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Quantum Mechanics In Phase Space

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Let $E = \mathbb{R}^{2d}$, $d \in \mathbb{N}$ denote an even dimensional real vector space and σ a symplectic (bilinear, antisymmetric, non-degenerate) form on E. We introduce the antisymmetric affine function

$$\phi(a,b,c) = \sigma(a,c) + \sigma(c,b) + \sigma(b,a) \qquad a,b,c \in E$$

and normalize Haar-measure on E such that a unit cube (with respect to an ortogonal structure compatible with σ, for details see [1], Section 2) has measure $(2\pi)^{-d}$. The twisted product of functions on E is defined by

$$(f\circ g)(v) = \iint_{EE} f(v')g(v'')e^{i\phi(v,v',v'')}dv'dv'' \qquad , v \in E$$

The twisted product leaves the Frèchet space $\mathcal{S}(E)$ of C^∞-functions rapidly decreasing at infinity invariant and satisfies

$$\overline{f\circ g} = \overline{g}\circ\overline{f} \ , \quad 1\circ f = f \ , \quad \int_E (f\circ g)(v)dv = \int_E f(v)g(v)dv$$

The twisted product can be regarded as a deformation of the usual pointwise product of functions in the following sense. Let for $\lambda \in \mathbb{R}_+$ the dilation operator R_λ be given by

$$(R_\lambda f)(v) = f(\lambda^{-\frac{1}{2}}v) \qquad , v \in E$$

and define

$$f \circ_\lambda g = R_\lambda(R_\lambda^{-1} f \circ R_\lambda^{-1} g)$$

We call $\circ_\lambda$ dilated twisted multiplication and note that $\circ_\lambda = \circ$ for $\lambda = 1$. Dilated twisted multiplication enjoys for all positive values of λ similar properties as twisted multiplication. For λ tending to 0, it becomes close to pointwise multiplication. More precisely, for $f,g \in S(E)$ we have

$$f \circ_\lambda g = f \cdot g + \frac{i\lambda}{2}\{f,g\} + \Gamma_\lambda \quad , \qquad \lambda \in \mathbb{R}_+$$

where $\{\cdot,\cdot\}$ denotes the Poisson bracket and $\lambda^{-1} \cdot \Gamma_\lambda$ tends to 0 in the topology of uniform convergence of all derivatives as λ tends to 0. An important difference between twisted multiplication and pointwise multiplication is that the former is continuous in the L^2-topology while the latter is not.

The Schwartz space $S(E)$, when equipped with twisted multiplication as product and complex conjugation as involution, is a topological involutive algebra which we shall denote by $\mathcal{U}$. Let $\Lambda \in S'(E)$ be a tempered distribution which is positive and faithful in the following sense

$$\Lambda(\bar{\xi} \circ \xi) > 0 \qquad \forall \xi \in \mathcal{U}, \quad \xi \neq 0$$

We define a positive definite inner product on $\mathcal{U}$ by

$$(\xi|\eta)_\Lambda = \Lambda(\bar{\xi}\circ\eta) \qquad \xi,\eta \in \mathcal{U}$$

and denote the $(\cdot|\cdot)_\Lambda$ - completion of $\mathcal{U}$ by H_Λ.

<u>Proposition</u> $\mathcal{U}$ equipped with the inner product $(\cdot|\cdot)_\Lambda$ as given above is a generalized Hilbert algebra in the sense of [2].

The left representation ℓ_Λ of $\mathcal{U}$ on H_Λ generates the left von Neumann algebra $\mathcal{L}_\Lambda(\mathcal{U})$.

<u>Theorem</u> Let $\Lambda \in S'(E)$ be positive and faithful, then

(i) $\mathcal{L}_\Lambda(\mathcal{U})$ is a type I factor with normalized trace τ given by

$$\tau(\ell_\Lambda(\xi)) = 2^{-d} \int_E \xi(v)dv \qquad \forall\xi \in \mathcal{U}$$

(ii) The Hilbert Schmidt operators in $\mathcal{L}_\Lambda(\mathcal{U})$ are exactly the operators of the form $\ell_\Lambda(\xi)$ with $\xi \in L^2(E,dv)$.

