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On the Decomposition of a Directed Graph with respect to Arborescences and Related Problems

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As is well known, an arborescence (or, more precisely, a spanning arborescence) on a diredted graph is considered as a maximum common independent set of the circuit matroid of the graph and a partition matroid. In this paper we apply the theory and the techniques which have been established for matroid and polymatroid intersections in [1], [3], [4] to this case. The main results are as follows: (i) We introduce the concept of 'degree of non-existence' of an arborescence which shows why and to what extent it is impossible to find an arborescence. (ii) In the case where there is an arborescence, we define a decomposition of the edge set into a partially ordered set so as to clarify the contribution of each edge to the reachability from the 'root' of the arborescences to the other vertices.

The relation of these results to the Hamiltonian-path problem, which is actually a three-matroid intersection problem, is also investigated to get a number of necessary conditions for the existence of a Hamiltonian path as well as a procedure of reducing the original Hamiltonian-path problem to smaller ones.

1. Preliminaries

In this section we shall outline the theory of principal partition [1], [3], [4]. Let P_1 and P_2 denote two polymatroids on a finite set E with rank functions μ_1 and μ_2, respectively.

The following equality is well-known: for each $\lambda \in [-1, 1]$,

$$\min \{ (1 - \lambda)\mu_1(A) + (1 + \lambda)\mu_2(E - A) \mid A \subset E \}$$
$$= \max \{ |u| \mid u \in (1 - \lambda)P_1 \text{ and } u \in (1 + \lambda)P_2 \}. \tag{1.1}$$

The collection of subsets which attain the minimum in the left-hand side of (1.1), to be denoted by $C(\lambda)$, constitutes a distributive lattice (i.e., it is closed under union and intersection), hence it uniquely defines a partition of the underlying set:

$$E = E_\lambda^+ \cup [\bigcup_{F \in F(\lambda)} F] \cup E_\lambda^- \tag{1.2}$$

with a partial order on $F(\lambda)$, where $F(\lambda)$ is the collection of difference sets of a maximal chain in $C(\lambda)$, and E_λ^+ and E_λ^- are the minimum and the complement in E of the maximum of $C(\lambda)$, respectively. The collection of maximum common independent vectors of the pair $((1-\lambda)P_1, (1+\lambda)P_2)$ is decomposed into a direct sum corresponding to (1.2).

Let $C_{all} = \bigcup_{-1 \le \lambda \le 1} C(\lambda)$. C_{all} also turns out to be a distributive lattice, and determines a partition of the underlying set:

$$E = \bigcup_{F \in F_{all}} F \tag{1.3}$$

with a partial order on F_{all}. We can prove that the partition (1.3) is a refinement of (1.2) for each $\lambda \in [-1, 1]$, and that the partially ordered set associated with the latter partition is homomorphic to that with the former. Furthermore,

$$F_{all} = \bigcup_{-1 \le \lambda \le 1} F(\lambda). \tag{1.4}$$

A simple argument shows that there exists a finite set $\Lambda \subset [-1, 1]$ such that

$$F_{all} = \bigcup_{\lambda \in \Lambda} F(\lambda), \tag{1.5}$$

$$\text{if } \lambda, \lambda' \in \Lambda \text{ and } \lambda \ne \lambda', \text{ then } E_\lambda^+ \ne E_{\lambda'}^+, \text{ and } E_\lambda^- \ne E_{\lambda'}^-. \tag{1.6}$$

where (1.5) is a disjoint union up to the "singular" blocks which are singletons and whose principal pair of minors (see below) are of null rank.

In terms of the Λ, (1.3) can be rewritten as

$$E = \bigcup_{\lambda\in\Lambda} [\bigcup_{F\in F(\lambda)} F] \tag{1.7}$$

which we shall call the *principal partition* of E with respect to (P_1, P_2) and the elements of Λ the *critical values* of (P_1, P_2).

