Title: 右自完集合半群とその周辺

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Citation: 数理解析研究所講究録 (1980), 395: 85-91

Issue Date: 1980-08

URL: http://hdl.handle.net/2433/105004

Type: Departmental Bulletin Paper

Publisher: Kyoto University
. RIGHT SELF-INJECTIVE SEMIGROUPS ARE ABSOLUTELY CLOSED

Kunitaka Shoji

Hinkle [3] has shown that the direct product of column-monomial matrix semigroups over groups is right self-injective. The author [12] has shown that the full transformation semigroup on a set (written on the left) is right self-injective and so every semigroup is embedded in a right self-injective regular semigroup. While absolutely closed semigroup has been first studied in Isbell[7]. In Howie and Isbell [5] and Scheiblich and Moore [8] it has been shown that inverse semigroups, finite cyclic semigroups, totally division-ordered semigroups, right [left] simple semigroups and full transformation semigroups are absolutely closed. In Section 1 we shall show that every right [left] self-injective semigroup is absolutely closed. This will give another proof of that right [left] simple semigroups, finite cyclic semigroups and full transformation semigroups are absolutely closed. Using a result of [5] we shall show that the class of right [left] self-injective [regular] semigroups has the special amalgamation property. In Section 2 we shall show that a commutative separative semigroup is absolutely closed if and only if it is a semilattice of abelian groups. As its result we will obtain that every self-injective commutative separative semigroup is a semilattice of abelian groups. Using a characterization of self-injective inverse semigroups [9] we shall give a structure theorem for self-injective commutative separative semigroups. The complete proofs are omitted and will be given in detail elsewhere. Throughout this paper we freely
use the terms "right S-system", "S-homomorphism", "right self-injective" and so on, which are referred to [12].

§1. Right self-injective semigroups. Let A, B be semigroups such that A is a subsemigroup of B. Then by Isbell [7] the set \( \{ b \in B \mid f(b) = g(b) \} \) for all semigroups C and for all homomorphisms \( f, g : B \to C \) such that \( f|A = g|A \) is called the dominion of A in B and is denoted by \( \text{Dom}_B(A) \). A semigroup S is called absolutely closed if \( \text{Dom}_T(S) = S \) for all semigroups T containing S as a subsemigroup.

Result 1. ([4, Isbell's zigzag theorem]) Let T be a semigroup and S a subsemigroup of T. Then for each \( d \in T \) \( d \in \text{Dom}_T(S) \) if and only if \( d \in S \) or there exist \( S_0, s_1, \ldots, s_{2m} \in S \) and \( x_1, \ldots, x_m, y_1, \ldots, y_m \in T \) such that \( d = s_0y_1, s_0 = x_is_1, s_{2i-1}y_{i+1} = s_{2i}y_{i+1}, x_is_{2i} = x_{i+1}s_{2i+1} (1 \leq i \leq m-1), s_{2m-1}y_m = s_{2m} \) and \( x_ms_{2m} = d \).

Theorem 1. Every right [left] self-injective semigroup is absolutely closed.

The next result follows from Theorem 1, and Corollary 1,2 of [12].

Corollary 1. I. ([8, H. Scheiblich and K. Moore]) Full transformation semigroups are absolutely closed.
II. The direct product of column [row]-monomial matrix semigroups over groups is absolutely closed.

According to [11] a semigroup S with 1 is called completely right injective if every right S-system is injective. It is clear that all the homomorphic images of a completely right injective
semigroup are completely right injective, of course, right self-injective.

Thus we have

Corollary 2. All the homomorphic images of a completely right injective semigroup are absolutely closed.

Remark 1. It easily follows from Isbell's zigzag theorem that a semigroup $S$ is absolutely closed if and only if $S_0 [S_1]$ is absolutely closed, where $S_0 [S_1]$ denotes the semigroup obtained from $S$ by adjoining a zero [an identity]. If a semigroup $S$ is right simple, then $S_0 [S_1] = (S_0 [S_1])$ is completely right injective. Thus it follows from Corollary 2 and the above that $S$ is absolutely closed. Also if a semigroup $S$ is finite and cyclic then we can show that $S_0 [S_1]$ is a self-injective semigroup (see[12]). Hence it follows from Theorem 1 and the above that $S$ is absolutely closed. These results have been obtained by Howie and Isbell[5].

Let $\mathcal{A}$ be any class of algebras. According to Hall [2], if for some index set $I$, $\{S_i : i \in I\}$ is an indexed set of algebras from $\mathcal{A}$ having a common subalgebra $U$ also in $\mathcal{A}$, then the list $(S_i : i \in I:U)$ is called an amalgam from $\mathcal{A}$. If there exist an algebra $W$ and morphisms $\phi_i : S_i \rightarrow W (i \in I)$ such that $\phi_i|U = \phi_j|U$ and $\phi_i(S_i) \cap \phi_j(S_j) = \phi_i(U)$ for all distinct $i, j \in I$, then the amalgam $(S_i : i \in I:U)$ is said to be strongly embeddable in $W$. If an amalgam of the form $(S,S:U)$ from $\mathcal{A}$ is strongly embeddable in an algebra from $\mathcal{A}$, then $U$ is said to be closed in $S$ (within $\mathcal{A}$). If $U$ is closed in $S$ within $\mathcal{A}$ for all $U, S \in \mathcal{A}$ with $U \subseteq S$, then $\mathcal{A}$ is said to have the special amalgamation property. If every amalgam from $\mathcal{A}$ is strongly embeddable in an algebra from $\mathcal{A}$,
then $\mathcal{A}$ is said to have the strong amalgamation property.

