
a parabolic subgroup of $G$, let $T$ be a maximal torus with $T \subset P$ and let $V := G/P$. Let $t$ be a regular element of infinite order in $T$ such that there exists a one-dimensional subtorus $S$ of $T$ passing through $t$. Let $S$ act on $V$ via left translations of $G$. Let $X$ be a holomorphic vector field on $V$ defined by the canonical Lie algebra homomorphism

$$\alpha_* : \mathfrak{g}_\mathfrak{g} := \text{Lie}(S) \to H^0(V, \mathcal{O}_V).$$

Then $\text{Zero}(X)$ is a finite set and $X$ has only simple zeros.

**Proof.** We claim that:

$(gP)$ is a fixed point of $S \xrightarrow{g^{-1}t g} P \xrightarrow{g} N(T)P$.

Indeed, $S$ is the closure of $\{ t^m | m \in \mathbb{Z} \}$, and hence $(gP)$ is a fixed point if and only if $g^{-1}t g \in P$. Then $t \in gP^{-1}$. Since $t$ is a regular element, $T \subset gP^{-1}$. Hence $g^{-1}T g \subset P$.

Therefore $g^{-1}T g = p^{-1}T p$ for some element $p \in P$. Hence $gP^{-1} \subset N(T)$. Since $\#(N(T)P/P) < +\infty$, there are only finitely many fixed points of $S$ on $V$. Let $(gP)$ be a fixed point of $S$. Let $\mathfrak{g}$ and $\mathfrak{p}$ be the Lie algebras of $G$ and $P$, respectively.

Now, $T_V,(gP)$ is identified with $\mathfrak{g}/\mathfrak{p}$ via $\mathfrak{g}, \star : \mathfrak{g}/\mathfrak{p} \xrightarrow{\sim} T_V,(gP)$.

Then the Lie derivative $L_X$ on $T_V,(gP)$ is identified with $Y \pmod{\mathfrak{p}} \xrightarrow{\text{Ad}(g^{-1}t g)(Y) \pmod{\mathfrak{p}}}$.

Noting that $g^{-1}t g \in P$, we know that $L_X$ is non-degenerate at $(gP)$.

(2) Let $V = \mathbb{P}^n_k$ with homogeneous coordinates $(x_0, x_1, \ldots, x_n)$. Let $\alpha_0, \ldots, \alpha_n$ be pairwise prime integers such that $\alpha_0 + \ldots + \alpha_n$
= 0. Let $G_m$ act on $V$ via
\[ t(x_0, x_1, \ldots, x_n) = (t^{a_0}x_0, t^{a_1}x_1, \ldots, t^{a_n}x_n). \]
Then the fixed points of $G_m$ on $\mathbb{P}^n$ are $0_i's$, where $0_i = (0, \ldots, 0, 1, 0, \ldots, 0)$. Let $u_j = x_j/x_i$ and $\xi_j = \frac{\partial}{\partial u_j}$ for $0 \leq j \leq n$ and $j \neq i$. Then we have
\[ T_{0_i} = \sum_{j=0}^{n} k^j \xi_j \text{ and } L_x(\xi_j) = (a_j - a_i) \xi_j. \]
Instead, consider the following action of $G_a$ on $\mathbb{P}^n_k$,
\[ G_a = \{ \exp(tA) \mid t \in k, A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \ddots & 0 \\ 0 & \cdots & 1 \end{pmatrix} \in M_{n+1}(k) \} \]
\[ t \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix} = \exp(tA) \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_n \end{pmatrix}. \]
Then $0 := (1, 0, \ldots, 0)$ is the unique fixed point of $G_a$. The holomorphic vector field $X$ on $\mathbb{P}^n$ defined by this action has the following Lie derivative $L_x$ on $T_{0_i} \mathbb{P}^n$:
\[ u_j = x_j/x_0, \quad \xi_j = \frac{\partial}{\partial u_j} \quad (1 \leq j \leq n), \]
\[ T_{0_i} = \sum_{j=1}^{n} k^j \xi_j, \]
\[ L_x(\xi_j) = 0 \text{ if } j = 1; = -\xi_{j-1} \text{ if } j > 1. \]
Hence the zero of $X$ at $0$ is not simple.