A "standard pair" of minors $(P_1[F], P_2[F])$ of (P_1, P_2) can be associated with each $F \in F(\lambda)$ $(\lambda \in \Lambda)$ as follows. Let $F^- = \bigcup F'$ for $F \prec F'$, $F \neq F'$ $(F' \in F_{all})$, and $F^+ = \bigcup F''$ for $F'' \prec F$, $F \neq F''$ $(F'' \in F_{all})$, where $\prec$ is the partial order on F_{all}, and then delete F^- and contract F^+ from P_1. The resulting minor is the direct sum of a polymatroid on F, denoted by $P_1[F]$, and polymatroids on the blocks incomparable with F. $P_2[F]$ is obtained from P_2 with the operation of deletion replaced by contraction and vice versa. For this pair,

$$(1 - \lambda)r(P_1[F]) = (1 + \lambda)r(P_2[F]) \tag{1.8}$$

holds ($r(P)$ being the rank of the polymatroid P), and the pair $((1 - \lambda)P_1[F], (1 + \lambda)P_2[F])$ has a common basis vector. The pair $(P_1[F], P_2[F])$ is called the *principal pair* of minors on $F \in F(\lambda)$.

The following lemma is easily deduced from the basic results of the theory of the principal partition.

Lemma 1.1. Let λ_{max} be the maximum of the critical values of (P_1, P_2), and $q \geq 0$. Then there exists a common independent vector of (P_1, qP_2) which is a basis vector of P_1 iff

$$q \geq \frac{1 + \lambda_{max}}{1 - \lambda_{max}} . \tag{1.9}$$

For details of the theory of the principal partition, we refer to [1], [2], [3], [4], and for matroid intersection problems and algorithms, to [5], [6], [7].

3. Arborescences on a directed graph

Let $G = (V, E)$ be a directed graph with a vertex set V and an edge set E. Throughout this paper, we assume G to be connected for the sake of simplicity. Let $\delta^- v$ denote the set of edges whose terminal vertex is v, and $\delta^+ v$ the set of edges with v as the initial vertex. A subset T of E is called an *spanning arborescence* or, simply, an *arborescence* if T is a tree on G (as an undirected graph) and if $|T \cap \delta^- v| \leq 1$ for every $v \in V$. An *inverse arborescence* is defined with the condition $|T \cap \delta^+ v| \leq 1$ instead of $|T \cap \delta^- v| \leq 1$. We shall slightly generalize the definitions. For a positive integer k, a *k-arborescence* is a tree T ($\subset E$) with $|T \cap \delta^- v| \leq k$ for every $v \in V$. A *k-inverse arborescence* is similarly defined. If T is an arborescence, there is a unique vertex v with $|T \cap \delta^- v| = 0$, which is called the *root* of T.

Theorem 2.1. A necessary and sufficient condition for an arborescence with its root $v \in V$ to exist in G is that all the other vertices of G are reachable from v through directed paths.

(Proof) The verification is a routine work. □

Theorem 2.1 indicates that a variant of shortest-path algorithm affords an efficient way for finding an arborescence.

Now let us define an arborescence in terms of matroids. G, as an undirected graph, determines the circuit matroid on E, which we denote by $\mathcal{G}$. For a positive integer k, the collection of subsets A of E such that $|A \cap \delta^- v| \leq k$ for every $v \in V$ satisfies the axioms of independent sets of a matroid, so that it defines a matroid which we denote by $\mathcal{P}_k^-$. We simply write $\mathcal{P}^-$ for $\mathcal{P}_1^-$. $\mathcal{P}_k^+$ and $\mathcal{P}^+$ are defined in a similar way with δ^+ instead of δ^-. Clearly, an arborescence is nothing but a common independent set of $(\mathcal{G}, \mathcal{P}^-)$ which is a base of $\mathcal{G}$, and a k-arborescence is a common independent set of (G, P_k^-) which is a base of G. Thus, we can apply the theory of the principal partition to arboresecnces. Suppose that the principal partition of E with respect to (G, P^-) is

