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(1,0,x)-REPRESENTATION OF A GRAPH

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1. INTRODUCTION

A new method to represent a graph is introduced. Following this method, each vertex is assigned a vector of element 1, 0 or x. The adjacency relation between two vertices follows the adjacency relation between corresponding two vectors which is defined here. For a given graph, the representation is not unique. Thus, the K-dimension is defined as the minimum dimension with which the graph can be represented. Discussions are made mainly with respect to K-dimension since it is considered as an essential characteristic number of the graph that is closely related to the symmetricity and distribution of cliques in the graph.

2. DEFINITIONS AND FUNDAMENATAL THEOREMS

A (1,0,x)-vector is a vector whose elements are 1, 0 or x. Let P^{ℓ} be the set of all (1,0,x)-vectors of dimension ℓ. A vector of P^{ℓ} that contains no x is called the cell. The domain D(v) of $v \in P^{\ell}$ is the set of all cells that are derived from v by letting each x of the vector to 1 or 0. Two vectors v_i and v_j of P^{ℓ} are defined to be adjacent if and only if $D(v_i) \cap D(v_j) \neq \emptyset$.

In the following, a graph always means an undirected graph without parallel edges. Definitions and notations such as 'union of two graphs', join of two graphs', 'complete graph', 'complete bipartite graph', 'complement', 'clique', and others that are not defined here follow Harary[1].

For a subset V of P^{ℓ}, a graph G=G(V) is defined as follows. See Fig.1(A). The vertices of G correspond one to one to V. Two vertices of G are adjacent if and only if the corresponding two vectors are adjacent. In this paper, the vector set V is, for simplicity, used to denote the vertex set of the graph G(V).

Theorem 1: For any graph G, there exists ℓ and $V \subset P^{\ell}$ such that G=G(V).

END

Lemma 1: Let G_1+G_2 and $G_1 \cup G_2$ denote the join and union of two graphs G_1 and G_2, respectively. Then, if $\ell=1$

$$G(V)=K_{n_x}+(K_{n_1} \cup K_{n_0}) \quad \text{and} \quad \overline{G(V)}=\overline{K_{n_x}} \cup (\overline{K_{n_1}}+\overline{K_{n_0}})$$

where n_x, n_1 and n_0 are the cardinalities of vertices assinged with x, 1 and 0, respectively.

END

Here we call such graphs G(V) and $\overline{G(V)}$ as in the lemma the co-complete bipartite graph and complete bipartite graph, respectively.

Consider two graphs with the same set of vertices, say $G_1=(V,X_1)$ and $G_2=(V,X_2)$. Then, their edge-intersection (edge-union) $G_1 \wedge G_2$ ($G_1 \vee G_2$) is $(V, X_1 \cap X_2)$ ($(V, X_1 \cup X_2)$).

In case $\ell>1$, each kth coordinate represents a co-complete bipartite graph which we denote as $G(V^k)$. Thus, we obtain the theorem.

Theorem 2:

$$G(V)=\bigwedge_{k=1}^{\ell} G(V^k) \quad \text{and} \quad \overline{G(V)}=\bigvee_{k=1}^{\ell} \overline{G(V^k)}$$

where $\bigwedge$ ($\bigvee$) means the edge-intersection (-union) taken over all graphs of k=1 to ℓ.

END

3. SOME THEOREMS ON K-DIMENSION

The minimum ℓ with which a graph G can be represented is called the K-dimension of G and denoted by K(G). By Theorem 2, to find the K-dimension for a given graph G=(V,X) is reduced to the problem of determining

(a) the minimum number of co-complete bipartite graphs by edge-intersection of which X can be covered,

or equivalently

(b) the minimum number of complete bipartite graphs by edge-union of which $\bar{X}$, the complement of X, can be covered.

Note that in (b), if 'complete bipartite graphs' were restricted to its special type '$K_{1,p}$', the problem would be no other than that well-known NP-class problem of determining the minimum set of vertices that cover the complement of X. If $\bar{\alpha}_0$ and γ denote the covering number of $\bar{G}$ and the clique number (the maximum cardinality of vertex set whose induced graph forms a clique) of G, respectively, $\bar{\alpha}_0+\gamma=p$ holds where p is the number of vertices of G. Thus, we obtain the theorem.

Theorem 3: $K(G)\ p-\gamma=\bar{\alpha}_0$.

END

Corollary of Th.3: If $\bar{G}$ contains no cycles of length 4, $K(G)=p-\gamma=\bar{\alpha}_0$

END

Consider a graph G=G(V), $V\subset P^{\ell}$. For a subset of $V_s \subset V$, define

$$D'(V_s)=\bigwedge_{v\in V_s} D(v).$$

Lemma 2: For V_s and $V_t \subset V$,

(a) $G(V_s)$ is a complete graph imbedded in G if and only if $D'(V_s)\neq\emptyset$.

(b) If $G(V_s)$ and $G(V_t)$ are two different cliques of G, then $D'(V_s)\cap D'(V_t)\neq\emptyset$.

(c) If the number of distictive maximal cliques of G is q, $2^{\ell}\geq q$.

END

(a) and (b) are easily obtained. (c) follows them since each clique occupies exclusively at least one cell, and the cardinality of possible cells of P^{ℓ} is 2^{ℓ}.

