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CHEVALLEY ALGEBRAS AND CHEVALLEY GROUPS

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1. Introduction  In this note, we introduce and discuss Chevalley algebras over commutative rings $R$ with identity, describe their arithmetic structure in the classical (i.e., non-Kac-Moody) cases, and relate that to the normal structure of Chevalley groups over $R$. In Section 3, we also describe recent work of Garland which leads to Chevalley algebras and groups associated with Kac-Moody Lie algebras. In several places, we discuss open questions and conjectures. The rest of this section is devoted to notational preliminaries.

Let $L$ be a finite-dimensional simple Lie algebra over the complex field, $H$ an $m$-dimensional Cartan subalgebra, $\Phi$ the set of roots of $L$ relative to $H$, and $\Pi = \{r_1, r_2, \ldots, r_m\}$ a simple system of roots. For $r \in \Phi$, let $L_r$ be the corresponding root space. Chevalley [5] established the following basic fact.

1.1 Theorem  There is a basis $B = \{\tilde{e}_r | r \in \Phi\} \cup \{\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_m\}$, where $\tilde{e}_r \in L_r$, $\tilde{r}_i \in H$, such that

(i) $[\tilde{r}_i, \tilde{r}_j] = 0$ for all $i$ and $j$,
(ii) $[\tilde{e}_r, \tilde{e}_{-r}] = \tilde{r}_r$, a certain $[26, \text{Lemma 1}]$ integral linear combination of $\tilde{r}_1, \tilde{r}_2, \ldots, \tilde{r}_m$.
(iii) If $r + s \neq 0$, then $[\tilde{e}_r, \tilde{e}_s] = \pm N_{rs} \tilde{e}_{r+s}$, where $N_{rs}$ is $0$ if $r + s \notin \Phi$, and otherwise is $p + 1$, where
p is the largest integer such that $s - pr \in \Phi$.

(iv) $[\bar{h}_r, \bar{e}_s] = \frac{2(s, r)}{(r, r)} \bar{e}_s = s(\bar{h}_r) \bar{e}_s$, where $(,)$ is the Killing form on the dual $H^*$ of $H$. We note that the Cartan integer $c(r, s) = \frac{2(s, r)}{(r, r)} = p - q$, where $q$ is the largest integer such that $s + qr \in \Phi$.

Denote by $L_Z$ the free abelian group on $B$. This is the Chevalley lattice of $L$ corresponding to $B$ and is closed under Lie products. Let $R$ be a commutative ring with identity.

1.2 Definition The Chevalley algebra of $L$ over $R$ is

$L_R = R \otimes_Z L_Z$.

This is uniquely determined up to isomorphism by $L$ [9, pp. 47-48].

Let $H_Z$ be the free abelian group on \{h_1, h_2, ..., h_m\}. Then we denote $R \otimes_Z H_Z$ by $H_R$. Similarly, if $E_Z$ is the free abelian group on \{\bar{e}_r | r \in \Phi\}, then $E_R$ stands for $R \otimes_Z E_Z$. Note that $H_R$ is a subalgebra of $L_R$, but $E_R$ is only an $R$-submodule.

2. Classical Chevalley algebras Results on the arithmetic structure of Lie algebras of Chevalley type tend to take the form of sandwich relations (cf. Equations (1) - (4) below). We consider in this section first the ideal structure of Chevalley algebras, and then the nature of orders in $L$ when the underlying ground ring is an integral domain.

Even though $L$ is simple over the complex field, $L_R$ is not in general simple. For an ideal $J$ of $R$ for instance, we can from the projection homomorphism $f_J : R \rightarrow R/J$, produce a homomorphism from $L_R$.
onto \( L_R / J \) with kernel \( JL_R \), which we can identify with \( L_J \). There are then ideals of \( L_R \) corresponding to ideals of the ring \( R \). A natural question then arises.

**Question 1** To what extent is the ideal structure of \( L_R \) determined by that of \( R \)?

This question is answered in [10] and [27], to which the reader is referred for proofs of the first two results below. Assume that \( 2 \) and \( 3 \) are not zero divisors in \( R \), and if \( L \) is of type \( A_m \) assume further that \( m + 1 \) is not a multiple of the characteristic of \( R \), or a 0-divisor.

**2.1 Theorem** Suppose that \( I \notin H_R \). Then there is an ideal \( J \) of \( R \) and a positive integer \( n \) such that

\[
(1) \quad nJL_R \subseteq I \subseteq JL_R.
\]

Here, \( n \) is a product of divisors of \( \det C \), where \( C = (c_{ij}) = (c(r_i, r_j)) \), and powers of \( k = (\lambda, \lambda)/(s, s) \) where \( \lambda \) is a long root and \( s \) is a short root of \( L \).

For fields of prime characteristic, Question 1 has also been answered by Hogewij [8], who determines all ideals of \( L_R \) even in case \( R \) is of characteristic 2 or 3 or in case its characteristic divides \( m + 1 \) in type \( A_m \). Using Theorem 2.1, one can obtain the following characterization of the circumstances under which all ideals of \( L_R \) arise from those of the ring \( R \), again retaining the assumptions on 2, 3, and \( m + 1 \).

