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ON A {1,2}-FACTOR OF A GRAPH

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Abstract.

A criterion for the existence in a graph of a spanning regular subgraph of degree 1 was found by Tutte [4], [2, Theorem 9.4]. We now give an analogous criterion for the existence in a graph of a spanning subgraph whose point degrees are 1 or 2.

1. DEFINITIONS AND NOTATION

A factor of a graph G is a spanning subgraph of G which is not totally disconnected. An n-factor is regular of degree n. We define a new factor of a graph called a {1,2}-factor of a graph, which is strongly related to a spnning linear forest of a graph.

Let G be a graph and H be a subgraph of G. A subgraph H is called a {1,2}-subgraph of G if $1 \leqq \deg_H v \leqq 2$ for every point v of H. Then H is called a {1,2}-factor of G if H is a factor of G. In other words, a {1,2}-factor of G is a special kind of a spanning linear forest of G having no isolated points. Every graph has, of course, a spanning linear forests, but not necessarily {1,2}-factor. We illustrate two graphs having no {1,2}-factors in Figure 1.

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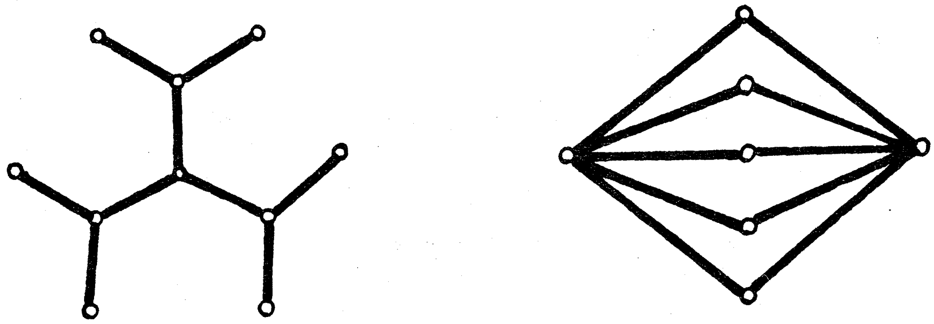

Figure 1. Two graphs having no {1,2}-factors.

Throughout this sections we denote by $S_0(G)$ a set of isolated points of G. A {1,2}-subgraph M of G is called maximal if the inequality $|V(M)| \geqq |V(M')|$ holds for any {1,2}-subgraph M' of G (M standing for maximal).

Let v, w be points of G and M be a maximal {1,2}-subgraph of G. Then a vw-path $P = [v = v_0, v_1, \cdots, v_\ell = w]$ in G is called a vw-alternating path with respect to M if the lines of P alternately lie in M, saying more precisely, $v_{2k}v_{2k+1} \notin M$ and $v_{2k+1}v_{2k+2} \in M$ for $k = 0, 1, \cdots, \{\ell/2\}-1$. In Figure 2, we illustrate a maximal {1,2}-subgraph M by the lines with slash and a vw-alternating path P with respect to M by the bold lines.

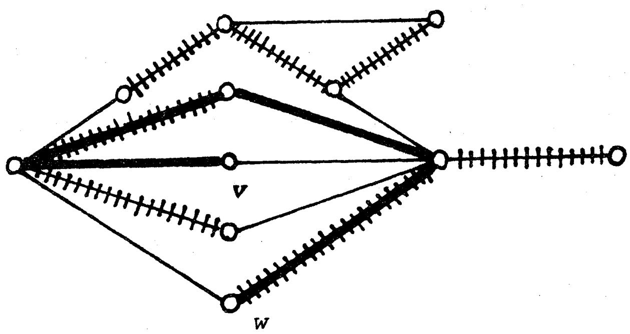

Figure 2 *A maximal {1,2}-subgraph M and a vw-alternating path with respect to M.*

2. CHARACTERIZATION

The following theorem gives a characterization for graphs possessing a {1,2}-facter. In general, this test for a {1,2}-factor is quite inconvenient to apply.

THEOREM 1 A graph G has a {1,2}-factor if and only if the following inequality holds:

$$|S_0(G - S) \leqq 2|S| \text{ for any point subset } S \text{ of } G.$$

NECESSITY OF THEOREM Suppose that G has a {1,2}-factor F. Denote by $F_1, F_2, \ldots, F_r$, the components of F. Let S be any point subset of G and

$$V_i = \{v | v \in S_0(G - S) \text{ and } v \in V(F_i)\}.$$

Then $\bigcup_{i=1}^{r} V_i = S_0\ (G - S)$, and $V_i \cap V_j = \emptyset$, $i \neq j$.

