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Uniformly Finite-to-one and Onto Extensions of Homomorphisms between Directed Graphs

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Introduction

For two directed graphs G_1 and G_2, a homomorphism h of G_1 into G_2 is, roughly speaking, a mapping of the set of arcs of G_1 into the set of arcs of G_2 that preserves the adjacency of arcs. The homomorphism h is naturally extended to a mapping h^* of the set of all paths in G_1 into the set of all paths in G_2, which is called the extension of h.

The main object of this paper is to establish two properties of uniformly finite-to-one and onto extensions of homomorphisms between strongly connected directed graphs. One of them is shown to be also a property of those between directed graphs with no restriction, from which the following result is immediately obtained. For two directed graphs G_1 and G_2, if there exists a homomorphism h of G_1 into G_2 such that the extension h^* of h is uniformly finite-to-one and onto, then the adjacency matrices $M(G_1)$ of G_1 and $M(G_2)$ of G_2 have the same maximal characteristic value but also the characteristic polynomial of $M(G_1)$ is divided by that of $M(G_2)$. The other property is stated as follows. If G_1 and G_2 are two strongly connected directed graphs such that their adjacency matrices have the same maximal characteristic value, then for any homomorphism h of G_1 into G_2, h^* is uniformly finite-to-one if and only if h^* is onto.

1. Preliminaries

Let A be a finite nonempty set. A sequence of finite length of elements of A is called a <u>string over A</u>. The sequence of length 0 is also a string and is denoted by Λ. For a string x, lg(x) denotes the length of x. The set of all strings over A is denoted by A^*. For a non-negative integer n, A^n is the set of all strings of length n over A. For $x, y \in A^*$, xy denotes the string obtained by concatenating the two strings x and y.

A <u>graph</u> (directed graph with labeled arcs and labeled points) G is defined to be a triple $\langle P, A, \zeta \rangle$ where P is a finite set of elements called <u>points,</u> A is a finite set of elements called <u>arcs</u> and ζ is a mapping of A into $P \times P$. If $\zeta(a) = (u, v)$ for $a \in A$ and $u, v \in P$, then u is called the <u>initial endpoint</u> of a, v is called the <u>terminal endpoint</u> of a, and we say that <u>a goes from u to v</u>.

Let $G = \langle P, A, \zeta \rangle$ be a graph. A string $x = a_1 \cdots a_p$ $(p \geq 1)$ over A with $a_i \in A$ $(i = 1, \cdots, p)$ is called a <u>path of length p</u> in G if the terminal endpoint of a_i is the initial endpoint of a_{i+1} for $i = 1, \cdots, p-1$. The initial endpoint u of a_1 is called the <u>initial endpoint</u> of x, the terminal endpoint v of a_p is called the <u>terminal endpoint</u> of x, and we say that <u>x goes from u to v</u>. Each point u of G is a <u>path of length 0</u> (going from u to itself). The set of all paths in G is denoted by $\Pi(G)$. The set of all paths of length $p (\geq 0)$ in G is denoted by $\Pi^{(p)}(G)$. Note that $\Pi^{(p)}(G) = A^p \cap \Pi(G)$ for $p \geq 1$.

A graph $G = \langle P, A, \zeta \rangle$ is said to be <u>strongly connected</u> if $p \neq \phi$ and for any $u, v \in P$, there exists a path from u to v in G. Of course, a graph consisting of exactly one point and no arc is strongly connected. But, for convenience, in what follows we assume, unless otherwise stated, that a strongly connected graph has at least one arc. However, we remark that all theorems, the proposition, and all lemmas concerning strongly connected graphs in this paper trivially hold for strongly connected graphs with one point and no arc.

Let $G_1 = \langle P, A, \zeta_1 \rangle$ and $G_2 = \langle Q, B, \zeta_2 \rangle$ be two graphs. A homomorphism $\mathfrak{h}$ of G_1 into G_2 is a pair (h, ϕ) of a mapping $h : A \to B$ and a mapping $\phi : P \to Q$ such that for any $a \in A$, if $\zeta_1(a) = (u, v)$ with $u, v \in P$, then $\zeta_2(h(a)) = (\phi(u), \phi(v))$.

If G_1 is strongly connected, then a homomorphism $\mathfrak{h} = (h, \phi)$ of G_1 into G_2 is uniquely determined by h. Therefore, when G_1 is strongly connected, we say that h is a homomorphism of G_1 into G_2 and we denote by ϕ_h the unique mapping ϕ such that (h, ϕ) is a homomorphsm of G_1 into G_2.

For a homomorphism $\mathfrak{h} = (h, \phi)$ of a graph $G_1 = \langle P, A, \zeta_1 \rangle$ into a graph $G_2 = \langle Q, B, \zeta_2 \rangle$ and a subgraph $G_1' = \langle P', A', \zeta_1' \rangle$ of G_1, we denote the subgraph $\langle \phi(P'), h(A'), \zeta_2' \rangle$ of G_2 by $\mathfrak{h}(G_1')$. (A graph $G' = \langle P', A', \zeta' \rangle$ is a subgraph of a graph $G = \langle P, A, \zeta \rangle$ if $P' \subset P$, $A' \subset A$, and $\zeta'(a) = \zeta(a)$ for all $a' \in A'$.) It is easy to see that if G_1' is strongly connected, then $\mathfrak{h}(G_1')$ is strongly connected. When G_1' is strongly connected, we often denote $\mathfrak{h}(G_1')$ by $h(G_1')$.

Let $G_1 = \langle P, A, \zeta_1 \rangle$ and $G_2 = \langle Q, B, \zeta_2 \rangle$ be graphs. Let $\mathfrak{h} = (h, \phi)$ be a homomorphism of G_1 into G_2. We define the extension $\mathfrak{h}^* : \Pi(G_1) \to \Pi(G_2)$ of $\mathfrak{h}$ as follows. For each $x \in \Pi(G_1)$, if $lg(x) = 0$, i.e., x is a point of G_1, then

$$\mathfrak{h}^*(x) = \phi(x)$$

and if $x = a_1 \cdots a_p$ $(p \geq 1)$ with $a_i \in A$ $(i = 1, \cdots, p)$, then

$$\mathfrak{h}^*(x) = h(a_1) \cdots h(a_p).$$

When G_1 is strongly connected, we often use h^* instead of $\mathfrak{h}^*$ and say that h^* is the extension of the homomorphism h.

A mapping $f : A \to B$ is said to be uniformly finite-to-one if there exists a positive number N such that $|f^{-1}(y)| \leq N$ for all $y \in B$.

Let G_1 and G_2 be two graphs. Let h be a homomorphism of G_1 into G_2. Two paths x and y in G_1 are said to be <u>indistingushable by h</u> if x and y have the same initial endpoint and the same terminal endpoint and $h^*(x) = h^*(y)$.

Proposition 1. Let $G_1 = \langle P, A, \zeta_1 \rangle$ be a strongly connected graph and let $G_2 = \langle Q, B, \zeta_2 \rangle$ be a graph. Let $h : A \to B$ be a homomorphism of G_1 into G_2. Then $h^* : \Pi(G_1) \to \Pi(G_2)$ is uniformly finite-to-one if and only if no two distinct paths in G_1 are indistinguishable by h.