**2.2 Theorem** Every ideal \( I \) of \( L_R \) has the form \( JL_R \) for \( J \) an ideal of \( R \) if and only if \( k \) and \( \det C \) are invertible in \( R \).

**Question 2** What is the situation over a general commutative ring \( R \)
with identity in which 2 or 3 or \( m+1 \) may be a zero-divisor?

Chevalley algebras have also been used to study orders in split simple Lie algebras \( L \) over a field \( F \) which is the field of fractions of an integral domain \( D \). Such algebras have a Chevalley basis over \( F \), and we can identify \( L \) with \( F \otimes_{\mathbb{Z}} L_{\mathbb{Z}} \). The results below generalize the principal theorems found in the Ph.D. dissertation of M. Harvey Hyman [16]. For a more complete discussion, consult [14]. We first give the basic definition.

2.3 Definition  An order in \( L \) is a lattice \( \text{i.e., a finitely generated } D\)-module whose \( F\)-span is \( L \) \( X \) which is closed under multiplication.

We can then regard \( X \) as a Lie algebra over \( D \). The Chevalley algebra \( L_D \) is, of course, a natural order to consider in \( L \), and is called in this context the Chevalley order. In the remainder of this section, \( X \) stands for an arbitrary order in \( L \).

2.4 Theorem  If \( X \cong L_D \), then there is an integer \( n \) as in Theorem 2.1 such that
\[
2) \quad n J L_D \subseteq X \subseteq L_D ,
\]
where \( J \) is the smallest \( D\)-submodule of \( F \) such that \( JL_D \cong X \). If \( D \) is Noetherian, then \( J \) is a fractional ideal.

Observe that \( J \) is well-defined, since we have \( X \subseteq J'L_D \) for the \( D\)-submodule \( J' \) of \( F \) generated by \( 1 \) and all coefficients of elements of \( X \) expressed as \( F\)-linear combinations of the Chevalley basis elements.

Let \( \tilde{D} \) denote the integral closure of \( D \) in \( F \) and \( L'_D = E_D \otimes H_D' \),
where $H_D'$ is the lattice of coroots,

$$H_D' = \{ h \in H \mid r(h) \in D \text{ for all } r \in \Phi \}.$$  

We have the following result.

2.5 Theorem  

(a) Suppose rank $L$ is at least 2 and $2$ has an inverse in $D$ in case $L$ is of type $B_m$ or $C_m$. Let $D$ be a Noetherian domain. Then for any order $X \supseteq L_D$,

$$L_D \subseteq X \subseteq L_D'.$$

(b) If $D$ is integrally closed and Noetherian, (e.g., a Dedekind domain), then for $n$ as in Theorem 2.1,

$$n L_D' \subseteq X \subseteq L_D'.$$

These results describe essentially the nature of orders which contain a certain fixed order $L_D$. It is perhaps worth noting that, even in the case of a Dedekind domain, infinite descending chains of orders are easily produced. If, for example, $a \in D$ is not invertible, then the chain

$$L_D \supseteq a L_D \supseteq a^2 L_D \supseteq a^3 L_D \supseteq ...$$

is an infinite descending chain of orders. It seems to be appropriate then to study orders which contain a fixed order such as $L_D$. Such orders were referred to by Hyman as comprising the superstructure of the order $L_D$. One can ask the following question, whose answer one would expect to be related to the ideal structure of $L'_R$ (cf. [11]).

**Question 3**  What is the superstructure of the order $L_D'$?


We continue the notation of preceding sections. Garland [6] considers Kac-Moody Lie algebras $L_{\mathcal{C}}$ associated with an $m+1$-by-$m+1$ affine Cartan matrix $	ilde{\mathcal{C}}$ obtained from a classical Cartan matrix $\mathcal{C}$, and shows that over the
complex field such algebras have an integral basis closely related to the Chevalley basis for $L$. We discuss this from the more general viewpoint of Moody [20] first, and then specialize to the affine case to state Garland's theorem on Chevalley bases and pose two questions which arise naturally from his construction.

We begin with an $n$-by-$n$ generalized Cartan matrix (GCM) $A = (a_{ij})$, that is, a matrix of integers such that for all $i$ and $j$,

- $a_{ij} \leq 0$ if $i \neq j$,
- $a_{ii} = 2$ for all $i = 1, 2, ..., n$,
- $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Let $K$ be any field of characteristic zero. Let $L_\perp = L_\perp(A)$ be the Lie algebra defined by a set $\{h_i, e_i, f_i\}_{i=1}^n$ of $3n$ generators with defining relations

$$\begin{align*}
[h_i, h_j] &= 0, \text{ for all } i \text{ and } j, \\
[e_i, f_j] &= \delta_{ij} h_i, \text{ for all } i \text{ and } j, \\
[h_i, e_j] &= a_{ij} e_j, \text{ for all } i \text{ and } j, \\
[h_i, f_j] &= -a_{ij} f_j, \text{ for all } i \text{ and } j,
\end{align*}$$

(5)

$$(\text{ad } e_i)^{-a_{ij}+1}(e_j) = 0 = (\text{ad } f_i)^{-a_{ij}+1}(f_j) \text{ for } i \neq j, \quad i, j = 1, 2, ..., n.$$ Thus, $L_\perp$ is the quotient of the free Lie algebra on these $3n$ generators factored by the ideal generated by the elements obtained by rewriting each equation as an expression equated to zero.