From the fact that every component F_i is either a path or a cycle, the following inequality follows at once:

$$2|S \cap V(F_i)\ | \geqq |V_i|,\ i = 1,2,\ldots,r.$$

Thus we obtain the inequality:

$$2|S| = \sum_{i=1}^{r} 2|S \cap V(F_i)| \geqq \sum_{i=1}^{r} |V_i| = |S_0(G - S)|.$$

We require three lemmas in order to prove the sufficiency of the theorem.

LEMMA 1 Let M be a maximal {1,2}-subgraph of G, $u=u_0$ be a point of G - M and $P = [u=u_0, u_1, \ldots, u_{2\ell}]$ be an alternating path. Then

$$\deg_M(u_{2i-1}) = 2 \quad \text{and} \quad \deg_M(u_{2i}) = 1,\ i = 1,2,\ldots,\ell.$$

PROOF. Suppose that $\deg_M\ (u_{2i-1}) = 1$ for some i.

Denote by M' the subgraph of G obtained from M by deleting lines $u_{2j-1}u_{2j}$, $j = 1,\ldots, i$ and adding lines $u_{2j}u_{2j+1}$, $j = 0,\ldots, i-1$ instead. Figure 3 illustrates a path $p = [u=u_0,\ldots\ldots, u_{12}]$ in M and M' when i = 4, respectively.

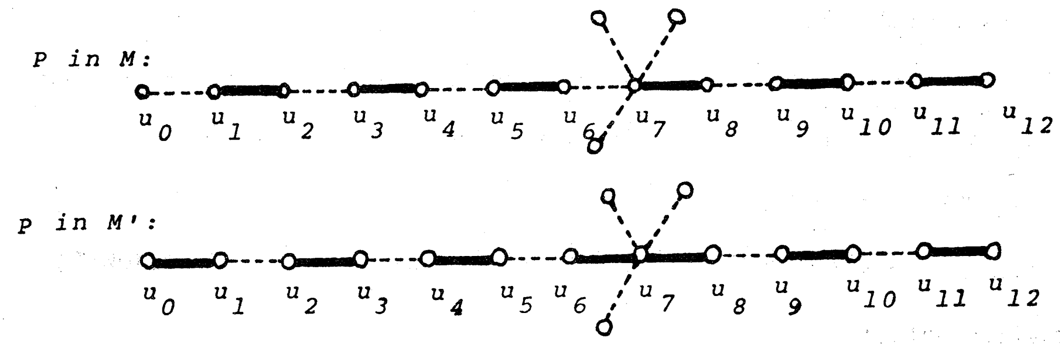

Figure 3   A step in the proof of Lemma 1

Then the following relations are easily verified:

$$\deg_{M'}(u_j) = \deg_M(u_j) \quad \text{for } j = 1,\dots, 2i-2.$$

and

$$\deg_{M'}(u_0) = 1, \quad \deg_{M'}(u_{2i-1}) = 2.$$

Furthermore,

$\deg_M(v) = \deg_{M'}(v)$ for every point v of M other than points u_i, $i = 0,1,\dots 2i-1$. Thus we see that M' is also a {1,2}-subgraph of G, contradicting the maximality of M since $|V(M')| > |V(M)|$.

In a quite similar way, we obtain that

$$\deg_M(u_{2i}) = 1 \text{ for } i = 1,2,\dots. \qquad \square$$

We denote by A(u) (or $A_M(u)$) the set of all points v of G such that there exists a uv-alternating path with respect to a maximal {1,2}-subgraph M. Note that $A(u) \subset V(M)$.

LEMMA 2 Let u be a point of G - M and $P = [u=u_0, u_1,\dots,u_k]$ be an alternating path. If a point w is adjacent to one of u_{2i} of G, then w is a point of A(u) and its degree in M is 2:
that is,

$$w \in A(u) \quad \text{and} \quad \deg_M(w) = 2.$$

PROOF. Suppose that $w \notin A(u)$ or "$w \in A(u)$ and $\deg_M(w) = 1$", then we could have another $\{1,2\}$-subgraph M' of order greater than $|V(M)|$ in a quite similar method applied in the proof of the previous lemma. This contradicts the maximality of M. □

LEMMA 3 Let u be a point og G - M and v be a point in A(u), then every component of M containing v is isomorphic to the path P_3.

PROOF. Let v be a point in A(u) and $P = [u = u_0, u_1, \cdots, u_k = v]$ be an uv-alternating path. We divide the proof into two cases depending on the parity of k.