Proof. Suppose that x_1 and x_2 are two distinct paths in G_1 such that they have the same initial endpoint, say u, and the same terminal endpoint, say v, and $h^*(x_1) = h^*(x_2)$. Since G_1 is strongly connected, there exists a path z going from v to u. For any positive integer N, we have $|(h^*)^{-1}((h^*(x_1)h^*(z))^N)| \geq 2^N$. Hence h^* is not uniformly finite-to-one.

Suppose that h^* is not uniformly finite-to-one. Then there exists a path $y \in \Pi(G_2)$ such that $|(h^*)^{-1}(y)| > |P|^2$. Since the number of paths x's in $(h^*)^{-1}(y)$ is gerater than the number of all possible pairs of the initial endpoint and the terminal endpoint of a path in G_1, there exist two distinct paths with the same initial endpoint and the same terminal endpoint in $(h^*)^{-1}(y)$. $\square$

For a graph $G = \langle P, A, \zeta \rangle$ with $P = \{u_1, \cdots, u_n\}$, the <u>adjacency matrix</u> M(G) is the square matrix (m_{ij}) of order n such that m_{ij} is the number of arcs going from u_i to u_j $(1 \leq i, j \leq n)$.

A matrix M with real elements is said to be <u>non-negative</u> if all elements of M are non-negative. By the Frobenius Theorem (cf. Gantmacher [2] or Nikaido [7]), any non-negative square matrix M has a non-negative real characteristic value which the moduli of all the other characteristic values of M do not exceed. We call that maximum real characteristic value the <u>maximal characteristic value</u> of M. For a graph G, we denote the maximal characteristic value of M(G) by r(G).

A square matrix M is said to be <u>irreducible</u> if there is no permutation matrix H such that $H^{-1}MH$ has the form

$$\begin{pmatrix} M_1 & 0 \\ M_2 & M_3 \end{pmatrix} \qquad (1.1)$$

where M_1 and M_3 are square matrices and 0 is a zero matrix. For a graph G, G is strongly connected if and only if M(G) is irreducible.

Let M be an irreducible non-negative square matrix of order n. Let r be the maximal characteristic value of M. By the Perron-Frobenius Theorem (cf. Gantmacher [2] or Nikaido [7]), $r > 0$ and to the maximal characteristic value r there corresponds a characteristic vector $w = (w_1, \cdots, w_n)$ with $w_i > 0$ for $i = 1, \cdots, n$. Let $D = (d_{ij})$ be the diagonal matrix of order n such that $d_{ii} = w_i$ $(i = 1, \cdots, n)$. Then the sum of all the coordinates of each column vector of DMD^{-1} is equal to r. (Cf. [2]). For any matrix K, let us denote the sum of all the elements of K by S(K). Then, since for each non-negative integer p, $M^p = D^{-1}(DMD^{-1})^p D$ and $S((DMD^{-1})^p) = nr^p$, we have

$$\alpha r^p \le S(M^p) \le \beta r^p \qquad (p = 0, 1, \cdots) \qquad (1.2)$$

where $\alpha = n \min_{1 \le i,j \le n} (w_i/w_j)$ and $\beta = n \max_{1 \le i,j \le n} (w_i/w_j)$.

Lemma 1. let G_1 and G_2 be two strongly connected graphs. Let h be a homomorphism of G_1 into G_2. Then the following two statements are valid.

(1) If h^* is uniformly finite-to-one, then $r(G_1) \le r(G_2)$.

(2) If h^* is onto, then $r(G_1) \ge r(G_2)$.

Proof. Assume that h^* is uniformly finite-to-one. Then there exists a positive number N such that $|(h^*)^{-1}(y)| \le N$ for all $y \in \Pi(G_2)$. Thus since for each non-negative integer p, $|\Pi^{(p)}(G_1)| = \sum_{y \in \Pi^{(p)}(G_2)} |(h^*)^{-1}(y)|$, it follows that

$$|\Pi^{(p)}(G_1)| \leq N|\Pi^{(p)}(G_2)| \qquad (p = 0, 1, \cdots) \qquad (1.3).$$

Since for $i = 1, 2$,

$$|\Pi^{(p)}(G_i)| = S((M(G_i))^p) \qquad (p = 0, 1, \cdots),$$

using (1.2) and (1.3) we have $r(G_1) \leq r(G_2)$.

Assume that h^* is onto. Then it follows that

$$|\Pi^{(p)}(G_1)| \geq |\Pi^{(p)}(G_2)| \qquad (p = 0, 1, \cdots).$$

Hence by the same argument as above, we have $r(G_1) \geq r(G_2)$. □

2. Uniformly finite-to-one and onto extensions.

Let $G = \langle P, A, \zeta \rangle$ be a graph with $A = \{a_1, \cdots, a_\ell\}$. Let Z be the ring of integers. We consider the polynomial ring $Z[a_1, \cdots, a_\ell]$ in indeterminates $a_1, \cdots, a_\ell$ over Z. Let $P = \{u_1, \cdots, u_n\}$. Let $\hat{M} = (\hat{m}_{ij})$ be the matrix of order n with elements in $Z[a_1, \cdots, a_\ell]$ such that $\hat{m}_{ij} = a_{p_1} + \cdots + a_{p_k}$ if $a_{p_1}, \cdots, a_{p_k}$ are all arcs from u_i to u_j in G, and $\hat{m}_{ij} = 0$ if there exists no arc from u_i to u_j $(1 \leq i, j \leq n)$. Then the matrix $\hat{M}$ is called the <u>representation matrix</u> of G and is denoted by $\hat{M}(G)$. Let X be an indeterminate not contained in A. Let $\hat{f}_G$ be the polynomial in $Z[a_1, \cdots, a_\ell, X]$ which is equal to the characteristic polynomial of $\hat{M}(G)$, i.e., let $\hat{f}_G$ be the polynomial defined by

$$\hat{f}_G(a_1, \cdots, a_\ell, X) = \det (XI_n - \hat{M}(G))$$

where I_n is the identity matrix of order n. Then $\hat{f}_G(a_1, \cdots, a_\ell, X)$ is homogeneous of degree n. Let $f_G(X)$ be the characteristic polynomial of the adjacency matrix M(G) of G. Then clearly

$$f_G(X) = \hat{f}_G(1, \cdots, 1, X).$$

In this section we shall prove the following theorem.

Theorem 1. Let $G_1 = \langle P, A, \zeta_1 \rangle$ and $G_2 = \langle Q, B, \zeta_2 \rangle$ be two strongly connected graphs with $A = \{a_1, \cdots, a_\ell\}$ and $B = \{b_1, \cdots, b_m\}$. Let $h : A \to B$ be a homomorphism of G_1 into G_2. Let g be the polynomial in $Z[b_1, \cdots, b_m, X]$ obtained from $\hat{f}_{G_1}(a_1, \cdots, a_\ell, X)$ by substituting $h(a_i)$ for a_i in $\hat{f}_{G_1}$ for $i = 1, \cdots, \ell$. Then, if h^* is uniformly finite-to-one and onto, then $r(G_1) = r(G_2)$ and $\hat{f}_{G_2}(b_1, \cdots, b_m, X)$ divides $g(b_1, \cdots, b_m, X)$ in $Z[b_1, \cdots, b_m, X]$.

Theorem 1 can be generalized to graphs with no restriction. Using Theorem 1, we can prove the following theorem. But the proof is omitted in this paper. (It is found in [5].)