For an $n$-tuple $(k_1, k_2, ..., k_n)$ of integers, we define subspaces $L_\perp(k_1, k_2, ..., k_n)$ as follows. $L_\perp(0, 0, ..., 0) = H(A) = \text{the abelian subalgebra of } L_\perp$ spanned by $\{h_1, h_2, ..., h_n\}$. If $(k_1, k_2, ..., k_n)$ consists of nonnegative (resp., nonpositive) integers, then $L_\perp(k_1, k_2, ..., k_n)$ is the subspace of $L_\perp$ spanned by all products $[e_{i_1}, [e_{i_2}, [e_{i_3}, ..., [e_{i_r},-1,}$
e_{i_r} \ldots \} \) (respectively, \([f_{i_1}, [f_{i_2}, \ldots [f_{i_{r-1}}, f_{i_r}]\ldots]\)) , where \(e_j\) (resp., \(f_j\)) occurs \(|k_j|\) times. For all other \(n\)-tuples, \(L_1(k_1, k_2, \ldots, k_n)\) is defined to be \(0\). Each of these subspaces is seen to be finite dimensional, and \(L_1\) is the sum of all the \(L_1(k_1, k_2, \ldots, k_n)\) over all members of \(Z^n\). This gives us a \(Z^n\)-gradation of \(L_1\). There is a unique graded ideal \(R_1\) maximal among all graded ideals which intersect the span of \(\{h_i, e_i, f_i\}_{i=1}^n\) only in zero.

3.1 Definition The Kac-Moody Lie algebra \(L_A\) is \(L_1/ R_1\).

Notice that if \(A\) is a classical Cartan matrix and \(K\) is the complex field, then \(R_1 = 0\) and \(L_A = L_C\) is a classical simple Lie algebra.

We denote the images of \(h_i, e_i, f_i, H(A)\), and \(L_1(k_1, k_2, \ldots, k_n)\) by \(h_i, e_i, f_i, H_A\), and \(L(k_1, k_2, \ldots, k_n)\) respectively. We define \(D_i : L_A \rightarrow L_A\) for each \(i = 1, 2, \ldots, n\), to be multiplication by the scalar \(k_i\) on \(L(k_1, k_2, \ldots, k_n)\). This is then a derivation of \(L_A\). Let \(D_0\) be the \(n\)-dimensional subspace of commuting derivations spanned by \(D_1, D_2, \ldots, D_n\). Let \(D\) be a subspace of \(D_0\) and form the semi-direct product algebra \(L^e = D \times L_A\) with component-wise addition and multiplication by scalars, and Lie product \([d + \lambda/, d' + \lambda'] = [d, d'] + (d(\lambda') - d'(\lambda) + [\lambda, \lambda']\). Let \(H_A^e = D \times H \subseteq L_A^e\), an abelian subalgebra which acts via scalar multiplication on \(L(k_1, k_2, \ldots, k_n)\). We further define \(a_1, a_2, \ldots, a_n \in (H^e)^*\) by

\[(6) \quad [h, e_i] = a_i(h) e_i, \quad \text{for } h \in H_A^e, \quad i = 1, 2, \ldots, n.\]

Thus \(a_j(h_i) = a_{ij}\), \(i, j = 1, 2, \ldots, n\). Henceforth we assume that \(D\) is so chosen that \(\{a_1, a_2, \ldots, a_n\}\) is a linearly independent set. This is possible since, for instance, \(D = D_0\) will serve, although it is often convenient to use a smaller such \(D\). Observe that \(a_i(D_j) = \delta_{ij}\) for \(i, j\)
ranging between 1 and n. We can now define the roots of $L_A$.

3.2 Definition Let $a \in (H^e)^*$. Then $L^a = \{x \in L_A \mid [h, x] = a(h)x$ for all $h \in H^e\}$. A root of $L_A$ relative to $H^e_A$ is a member $a$ of $(H^e)^*$ for which $L^a \neq 0$. The set of all roots is denoted by $\Delta = \Delta(A)$. The positive roots $\Delta_+ = \Delta_+(A)$ consist of all roots which are non-negative integral linear combinations of $a_1, a_2, \ldots, a_n$. The negative roots $\Delta_- = \Delta_-(A)$ are defined to be the negatives of the positive roots.

Notice that $L_A^0 = H_A$ and $L = H_A \oplus \sum a \in \Delta_+ L^a \oplus \sum a \in \Delta_- L^a$.

3.3 Definition The GCM $A$ is symmetrizable if there exist positive rational numbers $q_1, q_2, \ldots, q_n$ such that $\text{diag}(q_1, q_2, \ldots, q_n)A$ is a symmetric matrix.

Henceforth, we assume that $A$ is symmetrizable. Then we can define a symmetric bilinear form on the subspace of $(H^e)^*$ spanned by $\Delta$ by setting

$$ (a_i, a_j) = q_i a_{ij} , $$

for $i, j = 1, 2, \ldots, n$. Then $q_i = (a_i, a_i)/2$ and we set

$$ h_i' = \frac{1}{2}(a_i, a_i)h_i \in H , $$

for $i = 1, 2, \ldots, n$. For $\phi = \sum_{i=1}^n x_i a_i$, we also define

$$ h_i' = \sum_{i=1}^n x_i h_i' , $$

and use this to transfer $(\cdot, \cdot)$ to $H$ by defining $(h_i', h_j') = (a_i, a_j)$, for $i, j = 1, 2, \ldots, n$, and then $(h_a', h_b') = (a, b)$ for any $a$ and $b$ in the span of $\Delta$.