Result 2. ([4, theorem 2.4]) Let $U$, $S$ be semigroups such that $U$ is a subsemigroup of $S$. Then $U$ is closed in $S$ (within the class of semigroups) if and only if $\text{Dom}_S(U) = U$.

This follows from Theorem 1, Result 2, and Corollary 3 [12].

Theorem 2. The class of right [left] self-injective [regular] semigroups has the special amalgamation property.

The following example shows that the class of right [left] self-injective [regular] semigroups does not have the strong amalgamation property. This is constructed from an example in Imaoka [6].

Example. Let $U = \{0, e, f, g, l\}$, $V = \{0, e, f, g, h, l\}$ and $W = \{0, e, f, g, x, y, l\}$ be semigroups whose multiplicative tables are:

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By [11] $U$, $V$ and $W$ are completely right injective, of course, right self-injective and regular. Suppose now that the amalgam $(V, W:U)$ is embeddable in a semigroup $S$. But in $S$ we have $xh = (xe)h = x(eh) = xf = y$ and $xh = (xg)h = x(gh) = xg = x$. This is a contradiction. Hence the amalgam $(V, W:U)$ can not be embedded in any semigroup.
§2. Commutative separative semigroups. Let $S$ be a commutative separative semigroup. Then by [1, Theorem 4.18] $S$ is uniquely expressible as a semilattice $\Lambda$ of archimedean cancellative semigroups $S_\alpha$ ($\alpha \in \Lambda$) and $S$ can be embedded in a semigroup $T$ which is the same semilattice $\Lambda$ of abelian groups $G_\alpha$ ($\alpha \in \Lambda$) where $G_\alpha$ is the quotient group of $S_\alpha$ for each $\alpha \in \Lambda$, i.e. every element of $G_\alpha$ can be expressed in the form $ab^{-1}$ with $a$ and $b$ in $S_\alpha$.

Let $\xi, \psi$ be homomorphisms of $T$ to any semigroup $W$ such that $\xi|S = \psi|S$. Then for each $G_\alpha$, $\xi(G_\alpha)$ and $\psi(G_\alpha)$ are contained in a subgroup $H$ of $W$. Hence $\xi(a^{-1}) = \psi(a^{-1})$ for all $a \in S_\alpha$. Because that both $\xi(a^{-1})$ and $\psi(a^{-1})$ are inverses of $\xi(a)$ in the group $H$. Then it is clear that $\xi|G_\alpha = \psi|G_\alpha$. Therefore we have $\xi = \psi$. This implies that $\text{Dom}_T(S) = T$. Thus we have

Theorem 3. Let $S$ be a commutative separative semigroup. Then $S$ is absolutely closed if and only if $S$ is a semilattice of abelian groups.

In [10] we studied self-injective non-singular semigroups and showed that every self-injective non-singular semigroup is a semilattice of groups and every commutative non-singular semigroup is separative.

More generally by Theorem 1,3 we have

Theorem 4. Every self-injective separative commutative semigroup is a semilattice of abelian groups.

In [9] B. Schein characterized self-injective inverse semigroups as follows: Let $S$ be an inverse semigroup and $E_S$ the set of idempotents of $S$. A subset $B$ of $S$ is compatible if for
each \( b \in S \) there is \( e_b \in E_S \) with \( be_b = b \) and \( be_c = ce_b \) for all \( b, c \in B \). Define an order \( \preceq \) on \( S \) by \( a \preceq b \) (\( a, b \in S \)) if and only if \( a \in bE_S \). \( S \) is complete if every compatible set \( B \) of \( S \) has the least upper bound \( \bigvee B \) relatively to \( \preceq \). \( S \) is infinitely distributive if \( (\bigvee B)a = \bigvee Ba \) for any compatible set \( B \) of \( S \) and for any \( a \in S \). \( S \) is \( E_S \)-reflexive if \( st \in E_S \) implies \( st \in E_S \).

Result 3. ([9, 2.3 Theorem]) Let \( S \) be an inverse semigroup and \( E_S \) the set of idempotents of \( S \). Then \( S \) is self-injective if and only if \( S \) is complete, infinitely distributive and \( E_S \)-reflexive.

Here we can obtain the following:

Theorem 5. Let \( S \) be a commutative semigroup. Then \( S \) is self-injective and separative if and only if \( S \) is a semilattice \( \Lambda \) of abelian groups \( G_\alpha \) (\( \alpha \in \Lambda \)) satisfying the followings: (1) \( \Lambda \) is self-injective, (2) for any set \( \{ g_\alpha \}_{\alpha \in X} \) such that \( g_\alpha e_\beta = g_\beta e_\alpha \) (\( \alpha, \beta \in X, g_\alpha \in G_\alpha, g_\beta \in G_\beta, e_\alpha, e_\beta \) are identities of \( G_\alpha, G_\beta \), respectively) there exists \( g \in S \) such that \( ge_\alpha = g_\alpha \) for all \( \alpha \in X \).

REFERENCES


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