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Analysis of the Command Flow Numbers I
— Boolean Lattices —

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A partially ordered set is also called a poset. In the following, let a poset be finite. A series of our papers [5-9] on the inclusion-exclusion principle converged to the one on posets. The principle is a formula with the sum ranging over all chains in a given poset. Therefore, the number of chains in a poset seems to be a criterion of computational complexity in a problem to which the principle is applicable. So, motivated by a desire to consider the complexity, we introduced a recurrence method (without the use of the matrix operations) for counting the number of chains in a poset [10]. Then we called the cardinality of a class of chains in a poset a "command flow number" on a poset, and by applying the method to typical lattices we obtained some recurrence relations on the command flow numbers and presented some conjectures with some number tables computed by computer [11]. Roughly speaking, the computational complexity of the algorithm is $O(n^2)$ for the input = the Hasse diagram or the adjacency (or incidence) of a given poset P and n = the cardinality of P. The method can be easily extended to an algorithm for counting the number of paths in an acyclic flowgraph.

In this talk, we present the asymptotic formulas of the command flow numbers on Boolean lattices. We say x in a poset P <u>covers</u> y in P, written $x\downarrow y$ if there is no z in P such that $x<z<y$. One of our algorithms is as follows.

<u>Algorithm</u> (Value Assignment Form). Let P be a poset and N be the set of natural numbers. Then we define inductively an assignment $v:P\longrightarrow N$ ($v^*:P\longrightarrow N$) by

$$v(x) = \begin{cases} 1 & \text{for } x = \text{a minimal element in P} \\ (\sum_{x\downarrow y} v(y)) + 1 & \text{otherwise.} \end{cases}$$

$$(v^*(x)) \qquad ((\sum_{x>y} v^*(y)) + 1)$$

Proposition 1. (1) $v(x)(v^*(x))$ is the number of covering chains (chains) in P having x as their greatest element and called a (total) command flow number of x, written $c_P(x)(c_P^*(x))$.

(2) $\sum_{x\in P} v(x)(\sum_{x\in P} v^*(x))$ is the number of covering chains (chains) in P and called a (total) command flow number of P, written $\tilde{c}_P(\tilde{c}_P^*)$.

The name "command flow" means a series of commands from a higher person to lower persons in a system with an order relation. Now, we give a generalized value assignment form.

The Generalized Value Assignment Form. Let P be a poset and R be the set of real numbers. Then we define inductively an assignment $v^{(a,b)}:P \longrightarrow R$ $(v^{*(a,b)}:P \longrightarrow R)$ by

$$v^{(a,b)}(x) = \begin{cases} a & \text{for } x = \text{a minimal element in P} \\ (\sum_{x\downarrow y} v^{(a,b)}(y)) + b & \text{otherwise.} \end{cases}$$
$$(v^{*(a,b)}(x)) \qquad ((\sum_{x>y} v^{*(a,b)}(y)) + b)$$

We denote $\sum_{x\in P} v^{(a,b)}(x)$ $(\sum_{x\in P} v^{*(a,b)}(x))$ by $\tilde{v}^{(a,b)}(\tilde{v}^{*(a,b)})$.

Proposition 2. Let $v^{(a,b)}$ and $v^{*(a,b)}$ be the generalized value assignments. Then we have the following equalities.

(1) $v^{(a,b)} = av^{(1,0)} + bv^{(0,1)}$

(2) $\tilde{v}^{(a,b)} = a\tilde{v}^{(1,0)} + b\tilde{v}^{(0,1)}$

(3) $v^{*(a,b)} = av^{*(1,0)} + bv^{*(0,1)}$

(4) $\tilde{v}^{*(a,b)} = a\tilde{v}^{*(1,0)} + b\tilde{v}^{*(0,1)}$

Proposition 3. Let P be a poset with a unique minimal element $\mathbf{0}$ and $v^{*(a,b)}$ the value assignment. Then the following equalities hold for $x \neq \mathbf{0}$.