For $i = 1, 2, \ldots, n$, we define the Weyl reflection $w_i : (H^e)^* \to (H^e)^*$ by

$$ w_i(a) = a - a(h_i) a_i . $$
Thus, in particular, from (6) we see that \( w_i(a_j) = a_j - a_{ij}a_i \) for \( i, j = 1, 2, \ldots, n \). The Weyl group \( \text{W} \) of \( L_A \) is the subgroup of \( \text{Aut}(H^e)^* \) generated by all the \( w_i \). We define the set \( \Delta_R(B) \) of real roots to be \( \text{W}(r_1, r_2, \ldots, r_n) \), and the set of imaginary roots \( \Delta_I(B) \) to consist of all roots which are not real.

Now suppose that \( A \) is a classical \( m \)-by-\( m \) Cartan matrix \( C \). We take \( D = 0 \), so that \( H^e = H \), and \( L^e_C = L_C \) is a classical Lie algebra over \( K \). Our form \( ( , ) \) on \( H^*_C \) is just the usual transferred Killing form from \( L \). Using our notation \( \Phi \) for the set of roots of \( L_C \), the set \( \Pi \) of simple roots determines the positive roots \( \Phi_+(C) \). Let \( r_0 \in \Phi_+(C) \) be the highest root. We set \( r_{m+1} = -r_0 \), and form the affine Cartan matrix \( \tilde{C} \) where \( \tilde{c}_{ij} = 2(r_i, r_j)/(r_i, r_i) \), \( i, j = 1, 2, \ldots, m+1 \). Then \( \tilde{C} \) is a symmetrizable generalized Cartan matrix with associated Kac-Moody Lie algebra \( L_{\tilde{C}} \).

Next let \( K[t, t^{-1}] \) be the ring of Laurent polynomials over \( K \). We define the infinite dimensional Laurent polynomial Lie algebra

\[
\tilde{L} = K[t, t^{-1}] \otimes_K L_C
\]

with Lie product \( [f \otimes x, g \otimes y] = fg \otimes [x, y] \) for \( f, g \in K[t, t^{-1}] \) and \( x, y \in L_C \). Note that from the decomposition of \( L_C \) into \( H_C \) and the sum of the root spaces \( L^r \), we obtain

\[
\tilde{L} = K[t, t^{-1}] \otimes_K H_C \otimes \sum_{r \in \Phi} L^r \otimes \sum_{n \in \mathbb{Z}^+ \cup \{0\}} t^n \otimes K L_C.
\]

Now to avoid ambiguity, we write \( e_i^*, f_i^*, h_i^* \) for \( e_i, f_i, h_i \) in \( L_C \), \( i = 1, 2, \ldots, n \) and \( h_r^* \) for \( h_r \) in \( H_C \). For \( r_0 \), choose \( e_0^* \in L^r_0 \) and \( f_0^* \in L^{-r_0} \) so that \( [e_0^*, f_0^*] = 2h_r^* / (r_0, r_0) \). The following theorem of Kac [17] and Moody [21] helps to describe the set of roots of
In our next result, we identify $1 \otimes x$ in $\tilde{L}$ with $x$ in $L_{C}$.

3.4 Theorem There is a unique monomorphism $\tilde{\omega}: L_{C} \to \tilde{L}$ such that

$$
\tilde{\omega}(e_{i}) = e_{i}^{*}, \quad \tilde{\omega}(f_{i}) = f_{i}^{*}, \quad \tilde{\omega}(h_{i}) = h_{i}^{*}, \quad i = 1, 2, \ldots, m, \quad \tilde{\omega}(e_{m+1}) = t \otimes f_{0}^{*}, \quad \tilde{\omega}(f_{m+1}) = t^{-1} \otimes e_{0}^{*}, \quad \text{and} \quad \tilde{\omega}(h_{m+1}) = 2 h_{m+1}^{*} r_{0}/(r_{0}, r_{0}) .
$$

The kernel of $\tilde{\omega}$ is the one-dimensional center of $L_{C}$ and is spanned by $h_{i}^{*}$

$$
= \sum_{i=1}^{n} k_{i} h_{i}^{*} + h_{m+1}^{*}, \quad \text{where} \quad r_{0} = \sum_{i=1}^{m} k_{i} r_{i} .
$$

We define $D_{m+1}: L_{C} \to L_{C}$ to be the $(m+1)$-st degree derivation, and define $D$ to be the one-dimensional subspace of $D_{0}$ spanned by $D_{m+1}$. It is easy to check that $\{a_{1}, a_{2}, \ldots, a_{m+1}\}$ in the resulting $(H_{0})^{*}$ is then a linearly independent set [6, p. 487]. Note that $\tilde{\omega}$ isomorphically maps

$$
\sum_{a \in \Delta_{+}(C)} L^{a} \to \sum_{r \in \Phi_{\text{+}}} L^{r} \bigoplus \sum_{n \in Z^{+}} U Z^{+} t^{n} \otimes L_{C}
$$

and similarly for $\sum_{a \in \Delta_{-}(C)} L^{a}$. We thus identify the two sides of (7).