Theorem 1'. Let $G_1 = \langle P, A, \zeta_1 \rangle$ and $G_2 = \langle Q, B, \zeta_2 \rangle$ be two graphs with $A = \{a_1, \cdots, a_\ell\}$ and $B = \{b_1, \cdots, b_m\}$. Let $\mathsf{h} = (h, \phi)$ be a homomorphism of G_1 into G_2. Let g be the polynomial in $Z[b_1, \cdots, b_m, X]$ obtained from $\hat{f}_{G_1}(a_1, \cdots, a_\ell, X)$ by substituting $h(a_i)$ for a_i in $\hat{f}_{G_1}$ for $i = 1, \cdots, \ell$. Then if h^* is uniformly finite-to-one and onto, then $r(G_1) = r(G_2)$ and $\hat{f}_{G_2}(b_1, \cdots, b_m, X)$ divides $g(b_1, \cdots, b_m, X)$ in $Z[b_1, \cdots, b_m, X]$.

As a direct consequence of Theorem 1', we have the following result.

Corollary 1. Let G_1 and G_2 be two graphs. If there exists a homomorphism of G_1 into G_2 such that the extension h^* of h is uniformly finite-to-one and onto, then not only $r(G_1) = r(G_2)$ but also $f_{G_1}(X)$ is divided by $f_{G_2}(X)$.

To prove Theorem 1, we use Lemma 1 and furthermore four lemmas.

Lemma 2. Let $G_1 = \langle P, A, \zeta_1 \rangle$ and $G_2 = \langle Q, B, \zeta_2 \rangle$ be two strongly connected graphs with $A = \{a_1, \cdots, a_\ell\}$ and $B = \{b_1, \cdots, b_m\}$. Let $h : A \to B$ be a homomorphism of G_1 into G_2. Write $\hat{f}_{G_1} = \hat{f}_{G_1}(a_1, \cdots, a_\ell, X)$ and $\hat{f}_{G_2} = \hat{f}_{G_2}(b_1, \cdots, b_m, X)$. Let g be the polynomial in $Z[b_1, \cdots, b_m, X]$ defined by

$$g(b_1, \cdots, b_m, X) = \hat{f}_{G_1}(h(a_1), \cdots, h(a_\ell), X)$$

Then if h^* is uniformly finite-to-one and onto, then for any m positive integers $p_1, \cdots, p_m$, there exists a real number r such that

$$g(p_1, \cdots, p_m, r) = \hat{f}_{G_2}(p_1, \cdots, p_m, r) = 0.$$

Proof. Assume that $h^* : \Pi(G_1) \to \Pi(G_2)$ is uniformly finite-to-one and onto. Let $p_1, \cdots, p_m$ be any m positive integers. We construct two graphs G_1' and G_2' as follows. For each $i = 1, \cdots, \ell$, let $j(i)$ be the number such that $h(a_i) = b_{j(i)}$ and let $A_i = \{a_{i,1}, a_{i,2}, \cdots, a_{i,p_{j(i)}}\}$ where $a_{i,\nu}$, $\nu = 1, \cdots, p_{j(i)}$, are new distinct elements for every i. The graph G_1' is obtained from G_1 by replacing each arc a_i with the arcs consisting of the elements of A_i. That is, $G_1' = \langle P, A', \zeta_1' \rangle$ where $A' = \overset{\ell}{\underset{i=1}{\cup}} A_i$ and $\zeta_1' : A' \to P \times P$ is defined by

$$\zeta_1'(a_{i,\nu}) = \zeta_1(a_i) \qquad (\nu = 1, \cdots, p_{j(i)}, \ i = 1, \cdots, \ell).$$

For each $j = 1, \cdots, m$, let $B_j = \{b_{j,1}, \cdots, b_{j,p_j}\}$ where $b_{j,\nu}$, $\nu = 1, \cdots, p_j$, are new distinct elements for every j. The graph G_2' is obtained from G_2 by replacing each arc b_j with the arcs $b_{j,1}, \cdots, b_{j,p_j}$. That is, $G_2' = \langle Q, B', \zeta_2' \rangle$ where $B' = \overset{m}{\underset{j=1}{\cup}} B_j$ and $\zeta_2' : B' \to Q \times Q$ is defined by

$$\zeta_2'(b_{j,\nu}) = \zeta_2(b_j) \qquad (\nu = 1, \cdots, p_j, \ j = 1, \cdots, m).$$

Let $h' : A' \to B'$ be the mapping defined by

$$h'(a_{i,\nu}) = b_{j(i),\nu} \qquad (\nu = 1, \cdots, p_{j(i)}, \ i = 1, \cdots, \ell).$$

Then clearly, G_1' and G_2' are strongly connected and h' is a homomorphism of G_1' into G_2'. Furthermore, for any path $b_{j_1,\nu_1} \cdots b_{j_s,\nu_s}$ in G_2', $b_{j_1} \cdots b_{j_s}$ is a path of G_2 and it follows that

$$|((h')^*)^{-1}(b_{j_1,\nu_1} \cdots b_{j_s,\nu_s})| = |(h^*)^{-1}(b_{j_1} \cdots b_{j_s})|$$

Therefore, since h^* is uniformly finite-to-one and onto, so is $(h')^*$. Hence, by Lemma 1, $r(G_1') = r(G_2')$. Let $r = r(G_1')$. Then the characteristic polynomials $f_{G_1'}(X)$ of $M_{G_1'}$ and $f_{G_2'}(X)$ of $M_{G_2'}$ have r as their common root.

Write $A' = \{a_1', \cdots, a_{\ell'}'\}$ and $B' = \{b_1', \cdots, b_{m'}'\}$. Write $\hat{f}_{G_1'} = \hat{f}_{G_1'}(a_1', \cdots, a_{\ell'}', X)$ and $\hat{f}_{G_2'} = \hat{f}_{G_2'}(b_1', \cdots, b_{m'}', X)$. Then since for each arc b_j in G_2, if $\zeta_2(b_j) = (u_j, v_j)$ $(u_j, v_j \in Q)$, then there exist p_j arcs going from u_j to v_j in G_2', we have

$$\hat{f}_{G_2}(p_1, \cdots, p_m, X) = \hat{f}_{G_2'}(1, \cdots, 1, X).$$

Also, since for each arc a_i in G_1, if $h(a_i) = b_j$ and $\xi_1(a_i) = (s_i, t_i)$ $(s_i, t_i \in P)$, then there exist p_j arcs going from s_i to t_i in G_1', it follows that

$$g(p_1, \cdots, p_m, X) = \hat{f}_{G_1'}(1, \cdots, 1, X).$$

Since $\hat{f}_{G_1'}(1, \cdots, 1, X) = f_{G_1'}(X)$, $\hat{f}_{G_2'}(1, \cdots, 1, X) = f_{G_2'}(X)$ and $f_{G_1'}(r) = f_{G_2'}(r) = 0$, we have

$$g(p_1, \cdots, p_m, r) = \hat{f}_{G_2}(p_1, \cdots, p_m, r) = 0. \qquad \square$$

An arc a of a graph G is called a <u>loop</u> of G if the initial and terminal endpoints of a are the same.