(1) $v^{*(1,0)}(x) = v^{*(0,1)}(x)$

(2) $v^{*(a,b)}(x) = (a + b)v^{*(1,0)}(x)$

(3) $\tilde{v}^{*(a,b)}_{[0,x]} = 2v^{*(a,b)}(x) - b$

Proposition 2 shows that $v^{(1,0)}$ and $v^{(0,1)}$ are the bases of the value assignment $v^{(a,b)}$, and $v^{*(1,0)}$ and $v^{*(0,1)}$ also are same. Proposition 3 shows that the assignment $v^{*(a,b)}$ has only one base whenever P has a unique minimal element.

Remark 1. $v^{(1,1)}$ is the assignment v for counting the command flow number and $v^{*(1,1)}$ is the assignment v^* for counting the total command flow number.

The Command Flow Number on Boolean Lattices

Let B_n be a Boolean lattice with n atoms and $\mathbf{1}_{B_n}$ be a unique maximal element of B_n. Then we denote $v^{(a,b)}(\mathbf{1}_{B_n})$ and $\tilde{v}^{(a,b)}$ by $c_B^{(a,b)}(n)$ and $\tilde{c}_B^{(a,b)}(n)$ respectively, and $v^{*(a,b)}(\mathbf{1}_{B_n})$ and $\tilde{v}^{*(a,b)}$ by $c_B^{*(a,b)}(n)$ and $\tilde{c}_B^{*(a,b)}(n)$ respectively. The command flow numbers $c_B^{(1,1)}(n)$, $\tilde{c}_B^{(1,1)}(n)$, $c_B^{*(1,1)}(n)$ and $\tilde{c}_B^{*(1,1)}(n)$ on Boolean lattices are abbreviated to $c_B(n)$, $\tilde{c}_B(n)$, $c_B^*(n)$ and $\tilde{c}_B^*(n)$, respectively.

The Recurrence Relations. We have the following recurrence relations.

$$(1)\quad c^{(a,b)}(n) = \begin{cases} a & (n = 0) \\ nc^{(a,b)}(n-1) + b & (n \geq 1) \end{cases}$$

$$(2)\quad \tilde{c}^{(a,b)}(n) = \sum_{k=0}^{n} \binom{n}{k} c^{(a,b)}(k) \qquad (n \geq 0)$$

$$(3)\quad c^{*(a,b)}(n) = \begin{cases} a & (n = 0) \\ \sum_{k=0}^{n-1} \binom{n}{k} c^{*(a,b)}(k) + b & (n \geq 1) \end{cases}$$

$$(4)\quad \tilde{c}^{*(a,b)}(n) = \begin{cases} a & (n = 0) \\ 2c^{*(a,b)}(n) - b & (n \geq 1) \end{cases}$$

$$(5)\quad c^{*(a,b)}(n) = \begin{cases} a & (n = 0) \\ (a+b)c^{*(1,0)}(n) & (n \geq 1) \end{cases}$$

For a = b = 1, we have the following recurrence relations for the command flow numbers on Boolean lattices.

$$(6)\quad c_B(n) = \begin{cases} 1 & (n = 0) \\ nc_B(n-1) + 1 & (n \geq 1) \end{cases}$$

$$(7)\quad \tilde{c}_B(n) = \sum_{k=0}^{n} \binom{n}{k} c_B(k) \qquad (n \geq 0)$$

$$(8)\quad c_B^*(n) = \begin{cases} 1 & (n = 0) \\ \sum_{k=0}^{n-1} \binom{n}{k} c_B^*(k) + 1 & (n \geq 1) \end{cases}$$

(9) $\tilde{c}^*_B(n) = 2c^*_B(n) - 1 \qquad (n \geqq 0)$

$$(10)\quad c^*_B(n) = \begin{cases} 1 & (n = 0) \\ 2c_B^{*(1,0)}(n) & (n \geqq 1) \end{cases}$$ (by T. Ohya)

Remark 2. In the p.15 of Lovász [4], the number S_n is defined as follows, "S_n is the number of mappings $f: N_n \longrightarrow N_n$ such that if f takes a value i then it also takes each value j, $1 \leqq j \leqq i$, where $S_0 = 1$ and $N_n = \{1, \ldots, n\}$". The recurrence relation of S_n is just one of $c^{*(1,0)}(n)$.

