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RIGHT SELF-INJECTIVE SEMIGROUPS ARE ABSOLUTELY CLOSED

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Hinkle [3] has shown that the direct product of column-monomial matrix semigroups over groups is right self-injective. The author [12] has shown that the full transformation semigroup on a set (written on the left) is right self-injective and so every semigroup is embedded in a right self-injective regular semigroup. While absolutely closed semigroup has been first studied in Isbell[7]. In Howie and Isbell [5] and Scheiblich and Moore [8] it has been shown that inverse semigroups, finite cyclic semigroups, totally division-ordered semigroups, right [left] simple semigroups and full transformation semigroups are absolutely closed. In Section 1 we shall show that every right [left] self-injective semigroup is absolutely closed. This will give another proof of that right [left] simple semigroups, finite cyclic semigroups and full transformation semigroups are absolutely closed. Using a result of [5] we shall show that the class of right [left] self-injective [regular] semigroups has the special amalgamation property. In Section 2 we shall show that a commutative separative semigroup is absolutely closed if and only if it is a semilattice of abelian groups. As its result we will obtain that every self-injective commutative separative semigroup is a semilattice of abelian groups. Using a characterization of self-injective inverse semigroups [9] we shall give a structure theorem for self-injective commutative separative semigroups. The complete proofs are omitted and will be given in detail elsewhere. Throughout this paper we freely use the terms "right S-system", "S-homomorphism", "right self-injective" and so on, which are referred to [12].

§1. Right self-injective semigroups. Let A, B be semigroups such that A is a subsemigroup of B. Then by Isbell [7] the set $\{b \in B \mid f(b) = g(b)$ for all semigroups C and for all homomorphisms $f, g : B \longrightarrow C$ such that $f|A = g|A\}$ is called the <u>dominion</u> of A in B and is denoted by $\mathrm{Dom}_B(A)$. A semigroup S is called <u>absolutely closed</u> if $\mathrm{Dom}_T(S) = S$ for all semigroups T containing S as a subsemigroup.

Result 1. ([4, Isbell's zigzag theorem]) Let T be a semigroup and S a subsemigroup of T. Then for each $d \in T$ $d \in \mathrm{Dom}_T(S)$ if and only if $d \in S$ or there exist $s_0, s_1, \ldots, s_{2m} \in S$ and $x_1, \ldots x_m, y_1, \ldots, y_m \in T$ such that $d = s_0 y_1$, $s_0 = x_1 s_1$, $s_{2i-1} y_i = s_{2i} y_{i+1}$, $x_i s_{2i} = x_{i+1} s_{2i+1}$ $(1 \leqq i \leqq m-1)$, $s_{2m-1} y_m = s_{2m}$ and $x_m s_{2m} = d$.

Theorem 1. Every right [left] self-injective semigroup is absolutely closed.

The next result follows from Theorem 1, and Corollary 1,2 of [12].

Corollary 1. I. ([8, H. Scheiblich and K. Moore]) Full transformation semigroups are absolutely closed.

II. The direct product of column [row]-momonial matrix semigroups over groups is absolutely closed.

According to [11] a semigroup S with 1 is called <u>completely right injective</u> if every right S-system is injective. It is clear that all the homomorphic images of a completely right injective semigroup are completely right injective, of course, right self-injective.

Thus we have

Corollary 2. All the homomorphic images of a completely right injective semigroup are absolutely closed.

Remark 1. It easily follows from Isbell's zigzag theorem that a semigroup S is absolutely closed if and only if S_0 $[S^1]$ is absolutely closed, where S_0 $[S^1]$ denotes the semigroup obtained from S by adjoining a zero [an identity]. If a semigroup S is right simple, then S_0^1 (= $(S_0)^1$) is completely right injective. Thus it follows from Corollary 2 and the above that S is absolutely closed. Also if a semigroup S is finite and cyclic then we can show that S_0^1 is a self-injective semigroup (see[12]). Hence it follows from Theorem 1 and the above that S is absolutely closed. These results have been obtained by Howie and Isbell[5].

Let $\mathfrak{A}$ be any class of algebras. According to Hall [2], if for some index set I, $\{S_i : i \in I\}$ is an indexed set of algebras from $\mathfrak{A}$ having a common subalgebra U also in $\mathfrak{A}$, then the list $(S_i : i \in I : U)$ is called an <u>amalgam from</u> $\mathfrak{A}$. If there exist an algebra W and monomorphisms $\phi_i : S_i \longrightarrow W$ $(i \in I)$ such that $\phi_i | U = \phi_j | U$ and $\phi_i(S_i) \cap \phi_j(S_j) = \phi_i(U)$ for all distinct $i, j \in I$, then the amalgam $(S_i : i \in I : U)$ is said to be <u>strongly embeddable in</u> W. If an amalgam of the form $(S, S : U)$ from $\mathfrak{A}$ is strongly embeddable in an algebra from $\mathfrak{A}$, then U is said to be <u>closed in</u> S (within $\mathfrak{A}$). If U is closed in S within $\mathfrak{A}$ for all $U, S \in \mathfrak{A}$ with $U \subsetneq S$, then $\mathfrak{A}$ is said to <u>have the special amalgamation property</u>. If every amalgam from $\mathfrak{A}$ is strongly embeddable in an algebra from $\mathfrak{A}$, then $\mathfrak{A}$ is said to <u>have the strong amalgamation property</u>.

