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# Operators and operator algebras in Krein Spaces I.

## Spectral analysis in Pontrjagin space.

Minoru Tomita

## Introduction

Krein space is an infinite dimensional generalization of Minkowsky space, and more than fourty years ago several physists already awared the importance of studying such spaces. For instance one old idea is that a Hamiltonian in relativestic quantum mechaniques must be a selfadjoint operator in a Krein space.

In any sense the theory of operators and operator algebras in Krein spaces must be founded by developping the spectral analysis of selfadjoint operators in such spaces.

This paper is devoted to founding spectral analysis of self-adjoint operators in Pontrjagin spaces.

# Chapter 1. Notations and definitions

## §1.0. Preliminary images of Krein space and Pontrjagin space.

Consider the product space $L: = H_1 \times H_2$ of Hilbert spaces $H_1$ and $H_2$, and define a sequilinear form $\langle x,y\rangle$ in $L \times L$ by

$$\langle x,y\rangle = (x_1|y_1) - (x_2|y_2),$$

where $x: = (x_1,x_2)$ $y: = (y_1,y_2)$ are elements of $L$. If $H_1$ and $H_2$ are of finite dimensional. $L$ is known as a Minkowsky space. If $H_1$ and $H_2$ are Hilbert spaces, $L$ has been called a Krein space, or an indefinite metric space. If either $H_1$ or $H_2$ are of finite dimensional, $L$ has been called a Pontrjagin space.

In the following sections we shall introduce other definitions of Krein and Pontrjagin spaces, as we shall start from defining indefinite innerproduct space.

## §1.1. Indefinite innerproduct space.

In what follows "linear" always means" complex linear", $C$ denotes the complex numbers, and $R$ the real numbers.

1.1.1. (Definition) A linear space $L$ is called an indefinite innerproduct space if $L$ has an indefinite innerproduct $\langle x,y\rangle$ satisfying the following conditions I.1 and I.2.

I.1. $\langle x,y\rangle$ is a symmetric sequilinear form on $L \times L$. Namely, it is a complex valued functional on $L \times L$ satisfying

$$\langle x,y\rangle = \overline{\langle y,x\rangle} ,$$

$$\langle \alpha x_1+\beta x_2,y\rangle = \alpha\langle x_1,y\rangle + \beta\langle x_2,y\rangle .$$

$$\langle x,\alpha y_1+ y_2\rangle = \bar{\alpha}\langle x,y_1\rangle + \bar{\beta}\langle x,y_2\rangle .$$

I.2. There is a certain innerproduct-norm $\| \ \|$ (ie., a norm $\|x\|: = (x|x)$ defined by an innerproduct $(x|y)$ of L) which is selfpolar. Namely, it satisfies

$$\|x\| = \sup_{\|y\|\leq 1} |\langle x,y\rangle| .$$

A norm $\| \ \|$ satisfying I.2. in L is called a finite unitary norm of L.

1.1.2. (Definition). An indefinite innerproduct space L is called a Krein space if it becomes a Hilbert space under its certain finite unitary norm. The topology of L is that of the Hilbert space.

1.1.3. (Definition). The metrical completion L(p) of an indefinite innerproduct space L under a certain finite unitary norm p of L is a Krein space, which we call the Krein completion of L under p.

1.1.4. We must beware that the space L may have finite unitary norms which are not generally mutually equivalent on L, so that Krein completions of L under different norms cannot generally be identified to each other.

1.2. #-innerproducts in Krein spaces.

Let L be a Krein space. Then $\langle x,y\rangle$ denotes the indefinite innerproduct of L, and it is also called the #-innerproduct of L, to distinguish it from the *-innerproduct which is defined by a fixed anxiliary finite unitary norm * of L.

In what follows L and M denote Krein spaces.