1.4. Now, we shall be mainly interested in the unipotent group actions on complete algebraic varieties. A main problem is the Carrell conjecture, which we shall state below.
Let $G$ be a unipotent algebraic group defined over $k$. We shall summarize some of the known results on unipotent group actions.

1.4.1. **THEOREM.** (1) [Borel fixed point theorem] (cf. Fogarty [2], Horrocks [6]). *If a connected solvable affine algebraic group $G$ acts on a complete algebraic variety $V$ then the fixed point locus $V^G$ is nonempty. If $G$ is unipotent, $V^G$ is connected if and only if $V$ is connected.*

(2) *Let $G$ be a connected affine algebraic group. Then $G$ is unipotent if and only if, for any connected complete variety $V$ on which $G$ acts, $V^G$ is connected (cf. Fogarty [2]).*

(3) *Let $G$ and $V$ be the same as in (2) above. Then the canonical inclusion $\iota : V^G \hookrightarrow V$ induces an equivalence between the categories of etale coverings $\text{Et}(V)$ and $\text{Et}(V^G)$. In particular, the inclusion $\iota$ yields an isomorphism of algebraic fundamental groups, $\iota_* : \pi_1(V^G)_{\text{alg}} \cong \pi_1(V)_{\text{alg}},$ (cf. Horrocks [6]).*

1.4.2. We also recall the following result:

**THEOREM** (Matsumura [8]). *Assume that $V$ is a nonsingular complete variety. Then the group of all birational automorphisms of $V$, $\text{Bir}(V)$, contains an affine algebraic group of positive dimension if and only if $V$ is birationally equivalent to $\mathbb{P}^1 \times W$ ($V \sim \mathbb{P}^1 \times W$ as notation), where $W$ is a complete variety. Thus, if the Kodaira dimension $\kappa(V) > 0$, $\text{Bir}(V)$ cannot contain any affine algebraic group.*
1.4.3. Now, we consider the following:

**CARRELL CONJECTURE.** Assume that a connected unipotent group $G$ acts on a nonsingular projective variety $V$ in such a way that the fixed point locus $V^G$ consists of a single point. Then $V$ is rational.

1.4.4. A partial solution of the above conjecture is this:

**THEOREM.** Let $G$ and $V$ be the same as in the Carrell conjecture. Then we have:

1. If $\dim V \leq 2$, the Carrell conjecture is affirmative.

2. If $\dim V = 3$, $V$ is one of the following:
   
   (i) $V$ is rational,
   
   (ii) $V \cong \mathbb{P}^1 \times W$, where $W$ is a nonsingular projective surface with $\kappa(W) \geq 1$ and $p_g = q = 0$. Moreover, $W$ is simply connected.

**Proof.** Without loss of generality, we may assume that the action of $G$ is effective, i.e., the canonical homomorphism $G \rightarrow \text{Aut}(V)$ is injective.

1. The case $\dim V = 1$ is obvious by virtue of Matsumura's theorem. Suppose that $\dim V = 2$. If $\dim G = 1$, the action of $G$ on $V$ gives rise to a holomorphic vector field $X$ on $V$ such that $\text{Zero}(X) = V^G$, which consists of a single point. Then, by virtue of Corollary 1.2.2, $V$ is rational. Assume that $\dim G = 2$. Then $G$ is commutative, i.e., $G \cong G_1 \times G_2$ with $G_1 \cong G_2 \cong G_a$. By virtue of Matsumura's theorem, $V \cong \mathbb{P}^1 \times C$, where $C$ is a complete nonsingular model of $k(V^G)$. Then $G_2$ acts on $C$ effectively. Hence $C \cong \mathbb{P}^1$, and $V \cong \mathbb{P}^1 \times \mathbb{P}$. 

---
Namely, $V$ is rational.