Result 2. ([4, theorem 2.4]) Let U, S be semigroups such that U is a subsemigroup of S. Then U is closed in S (within the class of semigroups) if and only if $\mathrm{Dom}_S(U) = U$.

This follows from Theorem 1, Result 2 , and Corollary 3 [12].

Theorem 2. The class of right [left] self-injective [regular] semigroups has the special amalgamation property.

The following example shows that the classof right [left] self-injective [regular] semigroups does not have the strong amalgamation property. This is constructed from an example in Imaoka [6].

Example. Let U = {0, e, f, g, 1}, V = {0, e, f, g, h, 1} and W = {0, e, f, g, x, y, 1} be semigroups whose multiplicative tables are :

U	0	e	f	g	1
0	0	0	0	0	0
e	0	e	f	g	e
f	0	e	f	g	f
g	0	e	f	g	g
1	0	e	f	g	1

V	0	e	f	g	h	1
0	0	0	0	0	0	0
e	0	e	f	g	f	e
f	0	e	f	g	f	f
g	0	e	f	g	g	g
h	0	e	f	g	h	h
1	0	e	f	g	h	1

W	0	e	f	g	x	y	1
0	0	0	0	0	0	0	0
e	0	e	f	g	x	y	e
f	0	e	f	g	x	y	f
g	0	e	f	g	x	y	g
x	0	x	y	x	x	y	x
y	0	x	y	x	x	y	y
1	0	e	f	g	x	y	1

By [11] U, V and W are completely right injective, of course, right self-injective and regular. Suppose now that the amalgam (V, W:U) is embeddable in a semigroup S. But in S we have $xh = (xe)h = x(eh) = xf = y$ and $xh = (xg)h = x(gh) = xg = x$. This is a contradiction. Hence the amalgam (V, W:U) can not be embedded in any semigroup.

§2. Commutative separative semigroups. Let S be a commutative separative semigroup. Then by [1, Theorem 4. 18] S is uniquely expressible as a semilattice Λ of archimedean cancellative semigroups S_α $(\alpha \in \Lambda)$ and S can be embedded in a semigroup T which is the same semilattice Λ of abelian groups G_α $(\alpha \in \Lambda)$ where G_α is the quotient group of S_α for each $\alpha \in \Lambda$, i.e. every element of G_α can be expressed in the form ab^{-1} with a and b in S_α.

Let ξ, ψ be homomorphisms of T to any semigroup W such that $\xi|S = \psi|S$. Then for each G_α, $\xi(G_\alpha)$ and $\psi(G_\alpha)$ are contained in a subgroup H of W. Hence $\xi(a^{-1}) = \psi(a^{-1})$ for all $a \in S_\alpha$. Because that both $\xi(a^{-1})$ and $\psi(a^{-1})$ are inverses of $\xi(a)$ in the group H. Then it is clear that $\xi|G_\alpha = \psi|G_\alpha$. Therefore we have $\xi = \psi$. This implies that $\mathrm{Dom}_T(S) = T$. Thus we have

Theorem 3. Let S be a commutative separative semigroup. Then S is absolutely closed if and only if S is a semilattice of abelian groups.

In [10] we studied self-injective non-singular semigroups and showed that every self-injective non-singular semigroup is a semilattice of groups and every commutative non-singular semigroup is separative.

More generally by Teorem 1,3 we have

Theorem 4. Every self-injective separative commutative semigroup is a semilattice of abelian groups.

In [9] B. Schein characterized self-injective inverse semigroups as follows : Let S be an inverse semigroup and E_S the set of idempotents of S. A subset B of S is <u>compatible</u> if for each $b \in S$ there is $e_b \in E_S$ with $be_b = b$ and $be_c = ce_b$ for all b, $c \in B$. Define an order $\leqq$ on S by $a \leqq b$ $(a, b \in S)$ if and only if $a \in bE_S$. S is <u>complete</u> if every compatible set B of S has the least upper bound $\bigvee B$ relatively to $\leqq$. S is <u>infinitely distribute</u> if $(\bigvee B)a = \bigvee Ba$ for any compatible set B of S and for any $a \in S$. S is <u>E_S-reflexive</u> if $st \in E_S$ implies $st \in E_S$.

Result 3. ([9, 2.3 Theorem]) Let S be an inverse semigroup and E_S the set of idempotents of S. Then S is self-injective if and only if S is complete, infinitely distribute and E_S-reflexive.

Here we can obtain the following :

Theorem 5. Let S be a commutative semigroup. Then S is self-injective and separative if and only if S is a semilattice Λ of abelian groups G_α $(\alpha \in \Lambda)$ satisfying the followings : (1) Λ is self-injective, (2) for any set $\{g_\alpha\}_{\alpha \in X}$ such that $g_\alpha e_\beta = g_\beta e_\alpha$ $(\alpha, \beta \in X$, $g_\alpha \in G_\alpha$, $g_\beta \in G_\beta$, e_α, e_β are identities of G_α, G_β, respectively) there exists $g \in S$ such that $ge_\alpha = g_\alpha$ for all $\alpha \in X$.

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