1.2.1. (Definition). Let x and y belong to L. x is called #-orthogonal to y if $\langle x,y\rangle = 0$ holds. Let $\mathfrak{M}$ and $\mathfrak{N}$ be subsets of L. $\mathfrak{M}$ is called #-orthogonal if $\langle x,y\rangle = 0$ holds for $x \in \mathfrak{M}$ and $y \in \mathfrak{N}$. The elements of $\mathfrak{M}$ which are #-orthogonal to is denoted by $\mathfrak{M}^{\#\perp}$. If $\mathfrak{M}$ is a closed linear subspace of L, $\mathfrak{M}^{\#\perp}$ is called the #-orthogonal complement of $\mathfrak{M}$.

1.2.2. (Definition). $X : L \rightarrow M$ denotes a mapping whose domain $\mathcal{D}(X)$ is in L, and range R(X) is in M. X is called densely defined if $\mathcal{D}(X)$ is everywhere dense in L. X is called invertiblly densely defined of X and its inverse mapping $X^{-1} : M \rightarrow L$ are densely defined.

1.2.3. (Definition). Let $X : L \rightarrow M$ be a linear mapping. The #-adjoint, or the Krein adjoint, of X means a linear mapping $X^{\#} : M \rightarrow L$ which is determined by the following rule : Take any $x \in L$ and $y \in M$. $X^{\#}y$ is defined and $x = X^{\#}y$ holds iff it satisfies

$$\langle x,z\rangle = \langle y,Xz\rangle \quad \text{for all } z \in \mathcal{D}(X).$$

The #-adjoint $X^{\#}$ exists iff X is densely defined.

1.2.4. (Definition) (a). An operator $X$ in $L$ is called #-selfadjoint if $X = X^{\#}$ holds.

(b). An operator $X$ in $L$ is called #-positive if it is #-selfadjoint and satisfies $\langle Xx,x \rangle \geqq 0$ on $\mathcal{D}(X)$.

(c). An operator $X$ in $L$ is called a #-projection if it satisfies $X = X^{\#} = X^2$.

(d). A linear mapping $X : L \to M$ is called #-unitary if $X^{\#} = X^{-1}$ holds.

(e). A linear mapping $X : L \to M$ is called #-imaginary if $X^{\#} = -X^{-1}$ holds.

(f) A linear mapping $x : L \to M$ is called #-selforthogonal if $X^{\#}X$ vanishes on $\mathcal{D}(X)$.

1.2.5. Every #-selfadjoint operator is closed and densely defined, and so are every #-positive operator and #-projection. Every #-unitary mapping and every #-imaginary. Mapping are closed and invertiblly densely defined. Indeed, for instance, if $X : L \to M$ is #-unitary, $X^{\#} = X^{-1}$ means that $X$ is densely defined and the inverse mapping $X^{-1}$ is closed. Then $X$ is closed and hence $X^{\#}(=X^{-1})$ is densely defined. A mapping $X : L \to M$ is #-selforthogonal iff $X$ is densely defined and has the #-selforthogonal range. Indeed, $X$ is selforthogonal iff $X^{\#}$ is defined and satisfies $\langle Xx,Xy \rangle = 0$ for all $x$, $y$ in $\mathcal{D}(X)$.

1.2.6. (Remark). We must beware that #-projections and #-unitary mappings in our definition are not generally continuous.

§1.3. The *-norm in Krein space.

1.3.1. We consider a certain fixed finite unitary norm of L which makes L a Hilbert space. We call this norm the *-norm and denote it by $\| \ \|$ or by &. The innerproduct associate with the norm * is denoted by $(x|y)$, and is called the *-innerproduct of L. The definitions in the Section 1.2 are also applied to the *-innerproducts of the spaces L and M.

1.3.2. (Definition). Let x, y belong to L. x is *-orthogonal to y if $(x|y) = 0$ holds. Let $\mathfrak{M}$, $\mathfrak{N}$ be subsets of L. $\mathfrak{M}$ is *-orthogonal to $\mathfrak{N}$ if $(x|y) = 0$ holds for $x \in \mathfrak{M}$ and $y \in \mathfrak{N}$.