For $r \in \Phi$, $r = \sum_{i=1}^{m} n_{i} r_{i}$, $n_{i} \in Z^{+}$ or $n_{i} \in Z^{-}$ for all $i$, we define $a(r) \in (H_{0})^{*}$ by the formula $a(r) = \sum_{i=1}^{m} n_{i} a_{i}$. We define the Lie algebra derivation $D_{0}: L \to L$ by $D_{0}(t^{n} \otimes x) = n t^{n} \otimes x$ for $n \in Z$ and $x \in L_{C}$. Then [6, p. 487] $D_{0} \circ \tilde{\omega} = \tilde{\omega} \circ D_{m+1}$. Setting

$$
1 = \sum_{i=1}^{m} k_{i} a_{i} + a_{m+1} e \in (H_{0})^{*},
$$

it follows from Theorem 3.4 that

$$
\Delta_{+}(C) = \{a(r)\} \bigcup \{a(r) + n_{1}\} \bigcup \{n_{1}\} \quad \text{for} \quad r \in \Phi_{\text{+}}, \quad n_{1} \in Z^{+}
$$

3.5 Proposition (Kac [17, p. 287]) Let $A$ be a GCM. Then the root $a \in \Delta_{1}(A)$ if and only if $ja$ is a root for all integers $j \neq 0$. 


It now follows that \( \Delta_I(\mathbb{C}) = \{n\} \cap Z - \{0\} \) and \( \Delta_R(\mathbb{C}) = \{a(r) + n\} \cap Z, r \in \Phi \). Using our identification (7) above, the root spaces \( L^a \) of \( L_C \) are therefore \( L^a = t^n \otimes L^r \) (where \( a = n + a(r), r \in \Phi \) and \( n \in Z \)) and \( L^a = t^n \otimes H_C \) (where \( n \in Z - \{0\} \) and \( a = n \)).

Next suppose that \( K \) is the complex field. We take \( q_i = (r_i, r_i)/2 \), \( i = 1, 2, \ldots, m + 1 \), so that \( q_i > 0 \) for each \( i \). Then \( \text{diag}(q_1, q_2, \ldots, q_{m+1}) \mathcal{C} \) is a symmetric matrix with \( ij \)-entry \( (r_i, r_j) \) for \( i, j = 1, 2, \ldots, m + 1 \). Notice then that with this choice of \( q_i \), \((a_i, a_j) = (r_i, r_j)\), for \( i, j = 1, 2, \ldots, m + 1 \), and hence for \( a \in \Delta(\mathbb{C}) \), we have \( 2(a, a_i)/(a_i, a_i) \in Z \) for \( i = 1, 2, \ldots, m + 1 \). For each real root \( a = a(r) + n \), we define \( e_a \in L^a = t^n \otimes L^r \) by \( e_a = t^n \otimes \tilde{e}_r \), where \( \tilde{e}_r \) is as in Theorem 1.1. For each imaginary root \( n \), we define for \( i = 1, 2, \ldots, m \) and each nonzero integer \( n \), \( e_i(n) \in L^{n_i} = t^n \otimes H_{A_i} \) by \( e_i(n) = t^n \otimes h_i \). Note that \( \{e_i(n)\}_{i=1}^{m} \) is a basis for \( L^{n_i} = t^n \otimes H_C \) for each \( n \in Z \), and that \( \{h_1, h_2, \ldots, h_{m+1}\} \) is a basis for \( H_C \).

3.6 Definition The set \( \mathcal{B} = \{h_i\}_{i=1}^{m+1} \cup \{e_a\} a \in \Delta_R(\mathbb{C}) \cup \{e_i(n)\}_{i=1}^{m} \) is called a Chevalley basis for \( L_C \).

We now are in a position to state Theorem 4.12 of [6], which serves to explain and justify the terminology in the preceding definition.

3.7 Theorem \( \mathcal{B} \) is an integral basis for \( L_C \). In fact, the equations of structure are

(i) \( [e_a, e_b] = \pm (p + 1)e_{a+b} \) if \( a + b \not= 0 \), where \( a = a(r) + n \), \( b = b(s) + j \), \( r, s \in \Phi \), \( n, j \in Z \), and \( p \) as
in Theorem 1.1.

(ii) \([e_a, e_a] = h_a\), an integral linear combination of \(h_1, h_2, \ldots, h_{m+1}\) for all \(a \in \Delta_R(\mathcal{C})\).

(iii) If \(a = a(r) + n_1, b = a(-r) + j_1, r \in \Phi\), and \(n + j = z \neq 0\), then \([e_a, e_b] = t^z \otimes h_r\) is an integral linear combination of \(e_1(z), \ldots, e_m(z)\).

(iv) \([e_i(n), e_j(-n)] = n c_{i,j} 2h_i^* / (r_j, r_j)\) is an integral linear combination of \(h_1, h_2, \ldots, h_{m+1}\).