Lemma 3. Let $G_1 = \langle P, A, \xi_1 \rangle$ and $G_2 = \langle Q, B, \zeta_2 \rangle$ be two strongly connected graphs. Let $h : A \to B$ be a homomorphism of G_1 into G_2. Let $v \in Q$. Let G_2' be a graph obtained from G_2 by adding a loop $b_v (\notin B)$ from v to itself. Let G_1' be a graph obtained from G_1 by adding a loop a_u $(\notin A)$ from u to itself for every point u in $\phi_h^{-1}(v)$. (ϕ_h was defined in the preceding section.) Let h' be the mapping of $A \cup \{a_u \mid u \in \phi_h^{-1}(v)\}$ into $B \cup \{b_v\}$ defined by

$$h'(a) = \begin{cases} h(a) & \text{if } a \in A \\ b_v & \text{if } a = a_u \text{ with } u \in \phi_h^{-1}(v). \end{cases}$$

Then h' is a homomorphism of G_1' into G_2' and if h^* is uniformly finite-to-one and onto, then $(h')^*$ is also uniformly finite-to-one and onto.

Proof. Clearly h' is a homomorphism of G_1' into G_2'. Any path z in $\Pi(G_2')$ is written as $z = x_1y_1x_2y_2\cdots x_\ell y_\ell$ where $x_i \in \Pi(G_2)$ $(i = 1, \cdots, \ell)$, $x_1x_2\cdots x_\ell \in \Pi(G_2)$, and $y_i \in \{b_v\}^*$ for $i = 1, \cdots, \ell$. It is easily seen that $|((h')^*)^{-1}(z)| = |(h^*)^{-1}(x_1\cdots x_\ell)|$. Hence if h^* is uniformly finite-to-one and onto, then $(h')^*$ is also uniformly finite-to-one and onto. $\square$

Lemma 4. Let $G = \langle P, A, \zeta \rangle$ with $A = \{a_1, \cdots, a_m\}$ be a graph such that for every $u \in P$, there exists at least one loop going from u to itself. Then

(1) $\det \hat{M}(G)$ is an irreducible polynomial in $Z[a_1, \cdots, a_m]$ if (and only if) G is strongly connected, and

(2) $\hat{f}_G(a_1, \cdots, a_m, X)$ is an irreducible polynomial in $Z[a_1, \cdots, a_m, X]$ if (and only if) G is strongly connected.

Proof. We first note that for any homogeneous polynomial $f(X_1, \cdots, X_\ell)$ in indeterminates $X_1, \cdots, X_\ell$ over an integral domain k such that $f(0, X_2, \cdots, X_\ell) \neq 0$, if $f(0, X_2, \cdots, X_\ell)$ is irreducible, then $f(X_1, \cdots, X_\ell)$ is irreducible. (Since $f(X_1, \cdots, X_\ell)$ is homogeneous and $f(0, X_2, \cdots, X_\ell) \neq 0$, $\deg f(0, X_2, \cdots, X_\ell) = \deg f(X_1, \cdots, X_\ell)$. Assume that $f(0, X_2, \cdots, X_\ell)$ is irreducible but $f(X_1, \cdots, X_\ell)$ is reducible. Then there exist polynomials g_1 and g_2 such that $f(X_1, \cdots, X_\ell) = g_1(X_1, \cdots, X_\ell)g_2(X_1, \cdots, X_\ell)$ and $1 \leq \deg g_1 < \deg f$. Hence $f(0, X_2, \cdots, X_\ell) = g_1(0, X_2, \cdots, X_\ell)g_2(0, X_2, \cdots, X_\ell)$. Since $f(0, X_2, \cdots, X_\ell) \neq 0$, $g_1(0, X_2, \cdots, X_\ell) \neq 0$. Since $f(X_1, \cdots, X_\ell)$ is homogeneous, $g_1(X_1, \cdots, X_\ell)$ is homogeneous. (Cf. van der Waerden [9], § 23.) Hence $\deg g_1(0, X_2, \cdots, X_\ell) = \deg g_1(X_1, \cdots, X_\ell)$. Thus $1 \leq \deg g_1(0, X_2, \cdots, X_\ell) < \deg f(0, X_2, \cdots, X_\ell)$. Therefore, $f(0, X_2, \cdots, X_\ell)$ is reducible, which contradicts the assumption.) Therefore, since $\hat{f}_G(a_1, \cdots, a_m, 0) = \det(-\hat{M}(G))$, the if part of (2) follows from that of (1). Moreover, to prove the if part of (1), it suffices to show the following :

(3) det $\hat{M}(G)$ is an irreducible polynomial if $G = \langle P, A, \zeta \rangle$ is a graph such that (i) G is strongly connected, (ii) for any $u \in P$, there exists at least one loop from u to itself in G, and (iii) any graph obtained from G by deleting an arc of G does not satisfy both (i) and (ii).

Let $G = \langle P, A, \zeta \rangle$ be any graph. A path $z = a_1 \cdots a_\ell$ with $a_i \in A$ ($i = 1, \cdots, \ell$) is called a circuit of length ℓ if $a_i \neq a_j$ for any $i, j, 1 \leq i < j \leq \ell$, and the initial and terminal endpoints of z are the same. A circuit $z = a_1 \cdots a_\ell$ is said to be elementary if a_i and a_j have distinct initial endpoints for any $i, j, 1 \leq i < j \leq \ell$. (Each loop is an elementary circuit of length 1.) A set E of elementary circuits in G is called a circuit-cover of G if each point of G is on exactly one circuit in E.

Let $\mathcal{E}_G$ be the set of all circuit covers of G. Write $\hat{M}(G) = (a_{ij})$. Then det $\hat{M}(G)$ is written as

$$\det \hat{M}(G) = \sum_\sigma \varepsilon(\sigma)\, a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$$

where $n = |P|$, σ is a permutation on $\{1, \cdots, n\}$, and $\varepsilon(\sigma) = 1$ or -1 if permutation σ is even or odd, respectively. For every permutation σ such that $a_{1\sigma(1)} \cdots a_{n\sigma(n)} \neq 0$, the arcs (indeterminates) $a_{i\sigma(i)}$, $i = 1, \cdots, n$, constitute all elementary circuits in a circuit-cover of G. Conversely, for any circuit-cover E of G, the product of all arcs (indeterminates) that are on the circuits in E is equal to a term in det $\hat{M}(G)$ up to the sign. Precisely, we can write

$$\det \hat{M}(G) = \sum_{E \in \mathcal{E}(G)} t_E$$

where for any circuit cover $E \in \mathcal{E}(G)$, t_E is the monomial $\prod_{z \in E} t_z$ where for any elementary circuit $z = a_1 \cdots a_\ell$ ($a_i \in A$) in G, t_z is the monomial defined by $t_z = (-1)^{\ell+1} a_1 \cdots a_\ell$. Therefore, we shall prove the following by induction on the number n_G of the points of G.

(4) $p_G = \sum_{E \in \mathcal{E}(G)} t_E$ is an irreducible polynomial if G is a graph satisfying the conditions (i),(ii),and (iii).