$$E = \bigcup_{\lambda\in\Lambda^-} [\bigcup_{F\in F(\lambda)} F]. \tag{2.1}$$

Theorem 2.2. Let λ_{max} be the maximum value in Λ^-. Then a necessary and sufficient condition for a k-arborescence to exist in G is that

$$k \geq \frac{1 + \lambda_{max}}{1 - \lambda_{max}}. \tag{2.2}$$

(Proof) Lemma 1.1 implies that (G, kP^-) has a common independent vector which is a basis vector of G iff (2.2) holds. Although kP^- is not equal to P_k^-, the collection of common independent vectors of (G, kP^-) coincides with that of (G, P_k^-). □

A directed graph does not necessarily contain an arborescence. However, it is clear that a k-arborescence does exist if k is large enough. The maximum λ_{max} of the critical values of Λ^-, which determines the minimum of k for which a k-arborescence exists, can be considered as the 'degree of non-existence' of arborescences in G in the case where there is no arborescence in G.

The partition (2.1) possesses the following interesting property. As is easily seen,

$$k \geq \text{rank of } G \,/\, \text{rank of } P^- = (|V| - 1) \,/\, |\{v \in V: \delta^- v \neq \emptyset\}| \tag{2.3}$$

is a necessary condition for the existence of a k-arborescence, but not a sufficient condition in general. However, for the subgraphs (more exactly, subcontractions) corresponding to the principal pair of minors on the blocks of the partition (2.1), (2.3) is a sufficient condition as well. This is readily proved from basic properties of the principal pairs of minors.

Moreover, (2.1) is closely related to the decomposition of the graph into strongly connected components. Let $V = \bigcup_{i\in I} V_i$ be the decomposition of the vertex set V of G into strongly connected components.

<u>Theorem 2.3</u>. In the case where G has an arborescence, the partition (2.1) is a refinement of the partition $E = \bigcup_{i \in I} \delta^- V_i$ where $\delta^- V_i = \bigcup \delta^- v$ for $v \in V_i$.

(Proof) Let μ_1 and μ_2 denote the rank functions of G and P^-, respectively. From the assumption of the existence of an arborescence,

$$\min \{ \mu_1(A) + \mu_2(E - A) \mid A \subset E \} = |V| - 1.$$

Suppose $I^+ \subset I$ satisfies the condition: there is no pair (i, j), with $i \in I^+$ and $j \in I - I^+$, such that $V_j \prec V_i$ where $\prec$ is the natural partial order among the strongly connected components, i.e., $V_j \prec V_i$ implies that the vertices of V_i are reachable from those of V_j. Then, we have

$$\mu_1(\bigcup_{i \in I^+} \delta^- V_i) + \mu_2(\bigcup_{j \in I - I^+} \delta^- V_j) = |V| - 1,$$

as is readily shown from the definition of the rank functions. Hence, $\{ \bigcup_{i \in I^+} \delta^- V_i \mid I^+$ satisfies the above condition $\}$ is a subcollection of $\{ A \subset E \mid \mu_1(A) + \mu_2(E - A) = \text{minimum} \}$, and the assertion of the theorem directly follows. □

The implication of the partition (2.1) is further investigated. In the following, we assume that G contains a vertex s of null indegree, and that all the other vertices are reachable from s. (It is easy to transform the problem with a specified root into this form.) From Theorem 2.1, there is an arborescence in G, and every arborescence of G has the unique root s. Since in this case an arborescence is a common base of (G, P^-), zero is the only critical value of (G, P^-), so that (2.1) is reduced to

$$E = \bigcup_{F \in F(0)} F. \tag{2.4}$$

The elements of E can be classified into three parts $E_0(G, P^-)$, $E_1(G, P^-)$ and $E_2(G, P^-)$ as follows:

$$E_0(G, P^-) = \{ e \in E \mid \{e\} \in F(0), \text{ the rank of the principal pair of minors on } \{e\} = 0 \},$$

$$E_1(G, P^-) = \{ e \in E \mid \{e\} \in F(0), \text{ the rank of the principal pair of minors on } \{e\} = 1 \},$$

$$E_2(G, P^-) = \{ e \in E \mid e \in F \text{ for some } F \in F(0) \text{ such that } |F| \geq 2 \}.$$