(2) Assume that $\dim V = 3$. Consider first the case where $\dim G = 3$. Then $G$ has a central series of subgroups

$$G \supset G_1 \supset G_2 \supset (1),$$

such that $G/G_1 \cong G_1/G_2 \cong G_2 \cong G$. By Matsumura's theorem, $V \sim \mathbb{P}^1 \times W$, where $W$ is a nonsingular projective surface such that $W$ is a complete nonsingular model of $k(V^G)$ and the unipotent group $G/G_2$ acts effectively on $W$. By virtue of the above case where $\dim V = \dim G = 2$, we conclude that $W \sim \mathbb{P}^1 \times \mathbb{P}^1$; note that we did not use in the proof the assumption $V^G = \{\text{single point}\}$. Hence $V \sim \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Suppose next that $\dim G = 2$. By a similar reasoning as above, we know that $V \sim \mathbb{P}^1 \times \mathbb{P}^1 \times C$, where $C$ is a nonsingular complete curve. Since $V^G$ consists of a single point, we know that $\pi_1(V)_{\text{alg}} = (0)$ (cf. Theorem 1.4.1, (3)). Since $\pi_1(V)_{\text{alg}} \cong \pi_1(\mathbb{P}^1 \times \mathbb{P}^1 \times C)_{\text{alg}}$, we know that $\pi_1(C)_{\text{alg}} = (0)$, i.e., $C$ is simply connected. This implies that $C \sim \mathbb{P}^1$. Hence $V$ is rational. Suppose finally that $\dim G = 1$. By virtue of Matsumura's theorem, $V \sim \mathbb{P}^1 \times W$, where $W$ is a complete nonsingular model of $k(V^G)$. We may assume that $W$ is relatively minimal. Let $p_1$ and $p_2$ be the canonical projections from $\mathbb{P}^1 \times W$ to $\mathbb{P}^1$ and $W$, respectively. Then we have,

$$\Omega^1_{\mathbb{P}^1 \times W} \cong p_1^* \Omega^1_{\mathbb{P}^1} + p_2^* \Omega^1_W$$

$$\Omega^2_{\mathbb{P}^1 \times W} \cong p_1^* \Omega^1_{\mathbb{P}^1} \wedge p_2^* \Omega^1_W + p_2^* \Omega^2_W.$$

By virtue of Theorem 1.2.1, (1), we have $h^{1,0}(V) = 0$ for
i = 1, 2, because the action of G yields a holomorphic vector field X on V with \( \text{Zero}(X) = V^G = \{ \text{single point} \} \). Since \( h^{1,0}(V) \) is a birational invariant (cf. Griffiths-Harris [16; p.494]), we know that \( h^{1,0}(W) = h^{2,0}(W) = 0 \). Hence \( p_g = q = 0 \) for W. Moreover, since \( \pi_1(V)_{\text{alg}} = (0) \), we know that \( \pi_1(W)_{\text{alg}} = (0) \). Namely, W is simply connected. If W is rational, V is rational. Suppose that W is not rational. If \( \kappa(W) = 0 \) then \( p_g = q = 0 \) implies that W is an Enriques surface, which is, however, not simply connected. Hence \( \kappa(W) \geq 1 \).

Q.E.D.

1.5. We shall give the following result on the existence of a \( G_a \)-action.

**LEMMA** (cf. [9; p. 35]). Let W be a variety defined over k and let \( \pi : V \rightarrow W \) be a \( \mathbb{P}^1 \)-bundle over W. If there exists a nontrivial \( G_a \)-action on V whose orbits are contained in fibers of the projection \( \pi \), then the fixed point locus \( V^G \) contains a cross-section \( S \) of \( \pi \). Then there exists a locally free \( O_W \)-module \( E \) of rank 2 such that \( E \) is an extension of \( O_W \) by an invertible sheaf \( L \) on W, \( V \supset \mathbb{P}(E) \), and \( S \) is the cross-section corresponding to \( L \). Moreover, we have \( H^0(W,L^{-1}) \neq 0 \). Conversely, if \( H^0(W,L^{-1}) \neq 0 \), there exists a \( G_a \)-action on V along fibers of \( \pi \).

