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ON THE STRUCTURE OF EXPONENT SEMIGROUPS

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1. Introduction. Let S be a semigroup and let P be the multiplicative semigroup of all positive integers. The subset

$$E(S) = \{n \in P \mid (xy)^n = x^n y^n \text{ for all } x,y \in S\}$$

of P forms a subsemigroup of P and is called the exponent semigroup of S (Tamura [9]). If $m \in E(S)$ for some $m \geq 2$, we say S is an E-m semigroup. The structure of E-m semigroups has been studied by Nordahl [7] and Cherubini and Varisco [1]. However, the structure of E(S) itself had been veiled until a recent date except for the original results on order-bounded groups by Tamura [9]. Only recently, Clarke, Pfiefer and Tamura have proved that if $2 \in E(S)$, E(S) is equal to either P or $P \setminus \{3\}$ ([3]). Inspired by their work, Kobayashi [6] studied the case $3 \in E(S)$ and has determined the structure of such E(S) up to modulo 6.

On the analogy of the results on the case $3 \in E(S)$, we present in this paper two conjectures which describe the structure of the exponent semigroups containing m up to modulo m(m-1). Several results which support the validity of the conjectures are given. Above all, the structure of the exponent semigroups of finite semigroups is described. The exponent semigroups of separative (=right and left separative) semigroups, of 0-simple semigroups and of regular semigroups are completely determined.

Detailed proofs of the results will appear elsewhere.

2. Conjecture I. In this section an integer $m \geqq 2$ is fixed and S is always an E-m semigroup. In view of [6, Theorem 1], it is not unnatural to make the following conjecture:

Conjecture I. If $k \in E(S)$ for some $k \geqq 2$, then $\alpha m(m-1)+k \in E(S)$ for all $\alpha \geqq 0$.

For some special k the conjecture is true, for example,

Proposition 1. If $k \in E(S)$ for some $k \geqq 2$ such that $k \equiv 1 \pmod{m-1}$, then $\alpha m(m-1)+k \in E(S)$ for all $\alpha \geqq 0$.

Corollary (Cherubini and Varisco [1]). $\alpha m(m-1)+m \in E(S)$ for all $\alpha \geqq 0$.

Proposition 2. $\alpha m(m-1)+1 \in E(S)$ for all $\alpha \geqq 2$.

Using the propositions above, we can prove that the following weakened forms of Conjecture I are true.

Theorem 1. If $k \in E(S)$ for some $k \geqq m^2$, then $\alpha m(m-1)+k \in E(S)$ for all $\alpha \geqq 0$.

Theorem 1'. If $k \in E(S)$, then $\alpha m(m-1)+k \in E(S)$ for all $\alpha \geqq 2k$.

3. Conjecture II. S continues to be an E-m semigroup in this section. Naturally, Theorem 1' urges us to define the subset $\bar{E}_m(S)$ of $\mathbb{Z}_{m(m-1)}$ (the residue class ring modulo m(m-1) of the integers $\mathbb{Z}$) associated with S by

$$\bar{E}_m(S) = \{\bar{n} \in \mathbb{Z}_{m(m-1)} \mid n \in E(S),\ n \geqq 2\},$$

where $\overline{n}$ denotes the class of n modulo m(m-1). $\overline{E}_m(S)$ is a multiplicative subsemigroup of $\mathbb{Z}_{m(m-1)}$ and we call it the <u>exponent semigroup</u> mod m(m-1) of S. For an integer $n \geq 1$ we define two subsets M(n) and N(n) of P by

$$M(n) = \{kn+1,\ kn+n \mid k=0,1,2,\dots\},$$

$$N(n) = \{kn+1 \mid k=0,1,2,\dots\},$$

and two subsets $\overline{M}_m(n)$ and $\overline{N}_m(n)$ of $\mathbb{Z}_{m(m-1)}$ by