(v) \([h_i, e_a] = (2(a, a_i)/(a_i, a_i)) e_a\) for \(a \in \Delta_R(\mathcal{C}), i = 1, 2, \ldots, m+1\).

(vi) \([e_i(n), e_a] = (2(r, r_i)/(r_i, r_i)) e_{a+n_1}\), where \(i = 1, 2, \ldots, m, n \neq 0\) is in \(Z\) and \(a = a(r) + j_1, r \in \Phi, j \in Z\).

All other products of elements in \(B\) are zero.

This result raises immediately the following question, in view of the results set forth in Section 2.

**Question 4** If the free abelian group \((L_C^0)_Z\) is formed, and for a commutative ring \(R\) with identity \(1\), the Kac-Moody Chevalley algebra \((L_C^0)_R = R \otimes Z (L_C^0)_Z\) is constructed, then what is the ideal structure of \((L_C^0)_R\), and in particular, how is it related to the ideal structure of the Chevalley algebra \(L_R[t, t^{-1}]\)?

This question is currently under investigation by the author and J. Morita. Theorem 3.4 has been generalized to a form which appears useful in considering Question 4.
Garland [7] goes on to construct groups of automorphisms analogous to Chevalley groups of classical simple Lie algebras. In [6], he already constructs (Theorem 5.8) a Z-form U\(_Z\)(\(\tilde{\mathcal{C}}\)) of the universal enveloping algebra U(L\(_\mathbb{C}\)) of L\(_\mathbb{C}\) which is analogous to U\(_Z\) in [19]. He then proves (Theorem 11.3) the existence of a V\(_Z\)\(^\lambda\) which is invariant under U\(_Z\)(\(\tilde{\mathcal{C}}\)). He is able (Lemma 10.4) to exponentiate scalar multiples of e\(_a\), a \(\in\) \(\Delta_R\)(\(\mathcal{C}\)), the Chevalley basis elements of Theorem 3.7 above, and obtains automorphisms of V\(_\mathbb{C}\)\(^\lambda\), and can thus define Chevalley groups for L\(_\mathbb{C}\) over a commutative ring R with identity. This brings up another question.

**Question 5** What is the normal structure of these groups G and how does it relate to the ideal structure of (L\(_\mathbb{C}\))\(_R\) and of R?

Garland [6, p. 495] remarks that the groups G have an infinite dimensional completion \(\hat{G}\) which is a central extension of a Chevalley group with rational points in the field of formal Laurent series. Thus there may be some relation between Question 5 and recent work of Morita [22] on Chevalley groups over rings of Laurent polynomials. For a general idea of why one would expect some relationship between ideal structure of the Chevalley algebras (L\(_\mathbb{C}\))\(_R\) and the normal structure of the Chevalley groups G, refer to the next section.

4. **Chevalley groups over rings** We continue the notation of Sections 1 and 2, but remove the bars from the Chevalley basis elements in Theorem 1.1. Let U be the universal enveloping algebra of L. Let U\(_Z\) be the Z-algebra generated by all e\(_r\)^{m/m!}, r \(\in\) \(\phi\), m \(\in\) \(Z^+\) U \{0\}. Then [19, 26] under the adjoint representation, each generator of U\(_Z\) preserves L\(_Z\). If \(\rho\) is a faithful finite dimensional representation of L on a
complex vector space \( V \), then there is [26, p. 17] a lattice \( M \) invariant under \( U_Z \). Let \( \mathcal{L}_Z \) be the part of \( L \) which preserves \( M \). (If \( \rho = \text{ad} \), then \( \mathcal{L}_Z = L_Z \) of Section 1.) We can form the Chevalley algebra \( L_R = R \otimes_Z \mathcal{L}_Z \) as before, and let \( \exp(t e_r), t \in R, r \in \Phi \), act on \( R \otimes Z M \) in the natural \([26, \text{p. 21}]\) way, and we label the resulting automorphism \( x_r(t) \). The group \( E_\rho(\Phi, R) \) generated by all such \( x_r(t) \) is the elementary subgroup of the Chevalley group \( G_\rho(\Phi, R) \) of \( L \) over \( R \). The latter group consists of the points in \( R \) of a Chevalley-Demazure group scheme \([4, 55]\) associated with \( L \) and \( \rho \), which depends only on \( \Phi \) and the lattice of weights of \( \rho \). When this is the lattice of fundamental weights, \( G_\rho(\Phi, \cdot) \) is universal, and \( G_\rho(\Phi, \mathbb{C}) \) is simply connected \([26, \text{p. 89}]\) over the complex field \( \mathbb{C} \). In this case, \( G_\rho(\Phi, R) = E_\rho(\Phi, R) \) when \( R \) is a field, local or even semi-local ring, or a Euclidean ring.

While \( G_\rho(\Phi, K) \) is simple over a field \( K \) in almost all cases when \( \rho = \text{ad} \), \( G_\rho(\Phi, R) \) has normal subgroups arising from the ideal structure of \( R \) in a way that is reminiscent of the ideals of \( L_R \). For notational simplicity, let us fix \( \rho \) and \( \Phi \) and delete them from our notation for the groups. Let \( f_J : G(R) \to G(R/J) \) be the natural epimorphism induced by reduction of the ring \( R \) modulo the ideal \( J \). Then \( G(R, J) = \text{Ker} f_J \) is of course a normal subgroup of \( G(R) \), as is \( f_J^{-1}(\text{Center } G(R/J)) = G^*(R, J) \).