If $n_G = 1$, then clearly (4) holds. Let $n \geq 2$ and assume that (4) holds when $n_G \leq n-1$. Let $G = \langle P, A, \zeta \rangle$ be a graph with $n_G = n$ satisfying (i), (ii), and (iii). Since $n_G > 1$ and G is strongly connected, there exists an elementary circuit $z = a_1 \cdots a_\ell$ of length $\ell > 1$ in G where $a_i \in A$ $(i = 1, \cdots, \ell)$. Let $u_1, \cdots, u_\ell$ be the points on the circuit z. Since G satisfies the conditions (iii) and (i), if we delete all the arcs $a_1, \cdots, a_\ell$ from G, we obtain pairwise disconnected ℓ strongly connected subgraphs G_i, $i = 1, \cdots, \ell$, of G such that u_i is a point in G_i and each point of G is a point of exactly one of G_i's. Clearly each G_i is a graph satisfying (i), (ii), and (iii). For $i = 1, \cdots, \ell$, let H_i be the subgraph of G_i obtained by deleting the point u_i from G_i. Then we can write

$$p_G = (-1)^{\ell+1} a_1 \cdots a_\ell \; p_{H_1} \cdots p_{H_\ell} + p_{G_1} \cdots p_{G_\ell}$$

where if G_i is a graph consisting of the point u_i and a loop from u_i to itself, then we set $p_{H_i} = 1$. Clearly p_G is a polynomial of degree 1 with respect to a_1. By the induction hypothesis, p_{G_i} is an irreducible polynomial for $i = 1, \cdots, \ell$. For $i = 1, \cdots, \ell$, any indeterminate in p_{H_i} appears only in p_{G_i}. Hence any factor of the $a_2 \cdots a_\ell p_{H_1} p_{H_2} \cdots p_{H_\ell}$ can not divide $p_{G_1} \cdots p_{G_\ell}$. Thus we conclude that p_G is an irreducible polynomial.

Thus we have proved the if part of the theorem. (It is easy to see that the only if part of the theorem holds.) □

Lemma 5.† Let f and g be two polynomials in $\mathbb{Z}[a_1, \cdots, a_m, X]$. If f is irreducible in $\mathbb{Z}[a_1, \cdots, a_m, X]$ and if for any m positive integers $p_1, \cdots,$

† See Acknowlegement.

p_m, there exists a real number r such that

$$f(p_1, \cdots, p_m, r) = g(p_1, \cdots, p_m, r) = 0,$$

then f divides g in $Z[a_1, \cdots, a_m, X]$.

Proof. Assume that f is irreducible in $Z[a_1, \cdots, a_m, X]$ and assume that for any m positive integers $p_1, \cdots, p_m$, there exists a real number r such that $f(p_1, \cdots, p_m, r) = g(p_1, \cdots, p_m, r) = 0$. Suppose that g is not divided by f in $Z[a_1, \cdots, a_m, X]$. Let K be the quotient field of $Z[a_1, \cdots, a_m]$. We can consider $f(X) = f(a_1, \cdots, a_m, X)$ and $g(X) = g(a_1, \cdots, a_m, X)$ as polynomials in $K[X]$. Since $f(X)$ is irreducible and $g(X)$ is not divided by $f(X)$ in $Z[a_1, \cdots, a_m][X]$, $f(X)$ is irreducible and $g(X)$ is not divided by $f(X)$ in $K[X]$. (Cf. van der Waerden [9], §23.) Therefore, there exist $s(X)$ and $t(X)$ in $K[X]$ such that

$$f(X)s(X) + g(X)t(X) = 1 \qquad (3.1).$$

Since the coefficients of $s(X)$ and $t(X)$ are rational functions in indeterminates $a_1, \cdots, a_m$ over Z, there exists a nonzero polynomial u in $Z[a_1, \cdots, a_m]$ such that both us and ut are in $Z[a_1, \cdots, a_m, X]$. Let $\hat{s} = us$ and $\hat{t} = ut$. Then by equation (3.1) we have

$$f(a_1, \cdots, a_m, X)\hat{s}(a_1, \cdots, a_m, X) + g(a_1, \cdots, a_m, X)\hat{t}(a_1, \cdots, a_m, X) = u(a_1, \cdots, a_m). \qquad (3.2)$$

Since u is a nonzero polynomial, there exist m positive integers $p_1, \cdots, p_m$ such that $u(p_1, \cdots, p_m) \neq 0$. (Cf. van der Waerden [9], §21.) By hypothesis, there exists a real number r such that $f(p_1, \cdots, p_m, r) = g(p_1, \cdots, p_m, r) = 0$. Substituting $(p_1, \cdots, p_m, r)$ for $(a_1, \cdots, a_m, X)$ in the polynomials in equation (3.2), we are lead to a contradiction. Thus the lemma is proved. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Assume that h^* is uniformly finite-to-one and onto. By Lemma 1, we have $r(G_1) = r(G_2)$.

It follows from Lemma 3 that by adding new loops $a_1', \cdots, a_p'$ to G_1 and new loops $b_1', \cdots, b_q'$ to G_2 if necessary, we can obtain two strongly connected graphs G_1' and G_2' and a homomorphism h' of G_1' into G_2' satisfying the following conditions, from G_1, G_2, and h.

(1) For each point u of G_2', there exists at least one loop from u to u.

(2) $h'(a) = h(a)$ for all $a \in A$ and $h'(\{a_1', \cdots, a_p'\}) = \{b_1', \cdots, b_q'\}$.

(3) $(h')^*$ is uniformly finite-to-one and onto.

Let $\hat{f}_{G_1'}(a_1, \cdots, a_\ell, a_1', \cdots, a_p', X)$ and $\hat{f}_{G_2'}(b_1, \cdots, b_m, b_1', \cdots, b_q', X)$ be the polynomials which are equal to the characteristic polynomials of $\hat{M}(G_1')$ and $\hat{M}(G_2')$, respectively. Let g' be the polynomial in $Z[b_1, \cdots, b_m, b_1', \cdots, b_q']$ defined by

$$\begin{aligned} & g'(b_1, \cdots, b_m, b_1', \cdots, b_q', X) \\ = \ & \hat{f}_{G_1'}(h'(a_1), \cdots, h'(a_\ell), h'(a_1'), \cdots, h'(a_p'), X). \end{aligned}$$

Then, since G_2' is a strongly connected graph and condition (1) is satisfied, it follows from Lemma 4 that $\hat{f}_{G_2'}$ is irreducible in $Z[b_1, \cdots, b_m, b_1', \cdots, b_q', X]$. Since $(h')^*$ is uniformly finite-to-one and onto, it follows from Lemma 2 that for any $m + q$ positive integers $p_1, \cdots, p_m, p_1', \cdots, p_q'$, there exists a real number r such that

$$\begin{aligned} & g'(p_1, \cdots, p_m, p_1', \cdots, p_q', r) \\ = \ & \hat{f}_{G_2'}(p_1, \cdots, p_m, p_1', \cdots, p_q', r) = 0. \end{aligned}$$