Then, we have

<u>Theorem 2.4.</u>

< A > $e \in E_0(G, P^-)$ (A.1)

$\Longleftrightarrow$ No arborescence on G contains e (A.2)

$\Longleftrightarrow$ There is no elementary directed path in G starting from s and containing e. (A.3)

< B > $e \in E_1(G, P^-)$ (B.1)

$\Longleftrightarrow$ Every arborescence on G contains e (B.2)

$\Longleftrightarrow$ The deletion of e from G makes some vertices unreachable from s. (B.3)

< C > $e \in E_2(G, P^-)$ (C.1)

$\Longleftrightarrow$ Some of the arborescences on G contain e and some do not. (C.2)

(Proof) (A.1) $\Longleftrightarrow$ (A.2), (B.1) $\Longleftrightarrow$ (B.2) and (C.1) $\Longleftrightarrow$ (C.2) follow from the basic results of the theory of the principal partition. (See [4].) It is easy to show that an arbitrary elementary directed path starting from s can be augmented to an arborescence in G, so that (A.2) $\Longleftrightarrow$ (A.3). The proof of (B.2) $\Longleftrightarrow$ (B.3) is straightforward from Theorem 2.1. $\square$

In other words, the edges belonging to $E_0(G, P^-)$ are of no use at all for the reachability from s, whereas the edges belonging to $E_1(G, P^-)$ are indispensable. The edges belonging to $E_2(G, P^-)$ can be considered as the replaceable edges, and the relation of replacement is also decided by (2.4).

3. Related Problems

In this section we consider problems related to the Hamiltonian-path problem. Let G be a directed graph with a vertex s of null indegree and a vertex t of null outdegree. A *Hamiltonian path* in G is a directed path from s to t which runs through each of the vertices of G exactly once. In terms of G, P^- and P^+ (which are defined in the same way as in the previous section), a Hamiltonian path is nothing but a common base of G, P^- and P^+, so that the following conditions should be satisfied for the existence of a Hamiltonian path.

(N1) G and P^- have a common base, i.e., G has an arborescence,

(N2) G and P^+ have a common base, i.e., G has an inverse arborescence,

(N3) P^- and P^+ have a common base.

Each of these conditions can easily be checked by a matroid intersection algorithm. If one of them is violated, it is directly concluded that there is no Hamiltonian path in G. If they are all satisfied, construct the three principal partitions of E corresponding to the three pairs (G, P^-), (G, P^+) and (P^-, P^+), and determine the sets $E_i(G, P^-)$, $E_i(G, P^+)$ and $E_i(P^-, P^+)$ $(i = 0, 1, 2)$. (This can also be done by an efficient algorithm. See [4].) Put $E_0 = E_0(G, P^-) \cup E_0(G, P^+) \cup E_0(P^-, P^+)$, and $E_1 = E_1(G, P^-) \cup E_1(G, P^+) \cup E_1(P^-, P^+)$. If G has a Hamiltonian path, it must contain all the elements of E_1 and none of E_0. Hence,

(N4) $E_0 \cap E_1 = \emptyset$

should be satisfied. If (N4) does not hold, there is no Hamiltonian path. If (N4) is satisfied, the original Hamiltonian-path problem can be reduced to a smaller one as follows. Let $\bar{E}_1 = \bigcup_{e \in E_1} [\, \delta^-(\partial^- e) \cup \delta^+(\partial^+ e) \,]$, where $\partial^- e$ (resp., $\partial^+ e$) is the terminal (resp., initial) vertex of e. The edges of $\bar{E}_1 - E_1$ are contained in no Hamiltonian path (even if one exists). Consider the graph G' obtained from G by deleting the edges of $E_0 \cup (\bar{E}_1 - E_1)$ and contracting the edges of E_1. Then, it will be obvious that