The Exponential Generating Functions. We denote by $G(a_n; x)$ the exponential generating function of a sequence $\{a_n\}$ of numbers. Then the following equalities hold.

$$(1)\quad G(c^{(a,b)}(n);\ x) = \frac{be^x + (a - b)}{1 - x}$$

$$(2)\quad G(\tilde{c}^{(a,b)}(n);\ x) = \frac{be^{2x} + (a - b)e^x}{1 - x}$$

$$(3)\quad G(c^{*(a,b)}(n);\ x) = \frac{be^x + (a - b)}{2 - e^x}$$

$$(4)\quad G(\tilde{c}^{*(a,b)}(n);\ x) = \frac{be^{2x} + (a - b)e^x}{2 - e^x}$$

For a = b = 1, we have the following generating functions for the command flow numbers on Boolean lattices.

$$(5)\quad G(c_B(n);\ x) = \frac{e^x}{1 - x}$$

$$(6)\quad G(\tilde{c}_B(n);\ x) = \frac{e^{2x}}{1 - x}$$

$$(7)\quad G(c^*_B(n);\ x) = \frac{e^x}{2 - e^x}$$ (by H. Enomoto)

$$(8)\quad G(\tilde{c}^*_B(n);\ x) = \frac{e^{2x}}{2 - e^x}$$

Remark 3. For a = 1 and b = 0 in (3), remenbering that $c^{*(1,0)}(n) = S_n$, we have the following generating function.

$$G(S_n;\ x) = \frac{1}{2 - e^x}$$ (p.149 of Lovász [4])

The Formulas. We have the following formulas.

(1) $c^{(a,b)}(n) = (b\sum_{k=0}^{n}\frac{1}{k!} + (a-b))n!$

(2) $\tilde{c}^{(a,b)}(n) = (b\sum_{k=0}^{n}\frac{2^k}{k!} + (a-b)\sum_{k=0}^{n}\frac{1}{k!})n!$

(3) $c^{*(a,b)}(n) = \begin{cases} a & (n=0) \\ (a+b)\sum_{k=1}^{\infty}\frac{k^n}{2^{k+1}} & (n \geqq 1) \end{cases}$

(4) $\tilde{c}^{*(a,b)}(n) = \begin{cases} a & (n=0) \\ 2(a+b)\sum_{k=1}^{\infty}\frac{k^n}{2^{k+1}} - b & (n \geqq 1) \end{cases}$

For $a = b = 1$, the formulas on the command flow numbers are obtained.

(5) $c_B(n) = (\sum_{k=0}^{n}\frac{1}{k!})n!$ (7) $c_B^*(n) = \sum_{k=1}^{\infty}\frac{k^n}{2^k}$

(6) $\tilde{c}_B(n) = (\sum_{k=0}^{n}\frac{2^k}{k!})n!$ (8) $\tilde{c}_B^*(n) = \sum_{k=1}^{\infty}\frac{(k+1)^n}{2^k}$

Remark 4. By setting $a = 1$ and $b = 0$ in (3), we have the formula.

$$S_n = \sum_{k=0}^{\infty}\frac{k^n}{2k+1} \qquad \text{(p.15 of Lovász [4])}$$

The Asymptotic Formulas. We have the following asymptotic formulas for the command flow number on Boolean lattices.

(1) $e\cdot n! - c_B(n) = O\left(\frac{1}{n}\right)$ (2) $e^2\cdot n! - \tilde{c}_B(n) = O\left(\frac{2^n}{n}\right)$

(3) $\left|c_B^*(n) - (\frac{1}{\log 2})^{n+1}\cdot n!\right| < (\frac{1}{\log 2}\cdot\frac{n}{e})^n$

(by H.Narushima & T. Ohya)

(4) $\lim_{n\to\infty}(c_B^*(n)/((\frac{1}{\log 2})^{n+1}\cdot n!)) = 1$

(5) $\lim_{n\to\infty}(\tilde{c}_B^*(n)/(2(\frac{1}{\log 2})^{n+1}\cdot n!)) = 1$

Let $f(x) = (\log 2)^{n+1}(x^n/2^n)$. Then (3) is obtained by comparing $\int_0^{\infty}f(x)dx = \int_0^{\infty}t^n\cdot e^{-t}dt = \Gamma(n+1) = n!$

with $\sum_{k=1}^{\infty}f(k) = (\log)^{n+1}\cdot c_B^*(n)$. We now have a more detailed result than (3). The result is as follows.