(iii) $\|f\circ g\|_2 \le 2^{-\frac{d}{2}} \|f\|_2 \|g\|_2 \qquad \forall f,g \in \mathcal{U}$

In classical mechanics each real function $h \in \mathcal{U}$ is considered to be an observable. The corresponding quantum observable (obtained by quantization) is the left multiplication operator $\ell_\Lambda(h) \in \mathcal{L}_\Lambda(\mathcal{U})$ given by

$$\ell_\Lambda(h)\eta = h\circ\eta \qquad \eta \in \mathcal{U}$$

We note that $\ell_\Lambda(h)$ is bounded and selfadjoint. Unfortunately, the class of C^∞- function on E rapidly decreasing as infinity is much too small to exhaust the classical observables. We shall therefore extend the representation ℓ_Λ to a class of real functions on E more relevant to physics. For h belonging to such a class, we do no longer expect $\ell_\Lambda(h)$ to be a bounded operator. But it should, as a minimum requirement, be selfadjoint on H_Λ and affiliated with $\mathcal{L}_\Lambda(\mathcal{U})$. We propose two different ways of attacking this problem.

Since $\mathcal{L}_\Lambda(\mathcal{U})$ is semi-finite, there exists a unique selfadjoint operator H on H_Λ affiliated with $\mathcal{L}_\Lambda(\mathcal{U})$ such that

$$\Lambda(\xi) = \tau(e^{-H}\ell_\Lambda(\xi)) \qquad \forall\xi \in \mathcal{U}$$

We have not yet specified the positive faithful tempered distribution Λ . If $h \in L^2(E)$, we set

$$\mathrm{Exp}^\circ(-h) = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} h^{\circ n}$$

where $h^{\circ n} = h\circ\cdots\circ h$ (n-factors). The formula

$$\Lambda(\xi) = \int_E \mathrm{Exp}^\circ(-h)(v)\ \xi(v)dv \qquad \xi \in \mathcal{U}$$

defines a positive faithful distribution. With this choice of Λ , we obtain

$$H\xi = h\circ\xi \qquad \forall\xi \in \mathcal{U}$$

that is $H = \ell_\Lambda(h)$. We now drop the condition that $h \in L^2(E)$

but maintain that $\mathrm{Exp}^{\circ}(-h)$ is to be defined as a (positive faithful) tempered distribution. The representation ℓ_{Λ} (with $\Lambda = \mathrm{Exp}^{\circ}(-h)$) can now be extended to h simply setting $\ell_{\Lambda}(h) = H$. This procedure is very general but its feasibility do depend of a calculation (of $\mathrm{Exp}^{\circ}(-h)$) potentially difficult to carry out. It is therefore of interest to give criteria on h which ensure that $\mathrm{Exp}^{\circ}(-h)$ is a (positive faithful) tempered distribution. Our second approach to the extension problem will provide us with one such criterion.

We set $\Lambda(\xi) = \int_E \xi(v)dv$, $\xi \in \mathcal{U}$ and denote $\ell_{\Lambda}(\xi)$ and $\mathcal{L}_{\Lambda}(\mathcal{U})$ by $\ell_2(\xi)$ and $\mathcal{L}_2(\mathcal{U})$ respectively. The symplectic Fourier transformation F is defined by

$$(Ff)(v) = \tilde{f}(v) = \int_E e^{i\sigma(v,v')} f(v')dv' \quad , \quad v \in E$$

<u>Theorem</u> Let h be a real tempered distribution satisfying

$$\iint_{EE} \overline{\xi(v)}\, \xi(v')\, e^{i\sigma(v,v')}\, \tilde{h}(v-v')dvdv' \geq \lambda \, \|\xi\|_2^2 \qquad \forall \xi \in \mathcal{U}$$

for some constant $\lambda \in \mathbb{R}$, then

(i) The sesquilinear form $h(\eta \circ \bar{\xi})$ defined on $\mathcal{U}$ is symmetric and bounded from below with bound λ. The formclosure t with domain D(t) admits a representation

$$t(\xi,\eta) = (K\xi|\eta)_2 \qquad \forall \xi \in D(K)\ \forall \eta \in D(t)$$

where K is a selfadjoint operator on $L^2(E)$. The domain D(K) is a core for t.

(ii) K is affiliated with $\mathcal{R}_2(\mathfrak{U})$ and bounded from below with bound λ.

When $h \in L^2(E)$, we obtain $K = \ell_2(h)$. We have thus constructed an extention of ℓ_2 from $L^2(E)$ onto real tempered distributions satisfying the condition of the theorem. Incidentally, such distributions can be exponentiated and

$$\mathrm{Exp}\circ(-h)(\xi) = 2^d \tau(e^{-K} \ell_2(\xi)) \quad , \qquad \xi \in \mathfrak{U}$$

defines a positive faithful tempered distribution. The first mentioned procedure for quantization is hence avaible too.

References

1. F. Hansen, Quantum mechanics in phase space, Université de Genève, preprint
2. M. Takesaki, Tomita's theory of modular Hilbert algebras and its applications, Lecture Notes in Math. No. 128, Springer-Verlag, Berlin, Heidelberg, New York, 1970.