<u>Theorem 3.1</u>. A subset T of the edges of G is a Hamiltonian path in G iff $T \supset E_1$ and $T - E_1$ is a Hamiltonian path in G'. □

If G' is strictly "smaller" than G, we can apply the above procedure to G'. By repeating this procedure, we can sometimes reduce the original problem to a considerable extent. However, it may also happen that G' equals to G, i.e., we have no substantial reduction. In fact, we can expect no powerful procedure which is "always" effective, since the Hamiltonian-path problem is known to be NP-complete.

4. Examples

<u>Example 1</u>. The graph G_1 shown in Fig.1 does not contain an arborescence. The partition (2.1) for G_1 is

$$E = \bigcup_{\lambda \in \{3/7, 1/7, 0, -1/3\}} \Big[\bigcup_{F \in \mathcal{F}(\lambda)} F \Big],$$

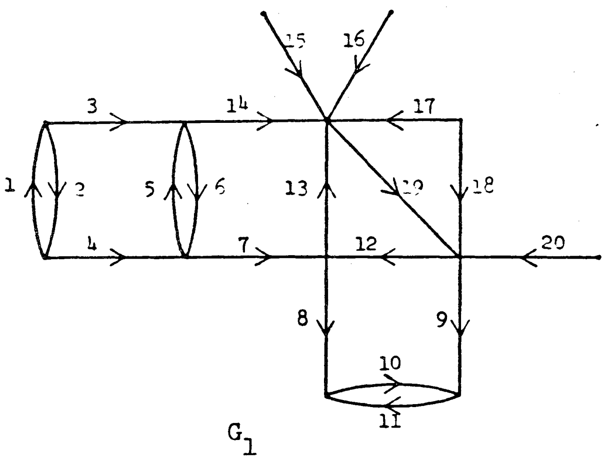


Fig. 1

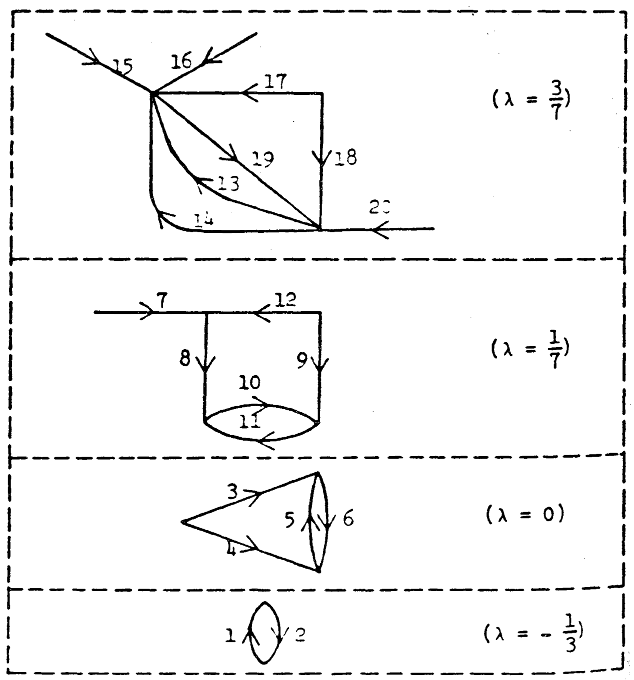


Fig. 2

where

$$F(3/7) = \{ \{13, 14, 15, 16, 17, 18, 19, 20\} \},$$

$$F(1/7) = \{ \{7, 8, 9, 10, 11, 12\} \},$$

$$F(0) = \{ \{3, 4, 5, 6\} \},$$

$$F(-1/3) = \{ \{1, 2\} \}.$$

From Theorem 2.2, a k-arborescence exists on G_1 iff $k \geq \frac{1 + 3/7}{1 - 3/7} = \frac{5}{2}$ (i.e., $k \geq 3$). And {1, 3, 6, 7, 8, 10, 12, 15, 16, 17, 19, 20} is an example of a 3-arborescence on G_1. Fig. 2 shows the subgraphs each corresponding to the principal pair of minors on a block of the partition (2.1).