Let X be an operator in L. X is *-selfadjoint if $X = X^*$ holds. X is *-positive if X is *-selfadjoint and satisfies $(Xx|x) \geqq 0$ on $\mathcal{D}(X)$. X is a *-projection if $X = X^* = X^2$ holds. A linear mapping $X : L \to M$ is *-unitary if $X^* = X^{-1}$ holds.

A subset $\mathfrak{M} \neq \phi$ of L becomes *-selforthogonal iff $\mathfrak{M} = 0$ holds. *-projections and *-unitary mappings are continuous. *-imaginary mapping $X : L \to M$ does not exist. A densely defined linear mapping $X : L \to M$ is *-selforthogonal iff X vanishes on the domain.

1.3.3. (Definition). An operator X in L is called biselfadjoint if it is #-selfadjoint and *-selfadjoint. X is called a biprojection if it is a #-projection and a *-projection. A linear mapping $X : L \to M$ is called biunitary if it is #-unitary and *-unitary.

§1.4. Defining operator of the *-norm.

The #-innerproduct $<x,y>$ is a continuous sequilinear form on $L \times L$ treating $L$ as a Hilbert space under the *-norm. Then

1.4.1. (Proposition). There is a certain continuous linear operator $J$ in $L$ satisfying

$$<x,y> = (Jx|y),$$

$$J = J^* = J^{-1}.$$

Indeed the last identity immediately follws from

$$<x,y> = \overline{<y,x>}$$

$$\|x\| = \sup_{\|y\| \leq 1} |<x,y>|.$$

1.4.2. (Definition). The operator $J$ in $L$ is called the defining operator of the *-norm. If necessary, we also denote $J$ by $J_L$.

The Definition 1.4.2 is based on the next theorem.

1.4.3. (Theroem). The defining operator $J$ of the *-norm of $L$ is a continuous #-positive #-unitary operator in $L$, and the *-norm of $L$ is determined by

$$\|x\|^2 = <Jx, x>.$$

It is also clear that :

1.4.4. (Proposition). Let $X : L \to M$ be a densely defined linear mapping. Then

$$X^{\#} = J_L X^* J_M, \quad X^* = J_L X^{\#} J_M.$$

## Chapter 2. Geometry in Krein space.

In Krein space, we may consider two types of geometries. One is the classical geometry which investigates the invariants of the group of continuous #-unitary operators in $L$. Another is the investigations of geometrical aspect of unbounded #-unitary operators in $L$. In this chapter we shall sketch certain aspects of the latter geometry based on the concept of the Cartan decomposition.

### §2.1. Cartan decomposition of #-unitary operators.

The Cartan decomposition of Lorentz groups in Minkowsky spaces is generalized to the case of #-unitary operators in Krein spaces.

2.1.1. (Theorem). Let $X : L \to M$ be a #-unitary mapping Then $X$ is represented in the form

$$X = K_2 U = U K_1.$$

$U : L \to M$ is a biunitary mapping. $K_1$ and $K_2$ are *-positive #-unitary operators in $L$ and $M$ respectively.

The above representation of $X$ is called the Cartan decomposition of $X$.

Proof. $X = K_2U = UK_1$ is the V. Neumann's polarization of the mapping $X : L \to M$ regarding L and M as Hilbert spaces. $U : L \to M$ is a *-unitary mapping, and $K_1$ and $K_2$ are *-positive operators in L and M, satisfying

$$K_1^2 = X^*X, \quad K_2^2 = XX^*.$$

We show that $X^*X$ and $K_1$ are #-unitary. Indeed, remarking that $X^{-1} = X^{\#} = JX^*J$, we have

$$J(X^*X)J = (JX^*J)(JXJ) = X^{-1}X^{-1*} = (X^*X)^{-1},$$

and $JK_1^2J = K_1^{-2}$. From this we have $JK_1J = K_1^{-1}$ and $K_1$ is #-unitary. Similarly we find that $K_2$ is #-unitary, and U satisfies

$$\langle K_1x, K_1y\rangle = \langle x,y\rangle = \langle Xx, Xy\rangle$$
$$= \langle UK_1x, UK_1y\rangle.$$

Since U is bijective, it is #-unitary, and the Theorem is proved.