**Proof.** Let \( V^G_1 \) be the union of irreducible components of \( V^G \) of codimension 1, and consider \( V^G_1 \) as a reduced effective divisor on V. Since \( G_a \) acts on V along fibers of \( \pi \), each general fiber contains one and only one fixed point. Hence
$(V^G \cdot \ell) = 1$, where $\ell$ is a general fiber of $\pi$. This implies that $V^G$ contains only one irreducible component $S$, which is a cross-section of $\pi$. Let $L = O_S(S)$ and let $E = \pi_* O_V(S)$. Then we have an exact sequence,

$$0 \rightarrow O_W \rightarrow E \rightarrow L \rightarrow 0.$$  

By construction, $S$ is the cross-section corresponding to $L$. The remaining part is proved in [9; p. 35]. Q.E.D.

§ 2. Unipotent group actions on affine varieties

2.1. Let $k$ be an algebraically closed field of characteristic $\geq 0$. Recall the following very well-known results:

2.1.1. THEOREM (Nagata [13], Haboush [5]). Let $R$ be a finitely generated $k$-algebra and let $G$ be a connected reductive algebraic group. Assume that $G$ acts on $R$ as $k$-automorphisms of $R$ in such a way that:

For every $f \in R$, a $k$-submodule $\Sigma_{g \in G} f^g_k$ of $R$ is a finite $k$-module; then we say that $G$ acts rationally on $R$.

Let $R^G$ be the subring consisting of $G$-invariant elements in $R$. Then $R^G$ is finitely generated over $k$.

2.1.2. THEOREM (Nagata [14]). There exists a unipotent algebraic group $G$ acting rationally on a polynomial ring $R := k[x_1, \ldots, x_n]$ such that $R^G$ is not finitely generated over $k$.

The writer believes that there should exist a rational action of the additive group $G_a$ on a polynomial ring $R = k[x_1, \ldots, x_n]$ such that $R^{G_a}$ is not finitely generated over $k$. If there is
such an action, we must have \( n \geq 4 \) by virtue of Zariski's theorem (cf. Nagata [14]), and the action is not linear by virtue of the following result of Weitzenböck:

2.1.3. THEOREM (cf. Seshadri [15]). Let there be given a linear action of \( G_\alpha \) on a polynomial ring \( R = k[x_1, \ldots, x_n] \), where \( \text{char}(k) = 0 \). Then \( R^G_\alpha \) is finitely generated over \( k \).

2.2. For the sake of simplicity, we assume that \( \text{char}(k) = 0 \).

THEOREM. Assume that \( G_\alpha \) acts non-trivially on a polynomial ring \( R = k[x_1, \ldots, x_n] \), where \( n \leq 3 \). Let \( A \) be the \( G_\alpha \)-invariant subring of \( R \). Then we have:

(1) \( A \) is a finitely generated over \( k \), and \( A \) is a unique factorization domain.

(2) If either \( n \leq 2 \) or \( A \) is regular then \( A \) is a polynomial ring over \( k \).

Proof. For the proof of the assertion (1) and the case \( n \leq 2 \) in the assertion (2), see Miyanishi [9; §§ 1, 3 of Chap.I]. We shall prove the assertion (2) in the case where \( n = 3 \) and \( A \) is regular.

(i) By virtue of Zariski's theorem [14; p. 52], \( A \) is finitely generated over \( k \). Moreover, \( A \) is a UFD and the set \( A^* \) of all invertible elements of \( A \) is \( k^* := k - (0) \). Let \( Y := \text{Spec}(R) \), let \( X := \text{Spec}(A) \) and let \( \pi : Y \longrightarrow X \) be the dominant morphism induced by the injection \( A \hookrightarrow R \). We shall prove that the logarithmic Kodaira dimension of \( X \) has value \( \overline{\kappa}(X) = -\infty \).

Then we can apply the following characterization of the affine plane (cf. Miyanishi-Sugie [12] and Fujita [3] as well as the
papers of Iitaka's given in the references of these papers):

Let $X = \text{Spec}(A)$ be a nonsingular affine surface. Then $X \cong \mathbb{A}^2_k$ if and only if $A$ is a UFD, $A^* = k^*$ and $\bar{k}(X) = -\infty$.

(ii) We claim that $\pi : Y \to X$ is a faithfully flat, equi-dimensional morphism of dimension 1.