Hence it follows from Lemma 5 that $\hat{f}_{G_2'}$ divides g' in $Z[b_1, \cdots, b_m, b_1', \cdots, X]$. Obviously,

$$\hat{f}_{G_2}(b_1, \cdots, b_m, X) = \hat{f}_{G_2'}(b_1, \cdots, b_m, 0, \cdots, 0, X)$$

and by condition (2),

$$g(b_1, \cdots, b_m, X) = g'(b_1, \cdots, b_m, 0, \cdots, 0, X).$$

Thus we conclude that $\hat{f}_{G_2}$ divides g in $Z[b_1, \cdots, b_m, X]$. □

Example 1. Let $G = \langle P, A, \zeta \rangle$ be a graph. For any non-negative integer p, we define a graph $L^{(p)}(G)$ as follows. $L^{(0)}(G) = G$. For $p > 1$, $L^{(p)}(G) = \langle \Pi^{(p)}(G), \Pi^{(p+1)}(G), \zeta^{(p)} \rangle$ where $\zeta^{(p)}(a_1 \cdots a_{p+1}) = (a_1 \cdots a_p, a_2 \cdots a_{p+1})$ for $a_1 \cdots a_{p+1} \in \Pi^{(p+1)}(G)$ with $a_i \in A$ $(i = 1, \cdots, p+1)$. (Recall that $\Pi^{(p)}(G) = A^p \cap \Pi(G)$ for $p \geq 1$.) We call $L^{(p)}(G)$ <u>the path graph of length p of G.</u> Especially, $L^{(1)}(G)$ is called <u>the line graph of G</u> and is denoted by L(G). (This is the same as the line digraph of G in Hemminger and Beinke [4] and the adjoint of G in Berge [1].) Clearly, if G is strongly connected, then $L^{(p)}(G)$ is strongly connected for all $p \geq 0$. Let p be any non-negative integer. We define mappings $h : \Pi^{(p+1)}(G) \to A$ and $\phi : \Pi^{(p)} \to P$ as follows. For any $a_1 \cdots a_{p+1} \in \Pi^{(p+1)}(G)$ with $a_i \in A$ $(i = 1, \cdots, p+1)$, $h(a_1 \cdots a_{p+1}) = a_{p+1}$ and for any $x \in \Pi^{(p)}(G)$, $\phi(x)$ is the terminal endpoint of x. Then clearly $\mathsf{h} = (h, \phi)$ is a homomorphism of $L^{(p)}(G)$ into G and h^* is uniformly finite-to-one. Clearly if G is strongly connected, then h^* is uniformly finite-to-one and onto. Let g be the polynomial obtained from $\hat{f}_{L^{(p)}(G)}$ by substituting h(y) for y for all indeterminates $y \in \Pi^{(p+1)}(G)$ in $\hat{f}_{L^{(p)}(G)}$. Then, whether G is strongly connected or not, we can show that

$$g = X^m \hat{f}_G , \qquad (3.3)$$

and hence we have

$$f_{L^{(p)}(G)}(X) = X^m f_G(X) ,$$

where $m = |\Pi^{(p)}(G)| - |P|$ and we assume that $\hat{f}_\phi = f_\phi = 1$ for the graph ϕ with no point. Note that m may be negative. Particularly we have

$$r(L^{(p)}(G)) = r(G).$$

Proof of Equation (3.3). It suffices to show the result for $p = 1$ because $L^{(p)}(G)$ is isomorphic to $L^p(G)$ for any $p \geq 0$† and hence the result for general p (≥ 0) is straightforwardly proved by induction. Thus we assume that $p = 1$.

Assume that G is strongly connected. Let $P = \{u_1, \cdots, u_n\}$ and let $A = \{a_1, \cdots, a_\ell\}$. Then $h : \Pi^{(2)}(G) \to A$ is a homomorphism of the strongly connected graph $L(G)$ into the strongly connected graph G. Since h^* is uniformly finite-to-one and onto, it follows from Theorem 1 that there exists $\alpha \in Z[a_1, \cdots, a_\ell, X]$ such that

$$g(a_1, \cdots, a_\ell, X) = \alpha(a_1, \cdots, a_\ell, X)\hat{f}_G(a_1, \cdots, a_\ell, X). \tag{3.4}$$

Let $\tilde{M}$ be the square matrix (m_{ij}) of order ℓ in which $m_{ij} = a_j$ if the terminal endpoint of a_i is the initial endpoint of a_j and $m_{ij} = 0$ otherwise. We can consider $g(X) = g(a_1, \cdots, a_\ell, X)$ as a polynomial in $K[X]$ where K is the quotient field of $Z[a_1, \cdots, a_\ell]$. Then $g(X)$ is the characteristic polynomial of $\tilde{M}$. By the construction of $\tilde{M}$, if a_i and a_j have the same terminal endpoint, then the i th and j th rows of $\tilde{M}$ are the same. Therefore, for each point u_k of G, if the number of arcs going to u_k is d_k, then d_k rows of $\tilde{M}$ are the same. Let $m = |A| - |P|$. Then it is easily shown that there exist m linearly independent row vectors V's such that $V\tilde{M} = \mathbf{0}$ where $\mathbf{0}$ is the zero vector. Thus 0 is a characteristic value of $\tilde{M}$ with at least m linearly independent characteristic vectors corresponding to it. Hence 0 is a root of $g(X)$ of multiplicity at least m. Thus $g(X)$ is divided by X^m.

We assume without loss of generality that (*) for each point u of G

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† We define $L^p(G)$ by $L^0(G) = G$ and $L^p(G) = L(L^{p-1}(G))$ $(p > 0)$.

there exists at least one loop going from u to itself. (For assume that a_1 is a loop. Let G' be the graph obtained from G by deleting the loop a_1. Let g' be the polynomial in $\mathbb{Z}[a_2, \cdots, a_\ell, X]$ defined for G' in the same way as g for G. Then it is easily checked that $g(0, a_2, \cdots, a_\ell, X) = Xg'(a_2, \cdots, a_\ell, X)$ and $\hat{f}_G(0, a_2, \cdots, a_\ell, X) = \hat{f}_{G'}(a_2, \cdots, a_\ell, X)$. Hence it follows that if (3.3) holds in case G satisfies (*), then it also holds in case G does not satisfy (*).) Thus since G is strongly connected, it follows from Lemma 4 that $\hat{f}_G$ is irreducible in $\mathbb{Z}[a_1, \cdots, a_\ell, X]$. Therefore, since $g(a_1, \cdots, a_\ell, X)$ is divided by X^m, so is $\alpha(a_1, \cdots, a_\ell, X)$ in (3.4). Since deg $\alpha = m$, we conclude that (3.3) holds.

For the general case where G is not necessarily strongly connected, the result is straightforwardly proved by induction on the number of the maximal strongly connected subgraphs† of G. Therefore, the remainder of the proof is omitted. □

It was mentioned in Hemminger and Beinke [4], p.298 that A.J. Hoffman had asked whether one can determine $f_{L(G)}(X)$ in terms of $f_G(X)$. The question has been solved by a result in Example 1.

3. The extension of a homomorphism between two strongly connected graphs G_1 and G_2 with $r(G_1) = r(G_2)$.

In this section, we prove that if G_1 and G_2 are strongly connected graphs such that their adjacency matrices have the same maximal characteristic value, then for any homomorphism h of G_1 into G_2, h^* is onto if and only if h^* is

† Here, "strongly connected graph" is used in the usual sence, that is, it includes " strongly connected graph with one point and no arc".

uniformly finite-to-one, and hence the surjectivity of h^* is equivalent to the nonexistence of two distinct paths which are indistinguishable by h, in G_1.

Let $G = \langle P, A, \zeta \rangle$ be a strongly connected graph. Let p be a non-negative integer. We define a mapping $\theta_{G,p} : \Pi(L^{(p)}(G)) \to \overset{\infty}{\underset{i=p}{\dot{\cup}}} \Pi^{(i)}(G)$ as follows. (Cf. Example 1.) If z is a path of length 0 in $L^{(p)}(G)$, then $z \in \Pi^{(p)}(G)$. For this case, we define

$$\theta_{G,p}(z) = z.$$

If z is a path of length $\ell(\geq 1)$ in $L^{(p)}(G)$, then z is of the form

$$z = (a_1 \cdots a_{p+1})(a_2 \cdots a_{p+1}) \cdots (a_\ell \cdots a_{\ell+p})$$

with $a_1, \cdots, a_{\ell+p} \in A$ such that $a_1 \cdots a_{\ell+p} \in \Pi^{(\ell+p)}(G)$. For this case, we define

$$\theta_{G,p}(z) = a_1 \cdots a_{\ell+p}.$$

Clearly, $\theta_{G,p}$ is one-to-one and onto.