Example 2. The graph G_2 of Fig. 3 has an arborescence, and the partition (2.4) for G_2 is

$$E = \bigcup_{F \in F(0)} F$$

where

$$F(0) = \{ \{1,2,3,4\}, \{5,6,7,8\}, \{9\}, \{10\}, \{11\}, \{12\}, \{13,14,15,16\} \}.$$

Fig. 4 shows the principal pairs on the blocks and the partial order among them. The classification of the edges is as follows;

$$E_0(G, P^-) = \{ 9, 11 \},$$

$$E_1(G, P^-) = \{ 10, 12 \},$$

$$E_2(G, P^-) = \{ 1,2,3,4,5,6,7,8,13,14,15,16 \}.$$

In Fig. 4, the broken lines indicate the partition of the edges corresponding to the decomposition of the graph into strongly connected components.

Example 3. Let us consider the Hamiltonian-path problem on the graph G_3 of Fig. 5. As is easily seen, G_3 satisfies (N1), (N2) and (N3). Fig. 7 shows the partitions (2.1) with respect to (G, P^-), (G, P^+) and (P^-, P^+) of G_3, and we have

$$E_0(G, P^-) = \{ 22 \},$$

$$E_0(G, P^+) = \{ 10, 14 \},$$

$$E_0(P^-, P^+) = \{ 1, 3, 4, 7, 11, 14, 15, 18, 22, 30 \},$$

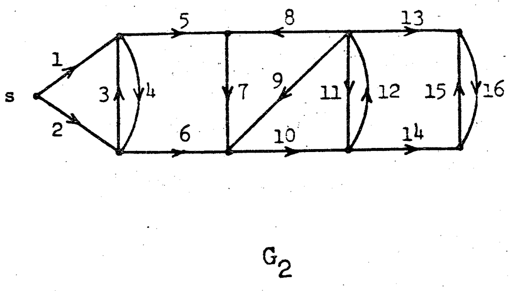


Fig. 3

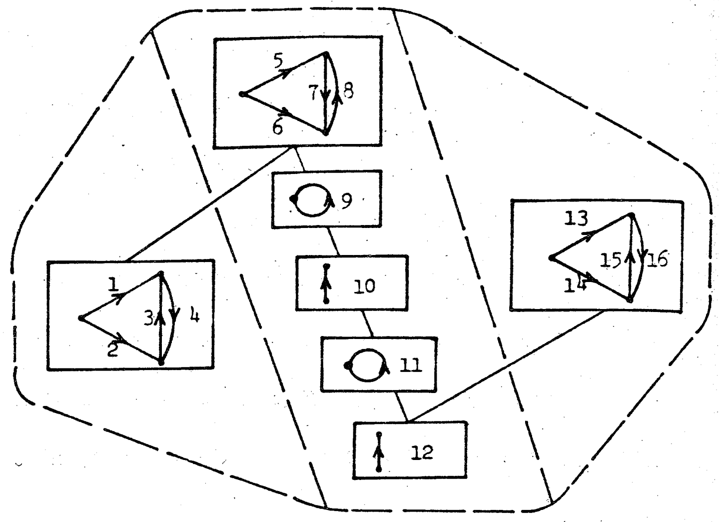


Fig. 4