4.1 Definition A congruence subgroup of \( G(R) \) is a subgroup \( N \) such that

\[ G(R, J) \subseteq N \subseteq G^*(R, J) \, . \]

Note the resemblance between (8) and (1). Also note that if \( N \) is any congruence subgroup of \( G(R) \), then \( N \) is necessarily a normal subgroup. For
letting $(X, Y)$ stand for the group generated by all commutators $(x, y) = xy x^{-1} y^{-1}$ for $x \in X$ and $y \in Y$, we have $(N, G) \leq (N, G^*(R, J)) \leq G(R, J) \leq N$ for any congruence subgroup $N$. Hence in particular, we have $G(R, J) \supseteq (G^*(R, J), G^*(R, J))$, which finishes the proof of the following basic result.

4.2 Corollary Every congruence subgroup is normal, and the factor group $G^*(R, J)/G(R, J)$ is an abelian group.

The study of normal subgroups of $G(R)$ has focused on congruence subgroups. We begin our description of the present status of this study by stating the following result of E. Abe [1].

4.3 Theorem Suppose $G(\phi, C)$ is simple and simply connected as a Lie group. Let $R$ be a local ring such that $R/M \neq GF(3)$ and $\text{char } R/M \neq 2$ if $L$ is of type $A_1$, $B_n$, $C_n$, $F_4$. Suppose also that $R/M \neq GF(2)$ or $GF(3)$ in types $B_2$ or $G_2$. (Here $M$ is the maximal ideal of $R$.) Then $G(R) = E(R)$ and the only normal subgroups of $G(R)$ are congruence subgroups.

4.4 Theorem [12, 13]. Let $R$ be any commutative ring with identity, with 2 and 3 not zero divisors in $R$ and $n + 1$ not a zero divisor if $L$ is of type $A_n$. Let $\rho = \text{ad}$. Then corresponding to an ideal $I$ of $L_R$ there is a normal subgroup $G_I$ of $E(R)$ generated by $x_r(t)$ such that $te_r \in I$ and by all iterated conjugates of $x_r(t)$ by elements of the form $x_{r_1}(u_1), x_{r_2}(u_2), x_{r_3}(u_3)$, etc. This $G_I$ is the normal closure in $E(R)$ of the subgroup generated by all $x_r(t)$ such that $te_r \in I$. If $L$ is not of type $C_n$, then $G_I = G_{I'}$ if and only if $I \cap E_R = I' \cap E_R$. 
The next theorem actually holds more generally [3], but for simplicity of statement, we restrict ourselves to the following version.

4.5 Theorem (Abe - Suzuki) Let $G$ be simple and simply connected as a complex Lie group, and have rank at least two. Let $R$ be a Noetherian ring or a direct product of fields. Let $\text{Spm}(R) = \{ M \mid M$ is a maximal ideal of $R \}$. If $\phi$ is of type $B_2$ or $G_2$, assume for all $M \in \text{Spm}(R)$ that $R/M \neq GF(2)$. If $\phi$ is of type $B_n$, $C_n$, or $F_4$, suppose for all $M \in \text{Spm}(R)$ that char $R/M \neq 2$, and if $\phi$ is of type $G_2$, suppose that char $R/M \neq 3$. Let $G_0(R)$ be the subgroup of $G(R)$ generated by all $\chi_r(t)$ for $t \in R$, $r \in \phi$ and by all $h(\chi) = \text{diag}(\chi(\lambda_1), \ldots, \chi(\lambda_n))$ for a certain [1, pp. 475-476] set $\{\lambda_1, \ldots, \lambda_n\}$ which generates the additive abelian group $P$ generated by the weights of $\rho$, and arbitrary $\chi \in \text{Hom}(P, C^*)$. Then any subgroup of $G_0(R)$ which is normalized by $E(R)$ (in particular, any normal subgroup of $E(R)$) satisfies for a unique ideal $J$ $E(R, J) \subseteq N \subseteq G^*(R, J)$, where $E(R, J)$ is the normal closure in $E(R)$ of all $\chi_r(t)$, $t \in R$, $r \in \phi$. That is, $E(R, J) = G_I$, where $I = JL_R$.

4.6 Theorem [13] Suppose $\phi$ has a single root length and rank at least two, with $\rho = \text{ad}$. Then $\chi_r(t)$ has normal closure $G_I$, where $I = JL_R$ for $J$ the principal ideal in $R$ generated by $t$. The same result holds if $\phi = B_n$, $n \geq 3$, or $F_4$ if $r$ is a long root.

Comparing the preceding result with [18, Satz 3], the following question is suggested.

Question 6 Is $E(R, J) = \text{Ker} f_J \mid E(R)$, at least in the single root length cases?
E. Abe has been able to answer Question 6 affirmatively in case the exact sequence \( 0 \to J \to R \to R/J \to 0 \) splits. It would be interesting to find other conditions on \( R \) which provide a positive answer. Swan [29] states without proof that Question 6 has a positive answer in the case of the stable group \( E(R) \) for any commutative ring with identity. Silvester [24] makes a similar statement in the nonstable case. (In both these claims, \( \phi = A_n \).)