For a real number r, {r} denotes the smallest integer not less than r.

Theorem 4: $K(G)\geq\{\log_2 q\}$.

END

Theorem 5: If the clique graph C(G) of G is a subgraph of the n-cube, $K(G)\leq n$.

END

Corollaries of Ths.4 and 5: If C(G) is a subgraph of the $\{\log_2 q\}$-cube,

$$K(G)=\{\log_2 q\}.$$

END

Example: A graph G and its clique graph C(G) are shown in Fig.1(A) and (B), respectively. Since q=6, $\{\log_2 6\}=3$ and C(G) is a subgraph of 3-cube as shown in Fig.1(C), a (1,0,x)-representation of $\ell=3$ in Fig.1(A) is the minimum representation.

END

4. SOME GENERAL EXAMPLES

Some results on K-dimension for certain graphs are presented.

Theorem 6: If G_s is an induced subgraph of G, $K(G_s)$ $K(G)$.

END

Theorem 7: For two graphs G_1 and G_2,

(A)$K(G_1+G_2) \leq K(G_1)+K(G_2)$.

(B)$K(G_1 \cup G_2) \leq Max(K(G_1),K(G_2))+1$.

END

Theorem 8: Let G_e and $G_{\bar{e}}$ be the graphs obtained from G by contracting and deleting the edge e, respectively. Then,

(A)$K(G_{\bar{e}}) \leq K(G)+1$.

(B)$K(G_e) \leq K(G)+\alpha$, where $\alpha=0$ if two endpoints of e cover all the vertices of G and $\alpha=0$ otherwise.

END

Let a path and a cycle of p vertices be P_p and C_p, respectively. Then, $C(P_p)=P_{p-1}$ and $C(C_p)=C_n$. And it is not difficult to show that $\{log_2 p\}$-cube has P_p as a subgraph, and that $\{log_2 2k\}$-cube C_{2k}. Thus we have the theorem.

Theorem 9:

(A) $K(P_p)=\{log_2 p\}$.

(B) $K(C_{2k}\}=\{log_2 2k\}$.

END

As for the cycle of odd length, we do not yet have definite result. Up to C_{10}, we will list $K(C_p)$.

$K(C_3)=1$, $K(C_4)=2$, $K(C_5)=3$, $K(C_6)=3$, $K(C_7)=4$, $K(C_8)=3$, $K(C_9)=4$, $K(C_{10})=4$.

Thus, $K(C_p)$ is not a non-decreasing function of p.

5. CONCLUSION

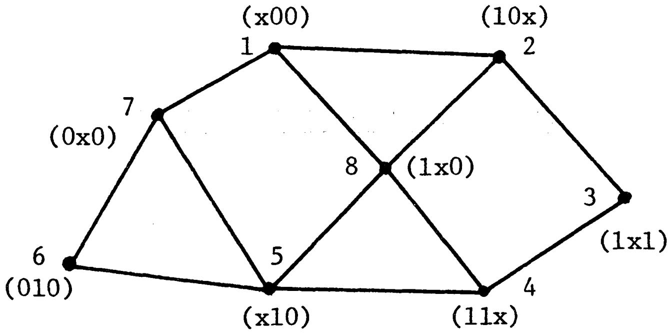
We introduced a new representation, called the (1,0,x)-representation, of a graph. The concept was first introduced in the problem of diagnosis of logic networks[2]. But herethe consideration was devoted mainly to the problem on K-dimension since the K-dimension is considered as an important characteristic number of the graph.

It is shown that for a given graph to get the (1,0,x)-representation with its K-dimension is the problem to find the minimum covering of the complement edges by the complete bipartite graphs.

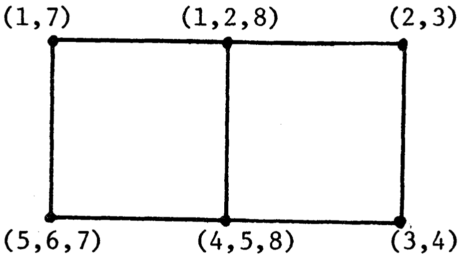
REFERENCES

[1] F. Harary: Graph Theory, Addison-Wesley, 1969.

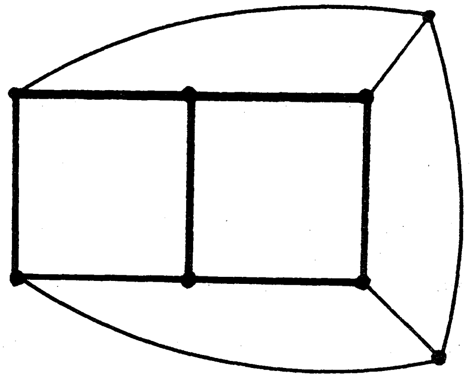
[2] K. Kojima, C. Tanaka and H. Kanada:" A Consideration of the Diagnosis for Sequential Circuits" , Trans. Inst. Elec. Comm. Engrg. of Japan, 55D, 1, p.39, 1972.



(A) graph G



(B) clique graph C(G)



(C) C(G) as a subgraph of Q_3

Fig.1 An example for Cor. of Ths. 4 and 5