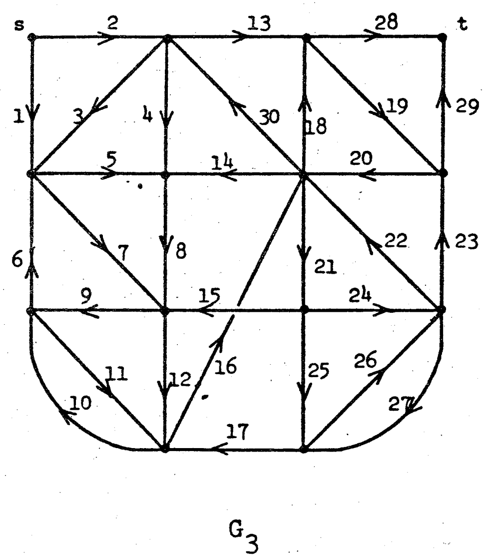


Fig. 5

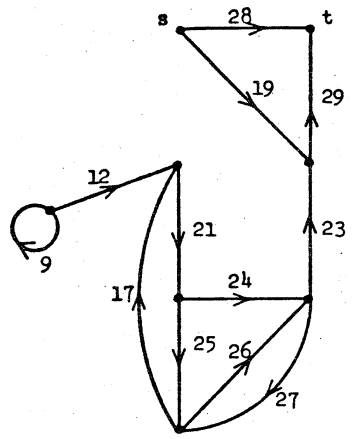


Fig. 6

$E_1(G, P^+) = \{ 21 \}$,

$E_1(G, P^+) = \{ 8, 16 \}$,

$E_1(P^-, P^+) = \{ 2, 5, 6, 8, 13, 21 \}$.

Since

$$E_0 \cap E_1 = \{1,3,4,7,10,11,14,15,18,22,30\} \cap \{2,5,6,8,13,16,21\} = \emptyset,$$

(N4) is also satisfied. Deletion of the edges of $E_0 \cup (\bar{E}_1 - E_1)$ and contraction of those of E_1 from G_3 will give the graph of Fig. 6, which does not posses a Hamiltonian path; in fact, it contains even no arborescence. Hence, from Theorem 4.1, it is seen that there is no Hamiltonian path in G_3.

Example 4. Next the Hamiltonian-path problem on the graph G_4 of Fig. 8 is examined. As for the graph G_4, we have

$E_0 = \{ 2, 3, 9, 13, 20, 23 \}$,

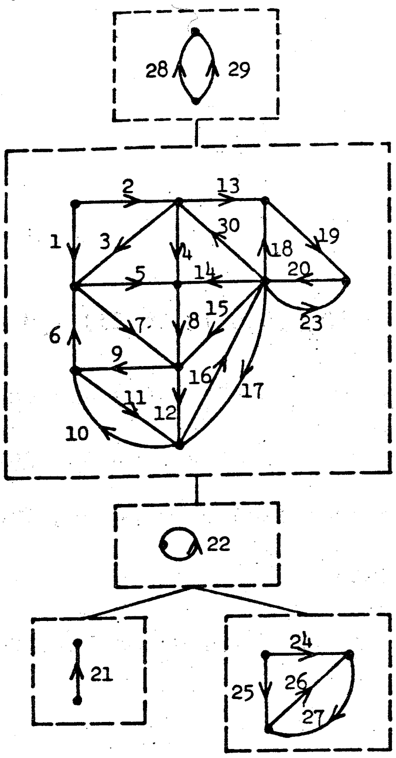
$E_1 = \{ 1, 6, 11, 15, 22 \}$.

By deleting the edges of E_0 and contracting the edges of E_1 from G_4, we obtain the graph of Fig. 9, to which the reduction procedure is again applicable. As for this graph, we have

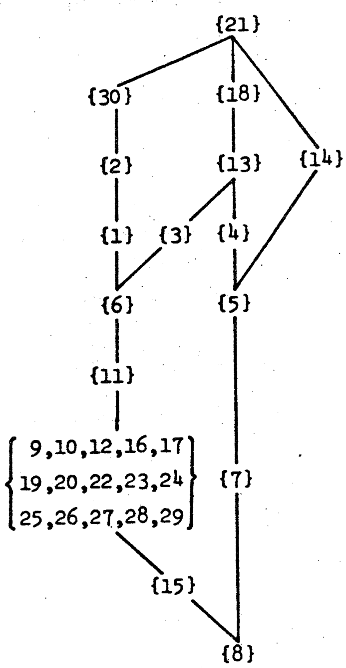
$E_0' = \{ 7, 16, 21 \}$,

$E_1' = \{ 4, 5, 8, 12, 17, 19 \}$.