4.7 Theorem [13] Under the hypotheses of Theorem 4.6 (first part),
the normal closure of a product \( x_r(t_1)x_s(t_2) \) where \( r \neq s \) in \( E(R) \) is \( G_I \) where \( I = JL_R \) for \( J \) the ideal in \( R \) generated by \( t_1 \) and \( t_2 \).

Theorem 4.7 also holds for products of three root elements, but in general no such fact is known. This leads to the following question.

**Question 7** Under what hypotheses on \( R \) and \( \phi \) can Theorem 4.7 be extended?

Suslin [28] has shown that \( E(R) \leq G(R) \) for \( \phi = A_n, n \geq 2 \), and in fact has shown that \( E(R, J) \leq G(R) \) in that case. He even showed normality in \( GL_n(R) \), a still larger group. This raises another natural question.

**Question 8** Under what hypotheses on \( R \) and \( \phi \) is \( E(R) \leq G(R) \) and \( E(R, J) \leq G(R) \)?

This question relates directly to the \( K_1 \)-functor on Chevalley groups of Stein [25]. Let \( \text{rank } \phi \geq 2 \). \( St(\phi, R) \) stands for the group generated by \( x_r(t), t \in R, r \in \phi \), subject to the relations

\[
(R1) \quad x_r(t)x_s(u) = x_r(t + u)
\]
\[(R2) \quad (x_r(t), x_s(u)) = \prod_{ir + js \in \Phi} x_{ir + js}(c_{ijrs} t^i u^j).\]

where \( r + s \in \Phi \) and the product is taken in some fixed order, with \( c_{ijrs} \in Z \) for all \( ir + js \in \Phi \).

Since the relations \((R1)\) and \((R2)\) hold, we have a mapping \( \pi : \text{St}(R) \to G(R) \) whose image is the elementary subgroup \( E(R) \). By definition, the group \( K_2(\Phi, R) \) is the kernel of the map \( \pi \), and \( K(\Phi, R) = \text{Cok} \ \pi = G(R)/E(R) \) as a homogeneous space. Question 8 can therefore be rephrased in the language of algebraic K-theory as follows.

\textbf{Question 8':} Under what conditions on \( \Phi \) and \( R \) is \( K_1(\Phi, R) \) a group?

In [25], Stein gives the following partial answer to this form of the question.

\textbf{4.8 Theorem} Let \( \Phi \) be one of the types \( A_n, B_n, C_n, D_n \), or \( G_2 \). Let \( R \) be a ring whose maximal ideal space is Noetherian of finite dimension \( d \). Suppose also that if \( \Phi \) is of type \( A_n \), then \( n \geq d + 1 \); if \( \Phi \) is of type \( B_n \), then \( n \geq d + 2 \); if \( \Phi \) is of type \( C_n \), then \( n \geq (d + 2)/2 \); if \( \Phi \) is of type \( D_n \), then \( n \geq d + 2 \); and if \( \Phi \) is of type \( G_2 \), then \( d \leq 1 \). Then \( E(R, J) \leq G(\Phi, R) \).

Finally, Silvester [23] considered the ring \( R = K\langle X \rangle \) freely generated over \( K \) by a set \( X \) of noncommuting indeterminates. For \( \Phi = A_n \), he considered \( GE(R) \), the subgroup of \( GL_n(R) \) generated by all \( x_r(t) \) and all \( h_i(z) = w_{r_i}(z)w_{r_i}^{-1} \), \( i = 1, 2, \ldots, n \), where \( w_r(u) = x_r(u)x_{-r}(-u^{-1})x_r(u) \). In addition to this group, he also considered \( GEU(R) \), the group generated by symbols \( x_r(t) \) and \( h_i(z) \) with the usual
relations [23, p. 37] in GE(R). He showed that the natural homomorphism 
\( f : GEU(R) \to GE(R) \) is an isomorphism, in which circumstance \( R \) is said to 
be universal for \( f \). Consideration of the analogous notions for general 
\( \phi \) leads naturally to the following question which is currently under joint 
investigation by the author and E. Abe.

**Question 9** Under what conditions on \( R \) and \( \phi \) is \( f \) an isomorphism 
for more general root systems? In particular, is it an isomorphism in the 
case of \( R = K[X] \), the free commutative \( K \)-algebra generated by a set \( X \) 
of indeterminates, at least in the single root length systems?

Silvester’s results for the case \( R = K[X] \) was a major tool in 
showing that \( K_2(A_n, K[X]) = K_2(A_n, K) \). Thus the answer to Question 9 
bears directly upon the question of computing \( K_2(\phi, K[X]) \) which is raised 
by E. Abe [2] in his article in these Proceedings.

In a similar vein, if \( R \) is a commutative ring with identity, then for 
\( \phi = A_n \), Center \( E_n(R) = \Omega_n \), the group of \( n \)-th roots of \( 1 \) in \( R \) [15]. 
Passing to the stable group, Center \( E(R) = 1 \). For more general \( \phi \), we 
can pose the following question.

**Question 10** What is the center of the group \( E(\phi_n, R) \)? What is the 
center of \( E_\rho(R) \), the direct limit of \( E_\rho(\phi_n, R) \) for the classical \( \phi_n \) ?

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