Theorem 2. Let $G_1 = \langle P, A, \zeta_1 \rangle$ and $G_2 = \langle Q, B, \zeta_2 \rangle$ be two strongly connected graphs with $r(G_1) = r(G_2)$. Then for any homomorphism $h : A \to B$ of G_1 into G_2, h^* is uniformly finite-to-one if and only if h^* is onto.

Proof. Let $h : A \to B$ be a homomorphism of G_1 into G_2. Let p be any non-negative integer. We define a mapping $h^{(p)} : \Pi^{(p+1)}(G_1) \to \Pi^{(p+1)}(G_2)$ by

$$h^{(p)}(x) = h^*(x) \qquad (x \in \Pi^{(p+1)}(G_1)).$$

Then $h^{(p)}$ is a homomorphism of the strongly connected graph $L^{(p)}(G_1)$ into the strongly connected graph $L^{(p)}(G_2)$. Moreover, we have, for each $z \in \Pi(L^{(p)}(G_1))$,

$$h^*(\theta_{G_1,p}(z)) = \theta_{G_2,p}((h^{(p)})^*(z)).$$

Therefore, since $\theta_{G_1,p}$ and $\theta_{G_2,p}$ are one-to-one and onto, h^* is uniformly finite-to-one [onto] if and only if $(h^{(p)})^*$ is uniformly finite-to-one [onto].

Assume that h^* is uniformly finite-to-one but not onto. Then there exists $y \in \Pi(G_2)$ with $lg(y) \geq 1$ such that $(h^*)^{-1}(y) = \phi$. Let $p = lg(y) - 1$. We consider the homomorphism $h^{(p)}$ of $L^{(p)}(G_1)$ into $L^{(p)}(G_2)$. Since h^* is uniformly finite-to-one, $(h^{(p)})^*$ is uniformly finite-to-one. Moreover, y is an arc of $L^{(p)}(G_2)$ such that $(h^{(p)})^{-1}(y) = \phi$. Let $H = h^{(p)}(L^{(p)}(G_1))$ (cf. Section 1 for this notation.). Then H is a strongly connected subgraph of $L^{(p)}(G_2)$. We define a mapping $(h^{(p)})' : \Pi^{(p+1)}(G_1) \to h^{(p)}(\Pi^{(p+1)}(G_2))$ by

$$(h^{(p)})'(x) = h^{(p)}(x) \qquad\qquad (x \in \Pi^{(p+1)}(G_1)).$$

Then clearly $(h^{(p)})'$ is a homomorphism of $L^{(p)}(G_1)$ into H. Since $(h^{(p)})^*$ is uniformly finite-to-one, $((h^{(p)})')^*$ is uniformly finite-to-one. Thus, by Lemma 1, we have

$$r(L^{(p)}(G_1)) \leq r(H).$$

On the other hand, the following result is known (e.g., Nikaido [13]).

(i) For any two distinct non-negative square matrices M_1 and M_2 of the same order, if M_1 is irreducible and $M_1 - M_2$ is non-negative, then the maximal characteristic value of M_1 is greater than that of M_2.

Since $h^{(p)}(\Pi^{(p+1)}(G_1)) \subset \Pi^{(p+1)}(G_2) - \{y\}$, it follows from (i) that the maximal characteristic value of $M(L^{(p)}(G_2))$ is greater than that of $M(H)$. Hence we have

$$r(L^{(p)}(G_2)) > r(H).$$

By Example 1 and hypothesis

$$r(L^{(p)}(G_1)) = r(G_1) = r(G_2) = r(L^{(p)}(G_2)).$$

Therefore, we have

$$r(L^{(p)}(G_1)) > r(H),$$

which is a contradiction. Thus we have proved that if h^* is uniformly finite-to-one, then h^* is onto.

To show the converse, we assume that h^* is onto but not uniformly finite-to-one. We construct two graphs G_1' and G_2' from $G_1 = \langle P, A, \zeta_1 \rangle$ and $G_2 = \langle Q, B, \zeta_2 \rangle$, respectively, as follows. Let $\bar{A} = \{\bar{a} \mid a \in A\}$ and let $\bar{B} = \{\bar{b} \mid b \in B\}$. The graph G_1' is obtained from G_1 by adding a new arc $\bar{a}$ having the same initial and terminal endpoints as a for every arc a of G_1. That is, $G_1' = \langle P, A', \zeta_1' \rangle$ where $A' = A \cup \bar{A}$ and ζ_1' is defined by $\zeta_1'(a) = \zeta_1'(\bar{a}) = \zeta_1(a)$ $(a \in A)$. In the same way, G_2' is obtained from G_2. That is, $G_2' = \langle Q, B', \zeta_2' \rangle$ where $B' = B \cup \bar{B}$ and ζ_2' is defined by $\zeta_2'(b) = \zeta_2'(\bar{b}) = \zeta_2(b)$ $(b \in B)$. Clearly, G_1' and G_2' are strongly connected and $M(G_i') = 2M(G_i)$ for $i = 1, 2$ so that $r(G_i') = 2r(G_i)$ for $i = 1, 2$. Since, by hypothesis, $r(G_1) = r(G_2)$, G_1' and G_2' are two strongly connected graphs with $r(G_1') = r(G_2')$. Define a mapping $h' : A' \to B'$ as follows. For each $a \in A$, $h'(a) = b$ and $h'(\bar{a}) = \bar{b}$ where $b = h(a)$. Then, since h is a homomorphism of G_1 into G_2, h' is a homomorphism of G_1' into G_2'. Let $d_1 \cdots d_\ell$ be any path of length ≥ 1 in G_2' with $d_i \in B'$ $(i = 1, \cdots, \ell)$. For $i = 1, \cdots, \ell$, let b_i be the element of B such that $b_i = d_i$ or $\bar{b}_i = d_i$. Then $b_1 \cdots b_\ell$ is a path in G_2 and

$$|((h')^*)^{-1}(d_1 \cdots d_\ell)| = |(h^*)^{-1}(b_1 \cdots b_\ell)|.$$

Therefore, since h^* is onto, $(h')^*$ is onto.

Since h^* is not uniformly finite-to-one, it follows from Proposition 1 that there exist two distinct paths x_1 and x_2 in G_1 which are indistingushable by h. Let $p = \lg(x_1) - 1$. For $i = 1, 2$, we write $x_i = a_{i1} \cdots a_{i(p+1)}$ with $a_{ij} \in A$ $(j = 1, \cdots, p+1)$. For $i = 1, 2$, let $x_i' = \bar{a}_{i1} a_{i2} \cdots a_{i(p+1)}$. Then x_1' and x_2' are two distinct paths in G_1' which are indistinguishable by h'.

Put $H_1 = L^{(p)}(G_1')$, $H_2 = L^{(p)}(G_2')$, and $g = (h')^{(p)}$. Then H_1 and H_2 are strongly connected graphs and g is a homomorphism of H_1 into H_2. Moreover,

x_1' and x_2' are distinct arcs of H_1. Let $\tilde{H}_1 = \langle R, E, \zeta \rangle$ be the maximal strongly connected subgraph of H_1 having the arc x_1' but not having the arc x_2'. Now we shall prove the following.