We shall first show that $\pi$ is surjective. Suppose $\pi$ is not surjective. Then there exists a maximal ideal $\mathfrak{m}$ of $A$ such that $\mathfrak{m}R = R$. Let $(\mathcal{O}, t\mathcal{O})$ be a discrete valuation ring of the quotient field $K$ of $A$ such that $\mathcal{O}$ dominates $A_{\mathfrak{m}}$. Let $R' := R \otimes \mathcal{O}$, which is identified with a subring of the field $L := k(x_1, x_2, x_3)$. Let $\Delta$ be a locally nilpotent derivation on $R$ associated with the given $G_a$-action on $Y$ (cf. [9; § 1, Chap. I]). Then $\Delta$ extends naturally to a locally nilpotent $\mathcal{O}$-derivation in $R'$, and $\mathcal{O}$ is the ring of $\Delta$-invariants in $R'$, i.e., $\mathcal{O} = \{r \in R'; \Delta(r) = 0\}$. By assumption, we have $tR' = R'$, where $t$ is a uniformisant of $\mathcal{O}$. Hence $tr = 1$ for some element $r \in R'$. Then $t\Delta(r) = 0$, whence $r \in \mathcal{O}$. This is a contradiction. Thus $\pi$ is surjective.

Secondly, we shall show that every irreducible component of a fiber of $\pi$ has dimension 1. Note that general fibers of $\pi$ are isomorphic to $\mathbb{A}^1_k$ (cf. [9; § 1, Chap. I]). Hence each irreducible component of a fiber has dimension $\geq 1$. Suppose that an irreducible component $T$ of a fiber $\pi^*(P)$ (with $P \in X$) has dimension 2. Since $R$ is a UFD, there exists an irreducible element $a \in R$ such that $T = \text{Spec}(R/aR)$. Since $T$ is $G_a$-stable, $a$ is $G_a$-invariant, i.e., $a \in A$. Let $C :=$
Spec(A/A). Since A is a UFD, C is an irreducible curve on X and \( \pi^{-1}(C) = T \subset \pi^{-1}(P) \). This is a contradiction because \( \pi \) is surjective. Thus \( \pi \) is an equi-dimensional morphism of dimension 1.

Finally, we shall show that \( R \) is flat over \( A \). Let \( q \) be a prime ideal of \( R \) and let \( p = q \cap A \). Then \( R_q \) dominates \( A_p \).

Since \( A_p \) is regular and \( R_q \) is Cohen-Macaulay, \( R_q \) is flat over \( A_p \) (cf. EGA [4; IV,15.4.2]). Hence \( \pi \) is faithfully flat.

(iii) Let \( U := \{ p \in X; \pi^*(p) \text{ is irreducible and reduced} \} \).

Then, by virtue of [9; Th.4.1.1,p.46], \( W := \pi^{-1}(U) \) is an \( \mathbb{A}^1 \)-bundle over \( U \). We claim that \( \overline{\kappa}(X) = -\infty \).

Let \( H \) be a hyperplane in \( Y = \mathbb{A}^3_k \) such that \( H \cap W \neq \emptyset \).

Suppose \( \overline{\kappa}(X) \geq 0 \). Let \( C \) be an irreducible curve on \( H \).

Consider a morphism:

\[
\varphi: C \times \mathbb{A}_k^1 \hookrightarrow H \times \mathbb{A}_k^1 = Y \xrightarrow{\pi} X,
\]

and assume that \( \varphi \) is a dominant morphism. Since \( \dim(C \times \mathbb{A}_k^1) = \dim V = 2 \), we have

\[
-\infty = \overline{\kappa}(C \times \mathbb{A}_k^1) \geq \overline{\kappa}(X) \geq 0,
\]

which is a contradiction. Hence \( \varphi \) is not a dominant morphism.