CASE 1. k: odd

It follows immediately from Lemma 1 that $\deg_M(v) = 2$, since $v = v_k$ and k is odd. We now suppose that the component of M containing v is not isomorphic to P_3. Then M would contain either a path $P_4 = w_1, v, w_2, w_3$ or a triangle $C_3 = w_1, v, w_2, w_1$. Again applying the same method as in the proof of Lemma 1 we could construct a bigger $\{1,2\}$-subgraph of G than M, contradicting the maximality of M.

CASE 2. k: even

Considering the fact that the line $u_{k-1}u_k \in M$ and tha $k - 1$ is of course odd, the theorem follows at once from Case 1. □

We are now ready to give the sufficiency of Theorem 1 by using the previous three lemmas.

SUFFICIENCY OF THEOREM

Suppose that G does not have a $\{1,2\}$-factor. Let M be a maximal $\{1,2\}$-subgraph, u be a point of G - M and S be a set defined by:

$$S = \{v \mid v \in A(u),\ \deg_M(v) = 2\}.$$

Noting that $\deg_M(v') = 1$ for any point v' of $A(u) - S$, we see that the length of every uv'-alternating path is even by Lemma 1. Thus every point w adjacent to v' belongs to S by Lemma 2, which implies that the removal of all the points in S from G results in v' isolated, that is, $\deg_{G-S} v' = 0$. Furthermore it follows at once from the definition of $A(u)$ that $\deg_{G-S}(u) = 0$. Hence the following relation holds:

$$S_0(G - S) \supset (A(u) - S) \cup \{u\}.$$

On the other hand, since every component of M containing $v \in A(u)$ is isomorphic to the path P_3 by Lemma 3, we obtain

$$2|S| = |A(u) - S|.$$

Therefore the following inequality holds:

$$|S_0(G - S)| \geqq 2|S| + 1,$$

completing the proof. □

<u>COROLLARY 1</u>. Every regular graph has a $\{1,2\}$-factor.

<u>PROOF</u>. Let G be r-regular and S be any point subset of G. Consider the following two numbers D_1 and D_2:

$$D_1 = \sum_{v \in S_0} \deg_G v = r|S|,$$

$$D_2 = \sum_{v \in S_0(G-S)} \deg_G v = r|S_0(G-S)|.$$

Then the inequality $D_1 \geqq D_2$ holds, since every point of $S_0(G - S)$ is adjacent to only points of S in G. Thus we obtain

$$2|S| > |S| \geqq |S_0(G - S)| \text{ for any point subset } S \text{ of } G.$$

The proof is completed by Theorem 1. □

This result follows at once from Tutte's Theorem [5].

3. {1,2}-FACTORIZATION

If G is the sum of {1,2}-factors, their union is called an {1,2}-factorization and G itself is {1,2}-factorable. Using this terminology, it has proved in [0] that every cubic graph is {1,2}-factorable.

A criterion for the decomposability of a graph into 2-factors was obtained by Petersen [3].

THEOREM A A graph is 2-factorable if and only if it is regular of even degree.

By applying this, we obtain the following result:

THEOREM 2. Every regular graph is {1,2}-factorable.

PROOF. Let G be r-regular. If r is even, then G is 2-factorable by Theorem A and thus trivially {1,2}-factorable.

We thus assume r odd. By Corollary 1, we see that G has a {1,2}-factor F. By G' we denote the graph obtained by deleting all lines of F from G. Note that the degree of every point in G' is either r - 1 or r - 2. By adding a set I of new points and applying the method in the proof of Theorem 1.2 in [1], G' can be embeded into some (r - 1)-regular graph G'' as an induced subgraph. Applying Theorem A again, G'' can be decomposed into (r - 1)/2 2-factors, F_i, $i = 1,\cdots, (r-1)/2$. Removing a set I of points from G'', every F_i rusults in a {1,2}-factor of G, since the deficiency $e(v) = r - \deg v$ of each point v of G is at most one. Thus, these {1,2}-factors F_i, $i = 1,\cdots, (r-1)/2$, together with F constitutes a {1,2}-factorization of G. □

Theorem 2 has the following corollary, which was independently proved by Tutte [5].

COROLLARY 2. Every r-regular graph G has a {k,k+1}-factor for every k, $1 \leqq k \leqq r$.

PROOF. The corollary is trivial when r is even, since G is 2-factorable by Theorem A. Thus we assume r odd. Then G can be decomposed into {r/2} {1,2}-factors F_1, F_2, $\cdots$, $F_{\{r/2\}}$ by Theorem 2. Note here thet for any point v of G there exists exactly one {1,2}-factor F_j such that the degree of v in F_j is one. Deleting all lines of F_1, $\cdots$, F_i from G, we obtain an {r -2i, r - 2i + 1}-factor of G, for any i, $1 \leqq i \leqq [r/2]$. The proof is thus completed. $\square$

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