(*) $\tilde{H}_1$ exists and for any $z \in \Pi(H_1) - \Pi(\tilde{H}_1)$, there exists $\tilde{z} \in \Pi(\tilde{H}_1)$ such that $g^*(\tilde{z}) = g^*(z)$.

Let $z \in \Pi(H_1) - \Pi(\tilde{H}_1)$. Since H_1 is strongly connected, there exists a circuit C in H_1 such that the arc x_2' is on C and z is a subpath of C. (For two paths z_1 and z_2, z_1 is a <u>subpath</u> of z_2 if there exists paths w_1 and w_2 such that $z_2 = w_1z_1w_2$.†) Let $D = \theta_{G_1',p}(C)$. Then D is a path in G_1' and x_2' appears in D at least once as a subpath of D. Hence we can write $D = w_1x_2'w_2$ with $w_1, w_2 \in \Pi(G_1')$. Let $D_1 = w_1x_1'w_2$. Then since x_1' and x_2' have the same initial endpoint and the same terminal endpoint and $(h')^*(x_1') = (h')^*(x_2')$, D_1 is a path in G_1' and $(h')^*(D_1) = (h')^*(D)$. Since $x_1' = \bar{a}_{11}a_{12}\cdots a_{1p}$ and $x_2' = \bar{a}_{21}a_{22}\cdots a_{2p}$, x_1' and x_2' do not intersect, that is, there exist no paths t_1, t_2, and s of length >0 in G_1' such that $x_1' = t_1s$ and $x_2' = st_2$ or $x_1' = st_1$ and $x_2' = t_2s$. Hence replacing any subpath x_2' in D by x_1' does not generate a new subpath x_2' in D_1. Therefore, by replacing every subpath x_2' in D by x_1', we can obtain a path $\tilde{D}$ in G_1' such that x_2' is not a subpath of $\tilde{D}$, x_1' is a subpath of $\tilde{D}$, and

$$(h')^*(\tilde{D}) = (h')^*(D).$$

$\tilde{D}$ has the form $\tilde{D} = \tilde{d}_1\cdots\tilde{d}_\ell\tilde{d}_1\cdots\tilde{d}_p$ where $\tilde{d}_i \in A'$ $(i = 1, \cdots, \ell)$, by additional replacements if necessary. (D is of the form $D = d_1\cdots d_\ell d_1\cdots d_p$ where $d_i \in A'$ $(i = 1, \cdots, \ell)$. If a part of one of the initial and terminal subpaths $d_1\cdots d_p$ of D is replaced by a subpath of x_1' in the above replacements, the corresponding part of the other subpath $d_1\cdots d_p$ of D must be replaced by the same subpath of x_1'.) Let $\tilde{C} = (\theta_{G_1',p})^{-1}(\tilde{D})$. Then $\tilde{C}$ is a circuit in H_1.

† We assume that $uy = yv = y$ for paths u and v of length 0, i.e. points u and v, and a path y going from u to v.

Moreover, $\tilde{C}$ passes through the arc x_1' but does not pass through the arc x_2'. Hence, $\tilde{H}_1$ exists and $\tilde{C}$ is a circuit in $\tilde{H}_1$. Furthermore, we have

$$g^*(\tilde{C}) = (\theta_{G_2',p})^{-1}((h')^*(\tilde{D})) = (\theta_{G_2',p})^{-1}((h')^*(D)) = g^*(C).$$

Since z is a subpath of C, there exists a subpath $\tilde{z}$ of $\tilde{C}$ such that $g^*(z) = g^*(\tilde{z})$. Of course, $\tilde{z}$ is a path in $\tilde{H}_1$. Thus we have proved (*).

Let $\tilde{g} : E \to \Pi^{(p+1)}(G_2)$ be the restriction of g. Then $\tilde{g}$ is a homomorphism of $\tilde{H}_1$ into H_2. Since $(h')^*$ is onto, $g^* = ((h')^{(p)})^*$ is onto. Therefore, it follows from (*) that $(\tilde{g})^*$ is onto. Thus it follows from Lemma 1 that

$$r(\tilde{H}_1) \geq r(H_2).$$

However, H_1 is a strongly connected graph and $\tilde{H}_1$ is a subgraph of H_1 which has not the arc x_2' of H_1. Hence it follows from (i) that the maximal characteristic value of $M(H_1)$ is greater than that of $M(\tilde{H}_1)$. Thus we have

$$r(\tilde{H}_1) < r(H_1).$$

From example 3, $r(G_i') = r(L^{(p)}(G_i')) = r(H_i)$ for i = 1, 2. Therefore since $r(G_1') = r(G_2')$, $r(H_1) = r(H_2)$. Thus we have

$$r(\tilde{H}_1) < r(H_2).$$

which is a contradiction. Thus we have proved that if h^* is onto, then h^* is uniformly finite-to-one. The proof of the theorem is completed. □

Corollary 2. Let $G_1 = \langle P, A, \zeta_1 \rangle$ and $G_2 = \langle Q, B, \zeta_2 \rangle$ be two strongly connected graphs with $r(G_1) = r(G_2)$. Let $h : A \to B$ be a homomorphism of G_1 into G_2. Then h^* is onto if and only if no two distinct paths in G_1 are indistinguishable by h.

Proof. This follows from Theorem 2 and Proposition 1. □

We remark that Theorem 2 and Corollary 2 are no longer true if either

G_1 or G_2 in them is not strongly connected. This is shown by the following examples.

Define graphs G_1, G_2 and G_3 as follows. $G_1 = \langle \{u_1, u_2\}, \{a_1, a_2, a_3\}, \zeta_1 \rangle$ where $\zeta_1(a_1) = (u_1, u_1)$, $\zeta_1(a_2) = (u_1, u_2)$, and $\zeta_1(a_3) = (u_2, u_2)$; $G_2 = \langle \{v\}, \{b\}, \zeta_2 \rangle$ where $\zeta_2(b) = (v, v)$; $G_3 = \langle \{w_1, w_2\}, \{c_1, c_2\}, \zeta_3 \rangle$ where $\zeta_3(c_1) = (w_1, w_1)$ and $\zeta_3(c_2) = (w_1, w_2)$. Then G_2 is strongly connected but G_1 and G_3 are not strongly connected. Clearly $r(G_1) = r(G_2) = r(G_3)$. Let $\mathfrak{h}_1 = (h_1, \phi_1)$ be the homomorphism of G_1 into G_2 defined by $h_1(a_1) = h_1(a_2) = h_1(a_3) = b$ and $\phi_1(u_1) = \phi_1(u_2) = v$. Let $\mathfrak{h}_2 = (h_2, \phi_2)$ be the homomorphism of G_2 into G_3 defined by $h_2(b) = c_1$ and $\phi_2(v) = w_1$. Then $\mathfrak{h}_1^*$ is onto but not uniformly finite-to-one because, for each $n \geq 1$, $h_1^*(a_1^i a_2 a_3^{n-i-1}) = b^n$ for any i with $0 \leq i \leq n-1$ so that $|(h_1^*)^{-1}(b^n)| \geq n$. Note that $a_1 a_2$ and $a_2 a_3$ are distinct and indistinguishable by $\mathfrak{h}_1$. Clearly $\mathfrak{h}_2^*$ is one-to-one but not onto.

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