$$\overline{M}_m(n) = \{\overline{kn},\ \overline{kn+1} \mid k=0,1,2,\dots\},$$

$$\overline{N}_m(n) = \{\overline{kn+1} \mid k=0,1,2,\dots\}.$$

<u>Conjecture</u> II. A subset $\overline{E}$ of $\mathbb{Z}_{m(m-1)}$ $(m \geq 2)$ is an exponent semigroup mod m(m-1) of some E-m semigroup if and only if $\overline{E}$ is expressed as

$$(\#) \qquad \overline{E} = \bigcap_{i=1}^{s} \overline{M}_m(n_i) \cap \overline{N}_m(n)$$

for a finite number of integers $n_1, \dots, n_s \geq 2$ and $n \geq 1$ such that

$$n_i \mid m \quad \text{or} \quad n_i \mid (m-1) \quad \text{for } i=1,\dots,s$$

and

$$n \mid (m-1).$$

The results [6] on E-3 semigroups support the conjecture. Cherubini and Varisco [2] have shown that the conjecture is true for $m \leq 9$. The "if" part of the conjecture is true (see Theorem 3 in §5). For that reason we say "Conjecture II is true for S" to mean "$\overline{E}_m(S)$ is expressible as (#) in Conjecture II for $m \in E(S)$ $(m \geq 2)$". One of the main results in this paper is that Conjecture II is true for finite semigroups (see Corollary 1 of Theorem 4 in §6).

4. Separative semigroups. Following Petrich [8], we call a semigroup S separative if $x^2=xy$ and $y^2=yx$ imply $x=y$, and $x^2=yx$ and $y^2=xy$ imply $x=y$, for all $x,y\in S$.

Theorem 2. Let S be a separative semigroup. Then E(S) is equal to either {1} or

$$(\#\#) \qquad \bigcap_{i=1}^{s} M(m_i)$$

for a finite number of integers $m_1,\dots,m_s \geqq 2$. Conversely, for any subset E of P given as (##), there is a finite group G such that E = E(G).

We mention here only the second assertion of the theorem. In an elementary way it is shown that for the assertion we may only consider the case E=M(m), where m is either a prime power or a product of two distinct primes. The following examples show the existence of desired groups in this case.

Example 1. Let p be a prime and $e \geqq 1$. Let G be the group of 3×3 matrices over $\mathbb{Z}_{p^e}$ given by

$$G = \left\{ \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a,b,c \in \mathbb{Z}_{p^e} \right\}.$$

Then we have

$$E(G) = \begin{cases} M(p^e) & \text{if } p \neq 2 \\ M(2^{e+1}) & \text{if } p=2. \end{cases}$$

Example 2. Let p and q be distinct primes. Let G be a group defined by a set $\{x_1,\dots,x_q,y\}$ of generators and the following defining relations:

(1) $x_1^p = \cdots = x_q^p = y^q = 1$,

(2) $x_i x_j = x_j x_i$ $(i,j=1,\ldots,q)$,

(3) $yx_1 = x_2 y$, $\ldots$, $yx_{q-1} = x_q y$, $yx_q = x_1 y$.

Then we have $E(G) = M(pq)$.

Corollary 1. Let S be a separative semigroup. If $m,n \in E(S)$ for some $m,n \geq 2$ such that $(m(m-1), n(n-1)) = 2$, then S is commutative.

A semigroup S is called finite-subdirectly irreducible if for any congruences ρ and σ on S, $\rho \cap \sigma = \iota$ (the equality relation) implies $\rho = \iota$ or $\sigma = \iota$. The result [9, Proposition 6.1] on order-bounded groups is generalized as follows.

Corollary 2. Let S be a finite-subdirectly irreducible separative semigroup. Then E(S) is either {1} or M(m) for some $m \geq 2$.

Remark. Since an E-m inverse semigroup is separative by [7, Corollary 1.12], the same conclusions as in Theorem 2 and its corollaries hold for inverse semigroups.

5. 0-simple semigroups. Using Theorem 2 and [7, Proposition 1.6], we can get

Theorem 3. Let S be a 0-simple semigroup. Then E(S) is equal to either {1} or

(###) $$\bigcap_{i=1}^{s} M(m_i) \cap N(m)$$

for a finite number of integers $m_1,\ldots,m_s \geq 2$ and $m \geq 1$. Conversely, for any subset E of P given as (###), there is a completely simple finite semigroup S such that $E = E(S)$.