Let \( D \) be the closure of \( \varphi(C \times \mathbb{A}_k^1) \) in \( X \). Then \( C \times \mathbb{A}_k^1 \subset \pi^{-1}(D) \). Suppose \( C \cap W \neq \emptyset \). Then the general fibers of \( \pi: \pi^{-1}(D) \to D \) are isomorphic to \( \mathbb{A}_k^1 \). This implies that \( \pi^{-1}(D) \) is irreducible and reduced. Since \( \dim(C \times \mathbb{A}_k^1) = \dim \pi^{-1}(D) = 2 \), we have \( C \times \mathbb{A}_k^1 = \pi^{-1}(D) \).

Let \( Q \) be a point on \( H \), and let \( C_1, \ldots, C_r \) be irreducible
curves on $H$ such that $C_1 \cap \ldots \cap C_r = \{Q\}$ and that $C_i \not\cap W \neq \emptyset$ for $1 \leq i \leq r$. For any point $Q$ on $H$, we can find such a set of irreducible curves. Indeed, $H$ is the affine plane $\mathbb{A}_k^2$ and $H \cap (Y-W)$ has dimension $\leq 1$; thus we have only to take a set of suitably chosen lines on $H$ passing through $Q$. Let $D_i$ be the irreducible curve which is the closure of $\pi(C_i \times \mathbb{A}_k^1)$ on $X$ for $1 \leq i \leq r$. Then $C_i \times \mathbb{A}_k^1 = \pi^{-1}(D_i)$ for $1 \leq i \leq r$. Since we have

$$(Q) \times \mathbb{A}_k^1 = (C_1 \cap \ldots \cap C_r) \times \mathbb{A}_k^1 = (C_1 \times \mathbb{A}_k^1) \cap \ldots \cap (C_r \times \mathbb{A}_k^1)$$

$$= \pi^{-1}(D_1) \cap \ldots \cap \pi^{-1}(D_r) = \pi^{-1}(D_1 \cap \ldots \cap D_r),$$

we know that $D_1 \cap \ldots \cap D_r = \{P\}$, $P$ being a point on $X$. The correspondence $Q \mapsto P$ defines a morphism $\psi : H \to X$ such that $(Q) \times \mathbb{A}_k^1 = \pi^{-1}(P)$. If $\psi$ is a dominant morphism, we have

$$-\infty = \overline{k}(H) \geq \overline{k}(X) \geq 0,$$

which is a contradiction. Hence $\psi$ is not a dominant morphism. Let $F$ be the closure of $\psi(H)$ in $X$. Then, for every point $P$ of $F$, we have $\dim \psi^{-1}(P) \geq 1$, and $\pi(\psi^{-1}(P) \times \mathbb{A}_k^1) = \psi(\psi^{-1}(P)) = P$. This contradicts the assertion proved in the step (ii). Therefore $\overline{k}(X) = -\infty.$ Q.E.D.

Natural as it is, the situation of $G_a$-actions on a polynomial ring $R = k[x_1, \ldots, x_n]$ becomes complicated and worse as $n$ increases. If $n \leq 2$, $R$ is a polynomial ring in one variable over the subring $A$ of $G_a$-invariants (cf. [9; §1, Chap.I]). However, this does not hold in the case where $n = 3$. Still, the property that $A$ be a polynomial ring seems to hold without the assumption that $A$ is regular. When $n = 3$, 

- 14 -
another criterion for $A$ to be a polynomial ring is that $A$
contains one of coordinates $x_1, x_2, x_3$. Thus, if $G_a$
acts linearly on $R = k[x_1, x_2, x_3]$, then $A$ is a polynomial ring.
Perhaps, $A$ no longer is a polynomial ring for a general $G_a$-
action on $R$ if $n \geq 4$.

2.3. Finally, we shall state the following result without proof:

THEOREM (cf. [11]). Assume that $\text{char}(k) = 0$. Let $X =$
$\text{Spec}(A)$ be a normal affine surface defined over $k$, possessing
a non-trivial action of the additive group $G_a$. Then every
singular point of $X$ is a cyclic quotient singularity.

The result no longer holds if $X$ is not affine.

REFERENCES

   35-51.
   28 (1966).
5. Haboush, W.J.: Reductive groups are geometrically reductive:
   A proof of the Mumford conjecture. Ann. of Math. 102
   (1975), 67-83.
6. Horrocks, G.: Fixed point schemes of additive group