(6) For $\log 2 < a$, $0 < c$ and $2\pi m < b < 2\pi(m+1)$ $(m \geqq 0)$,

$$\left| c_B^*(n) - n!\left(\left(\frac{1}{\log 2}\right)^{n+1} + \sum_{k=1}^{m}(z_k^{n+1} + \bar{z}_k^{n+1})\right)\right| \leqq n!\alpha(a,b,c,n)$$

where $z_k = \dfrac{1}{\log 2 + 2\pi k i}$

$$\alpha(a,b,c,n) = \frac{1}{\pi}\left(\frac{2(a+c)}{\sqrt{h_0}}\cdot\frac{1}{b^{n+1}} + \frac{2b}{|e^{-c}-2|}\cdot\frac{1}{c^{n+1}} + \frac{2b}{|e^{a}-2|}\cdot\frac{1}{a^{n+1}}\right)$$

$$h_0 = \begin{cases} e^{-2c} - 4e^{-c}\cos b + 4 & \left(-1 \leqq \cos b \leqq \dfrac{1}{2e^{c}}\right) \\ 4\sin^2 b & \left(\dfrac{1}{2e^{c}} \leqq \cos b \leqq 1,\ \sin b \neq 0\right). \end{cases}$$

(7) $$c_B^*(n) = n!\left(\left(\frac{1}{\log 2}\right)^{n+1} + 2\sum_{k=1}^{\infty}\mathrm{Re}(z_k^{n+1})\right)$$

where

$$\mathrm{Re}(z_k^{n+1}) = \frac{1}{\left(\sqrt{(\log 2)^2 + (2\pi k)^2}\right)^{n+1}}\cdot\cos(n+1)\theta_k$$

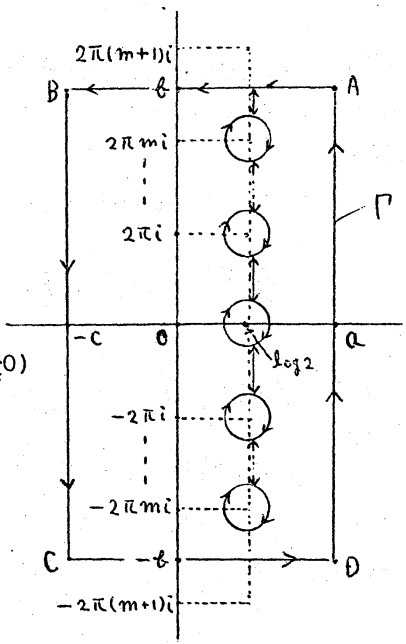
$$\tan\theta_k = (2\pi k)/\log 2.$$

(8) $$\limsup_{n\to\infty}\left(c_B^*(n) - \left(\frac{1}{\log 2}\right)^{n+1}\cdot n!\right) = \infty$$

$$\liminf_{n\to\infty}\left(c_B^*(n) - \left(\frac{1}{\log 2}\right)^{n+1}\cdot n!\right) = -\infty$$

(by H. Narushima & T. Hilano: see [12] for the more informations).

Since $c_B^*(n) = 2c^{*(1,0)}(n) = 2S_n$ and $S_n = G^{(n)}(S_n;\ 0)$, by using Cauchy's integral formula with a path Γ of integration in the right side Fig, we obtain the ineqaulity (6).



<u>Remark</u> 5. If we use a circle with radius r such that $\sqrt{(\log 2)^2 + (2\pi m)^2} < r < \sqrt{(\log 2)^2 + (2\pi(m+1))^2}$ $(m \geqq 0)$ instead of the outside rectangle path ABCD in the right side Fig, then the right side of (6) results in $\frac{c}{r^n}\cdot n!$ for some constant c. But we can not decide the value of c explicitly. For m = 0, the

following formulas are derived.