§2.2. Unitary norms and #-unitary operators.

2.2.1. (Definition). Let $X : L \to M$ be a linear mapping. Then X induces a (unbounded) seminorm $p_X : x \to \|Xx\|$ defined on L, where we set $\|Xx\| = +\infty$ if Xx is not defined. If X is injective, $p_X$ is called a (unbounded) norm in L.

Now we characterize unbounded norms in L induced from #-unitary mappings.

2.2.2. (Definition). A functional $p$ on $L$ is called a unitary norm of $L$ if the following U0, U1, U2 are satisfied :

U0. $0 \leqq p(x) \leqq +\infty$.

U1. The finite part $\mathcal{D}(p) := \{x \in L : p(x) < +\infty\}$ of $p$ is a linear subspace of $L$, and $p$ is an innerproduct norm on it. Namely, $p(x)^2 = (x|x)_p$ holds for a certain innerproduct $(x|y)_p$ in $\mathcal{D}(p)$. The innerproduct $(x|y)_p$ is called the p-innerproduct in $L$.

U2. The norm $p$ is selfpolar on $L$. Namely, it satisfies

$$p(x) = \sup_{p(y) \leqq 1} |\langle x,y \rangle|.$$

2.2.3. (Theorem). A functional $p$ on $L$ is a unitary norm of $L$ iff $p$ is induced from a certain #-unitary mapping $X : L \to K$, where $K$ is a certain Krein space.

Proof. Suppose taht $p$ is induced from a certain #-unitary mapping $X : L \to K$. Then the finite part of $p$ is identical to the domain $\mathcal{D}(X)$ of $X$ and $p$ is an innerproduct norm on it. $p$ is self-polar. Indeed,

$$\|Xx\| = \|X^{-1\#}x\| = \sup_{\|y\| \leqq 1} |\langle x, X^{-1}y \rangle|$$

$$= \sup_{\|Xz\| \leqq 1} |\langle x,z \rangle|.$$

The identity is valid even if $Xx$ is not defined. Then $p$ is a unitary norm of $L$.

Conversely, suppose that $p$ is a unitary norm of $L$. $p$ is an innerproduct norm of the finite part $\mathcal{D}(p)$ of $p$, and $\mathcal{D}(p)$ has an indefinite innerproduct $\langle x,y\rangle$ and a finite unitary norm $p$. Let $L(p)$ be the Krein completion of $\mathcal{D}(p)$ under the norm $p$, and $X : x \to x$, $L(p) \to L$ be the identity mapping on $\mathcal{D}(p)$. $X$ is injective, densely defined, and $X^{\#} \supseteqq X^{-1}$. We show that $X^{\#} \subseteq X^{-1}$. Namely, if a pair $x \in L$, $y \in L(p)$ satisfies

$$\langle x,Xz\rangle = \langle y,z\rangle \quad \text{for} \quad z \in \mathcal{D}(p),$$

then $x = y$ holds. If $x$ and $y$ are as above, then the norm $p(y)$ of $y$ in $L(p)$ is determined by

$$\begin{aligned} p(y) &= \sup_{p(z)\leqq 1, z\in\mathcal{D}(p)} |\langle y,z\rangle| \\ &= \sup_{p(z)\leqq 1} |\langle x,z\rangle| \\ &= p(x). \end{aligned}$$

Therefore $x$ belongs to $\mathcal{D}(p)$, and remarking

$$\langle x,z\rangle = \langle y,z\rangle \quad \text{for} \quad z \in \mathcal{D}(p),$$

we have $x = y$. Then $X$ is #-unitary and induces $p$.

2.2.4. (Theorem). For every unitary norm $p$ of $L$ there is a unique *-positive #-unitary oeprator $K$ in $L$ which induces $p$.