By the second assertion of Theorem 3, we can say that every type of exponent semigroups supposed in Conjecture II comes from completely simple finite semigroups.

Corollary. Let S be a simple semigroup. Then

(1) If $m,n \in E(S)$ such that $(m-1, n-1) = 1$, then S is a rectangular group.

(2) If $m,n \in E(S)$ such that $(m(m-1), n(n-1)) = 2$ and $2|mn$, then S is a rectangular abelian group.

6. Finite semigroups. To prove that Conjecture II is true for finite semigroups, we need to study the exponent semigroups of two special ideal extensions.

Proposition 3. Let N be a null semigroup and T be any semigroup with 0. Let S be an ideal extension of N by T. If S is an E-m semigroup, then $\overline{E}_m(S) = \overline{E}_m(T)$.

Theorem 4. Let U be a [0-]simple semigroup and T be any semigroup with 0. Let S be an ideal extension of U by T. Then, either $E(S) = \{1\}$ or

$$E(S) = E(U) \cap E(T) \cap N(\ell)$$

for some $\ell \geqq 1$.

The proof of Theorem 4 is done by a calculation using the normalized expression of the translational hull of a completely simple semigroup due to Clifford and Petrich [4].

By a <u>principal series</u> of a semigroup S, we mean a finite chain

$$S=S_1 \supsetneq S_2 \supsetneq \cdots \supsetneq S_r \supsetneq S_{r+1}=\phi$$

of ideals S_i of S such that there is no ideal of S strictly between S_i and S_{i+1}. The Rees quotient S_i/S_{i+1} is either [0-]simple or null. If S_i/S_{i+1} is [0-]simple (resp. completely [0-]simple) for every i, then we say S is <u>semisimple</u> (resp. <u>completely semisimple</u>) (see [5, Chapter 2.6 and Chapter 6.6]).

<u>Corollary</u> 1. Conjecture II is true for semigroups with principal series, especially for finite semigroups.

<u>Corollary</u> 2. Let S be a semisimple semigroup with a principal series. If $E(S) \neq \{1\}$, then S is completely semisimple and E(S) is expressed as (###) in Theorem 3 in §5.

7. <u>Regular semigroups</u>. Using Corollary 2 of Theorem 4, we can prove

<u>Theorem</u> 5. Let S be a semilattice of simple semigroups. Then E(S) is either equal to {1} or expressed as (###) in Theorem 3.

Since an E-m regular semigroup is a semilattice of simple semigroups by [7, Corollary 1.11], we get

<u>Corollary</u>. Let S be a regular semigroup, then E(S) is either {1} or expressed as (###) in Theorem 3.

References

[1] A.Cherubini Spoletini and A.Varisco, Some properties of E-m semigroups, Semigroup Forum 17 (1979), 153-161.

[2] ———, The exponent semigroup of an E-m semigroup, preprint.

[3] J.Clarke, R.Pfiefer and T.Tamura, Identities E-2 and exponentiality, Proc. Japan Acad. 55 (1979), 250-251.

[4] A.H.Clifford and M.Petrich, Some classes of completely regular semigroups, J. of Algebra 46 (1977), 462-480.

[5] A.H.Clifford and G.B.Preston, The algebraic theory of semigroups, Math. Surveys 7, Amer. Math. Soc., Providence, Vol.I (1961), Vol.II (1967).

[6] Y.Kobayashi, The exponent semigroup of a semigroup satisfying $(xy)^3=x^3y^3$, Semigroup Forum 19 (1980), 323-330.

[7] T.Nordahl, Semigroups satisfying $(xy)^m=x^my^m$, Semigroup Forum 8 (1974), 332-346.

[8] M.Petrich, Introduction to semigroups, Merrill Publishing Co., Columbus, 1973.

[9] T.Tamura, Complementary semigroups and exponent semigroups of order bounded groups, Math. Nachr. 49 (1973), 17-34.

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