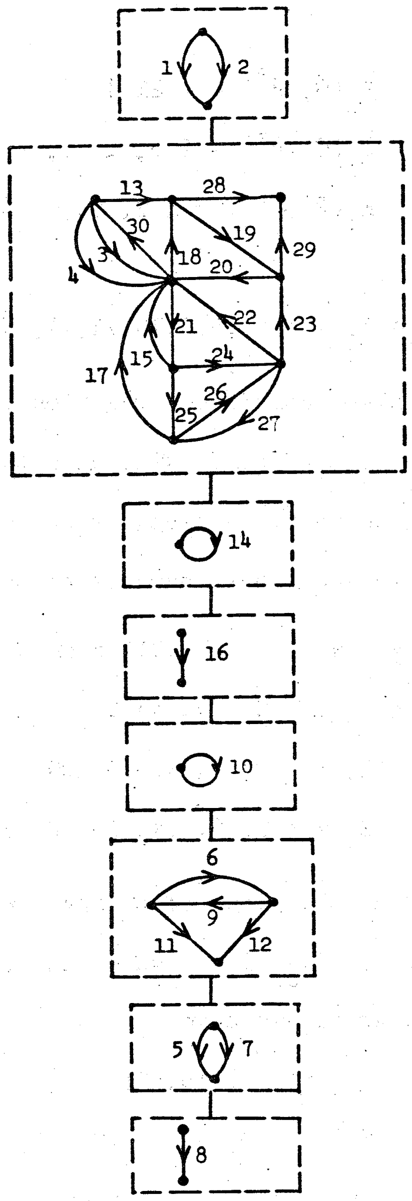
Finally, the original Hamiltonian-path problem on G_4 is reduced to the trivial problem on the graph of Fig. 10. Hence, from Theorem 3.1, a subset T of the edges of G_4 is a Hamiltonian path in G_4 iff $T = E_1 \cup E_1' = \{1,4,5,6,8,11,12,15,17,19,22\}$, i.e., $\{1,4,5,6,8,11,12,15,17,19,22\}$ is the unique Hamiltonian path in G_4.



Partition (2.1) with respect to (G, P^-)



Partition (2.1) with respect to (P^-, P^+)



Partition (2.1) with respect to (G, P^+)

Fig. 7

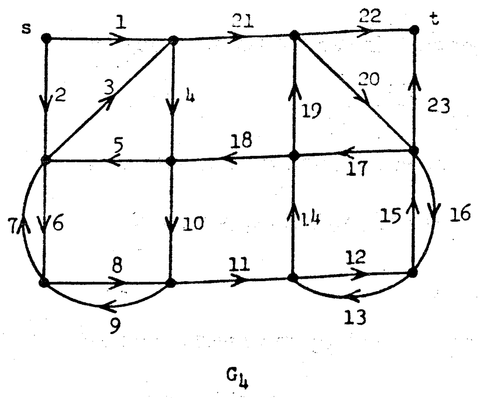


Fig. 8

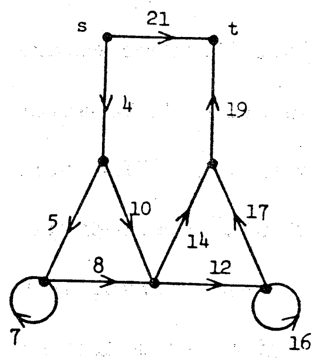


Fig. 9

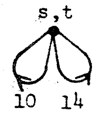


Fig. 10

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