(1) $\left|c_B^*(n) - (\frac{1}{\log 2})^{n+1}\cdot n!\right| < \frac{c}{r^n}\cdot n!$

(2) $\left|S_n - \frac{1}{2}(\frac{1}{\log 2})^{n+1}\cdot n!\right| < \frac{c}{2\cdot r^n}\cdot n!$

where $\log 2 < r < \sqrt{(\log 2)^2 + (2\pi)^2}$

(3) $\lim_{n\to\infty}(S_n/(\frac{1}{2}(\frac{1}{\log 2})^{n+1}\cdot n!)) = 1$

((2)&(3):p.150 of Lovász [4])

Note that $\lim_{n\to\infty}(\frac{c}{r^n}\cdot n!)/(\frac{1}{\log 2}\cdot\frac{n}{e})^n = 0$.

Problem. Decide whether or not there exists $n \geq 17$ such that $|c_B^*(n) - (1/\log 2)^{n+1}\cdot n!| < 1$.

The Asymptotic Relationships. We have the asymptotic relationships between B_n, the command flow numbers on B_n and $|2^{B_n}|$.

$$|B_n| \xrightarrow[(1)]{\infty} c_B(n) \xrightarrow{e} \tilde{c}_B(n) \xrightarrow[(2)]{\infty} c_B^*(n) \xrightarrow{2} \tilde{c}_B^*(n) \xrightarrow[(3)]{\infty} |2^{B_n}|$$

where "$f(n) \xrightarrow{a} g(n)$" denotes "$\lim_{n\to\infty}\frac{g(n)}{f(n)} = a$".

(1) $\frac{c_B(n)}{|B_n|} \sim e\cdot(\frac{n}{2e})^n\cdot\sqrt{2\pi n}$

(2) $\frac{c_B^*(n)}{\tilde{c}_B(n)} \sim \frac{1}{e}(\frac{1}{\log 2})^{n+1}$

(3) $\frac{|2^{B_n}|}{\tilde{c}_B^*(n)} \sim (\frac{\log 2}{2})\cdot\frac{2^{(2^n)}}{(\frac{n}{e\log 2})^n\sqrt{2\pi n}}$

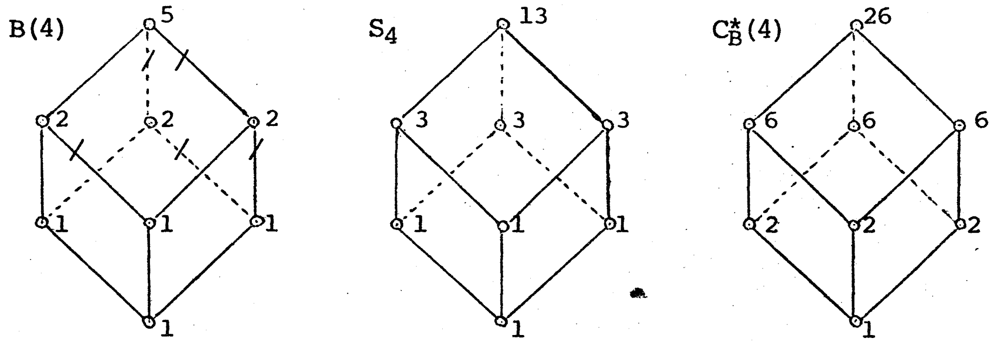
Remark 6. Let $S(n,k)$ be Stirling's number of second kind and $B(n)$ be Bell's number. Then we have the following relationships.

(1) $B(n) = \sum_{k=1}^{n} S(n,k) = \sum_{k=0}^{n-1}\binom{n-1}{k}B(k)$, $B(0) = 1$

(2) $S_n = \sum_{k=1}^{n} k!S(n,k) = \sum_{k=0}^{n-1}\binom{n}{k}S_k$, $S_0 = 1$

(3) $c_B^*(n) = 2S_n = \sum_{k=0}^{n-1}\binom{n}{k}c_B^*(k) + 1$, $c_B^*(0) = 1$

Each assignment for B_4 is demonstrated in the following.



("/" denote "cut")

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