Proof. We take a certain #-unitary mapping $X : L \to M$ which induces the norm $p$, and consider the Cartan decomposition

$X := UK$ of $X$. $U : L \to M$ is biunitary. $K$ is a x-positive #-unitary operator in $L$ and induces $p$. Suppose also that $p$ is induced from another *-positive #-unitary operator $S$ in $L$. Then $\|K<x\| = \|Sx\|$ holds on $\mathcal{D}(p) := \{x \in L : p(x) < +\infty\}$. Therefore $K = S$ holds, and the Theorem is proved.

§2.3. Defining operator of unitary norm.

Proposition 1.4.1 is now generalized as follows :

2.3.1. (Theorem). Let $p$ be a unitary norm of $L$. Then there is a unique #-positive #-unitary operator $J_p$ in $L$ satisfying

$$p(x)^2 = \langle J_p x, x \rangle.$$

for all $x$ in $\mathcal{D}(J_p)$.

Proof. $p$ is induced from a certain *-positive #-unitary operator $K$ in $L$. We set

$$J_p = JK^2.$$

Then $J_p$ is #-unitary. We show that $J_p$ is #-positive. Indeed, it is #-selfadjoint and

$$\langle J_p x, x \rangle = (K^2 x \mid x) = \|Kx\|^2 = p(x)^2,$$

and we have the Theorem. (For the uniqueness of $J_p$, refer the next Theorem 2.3.3).

2.3.2. (Definition). The operator $J_p$ in Theorem 2.3.1 is called the defining operator of the norm p.

2.3.3. (Theorem). Let S be a #-positive #-unitary operator in L, then S defines a certain unitary norm p in L.

Proof. JS is a *-positive #-unitary operator in L. Indeed, it is *-selfadjoint, #-unitary and

$$(JSx|x) = \langle Sx,x\rangle \geqq 0.$$

holds on $\mathscr{D}(S)$. Let $K = (JS)^{\frac{1}{2}}$. Then K is also *-positive #-unitary and $S = JK^2$ holds, and induces the norm $p_K$.

## §2.4. Canonical partition and canonical quasi-partition.

2.4.1. (Definition). Let $L_1$ and $L_2$ be subspaces of L such that :

(a). $L_1$ and $L_2$ are the #-orthogonal complements of each other.

(b). If $0 \neq x \in L_1$ then $\langle x,x\rangle > 0$. If $0 \neq y \in L_2$ then $\langle y,y\rangle < 0$.

(c). The linear sum of $L_1$ and $L_2$ is everywhere dense in L.

Then the representation of $L$ : $L = L_1 + L_2$ is called a canonical quasi-partition of L. Inparticular, if $L = L_1 + L_2$ holds, the representation $L = L_1 + L_2$ is called a canonical partition of L.

2.4.2. (Definition). Let p be a unitary norm of L. Let $L^+(p)$ be the elements x of L satisfying $p(x)^2 = \langle x,x\rangle$, and $L^-(p)$ the elements $x$ of $L$ satisfying $p(x)^2 = -\langle x,x\rangle$. We call the representation $L := [L^+(p) + L^-(p)]$ the p-quasi-partition of $L$.

2.4.3. (Theorem). Let $p$ be a unitary norm of $L$. Then the p-quasi-partition $L := [L^+(p) + L^-(p)]$ is a canonical quasi-partition of $L$. Let $J_p$ be the defining operator of $p$. Then $L^+(p)$ and $L^-(p)$ are eigen-spaces for eigenvalues $1$ and $-1$ of $J_p$, and we have

$$\mathcal{D}(J_p) = L^+(p) + L^-(p).$$

Proof. First we consider the case that $p$ is the *-norm $\|x\|$. Then the defining operator $J$ of the *-norm satisfies $J = J^* = J^{-1}$, and hence is represented in the form : $J = J^+ - J^-$, where $J^+$ and $J^-$ are *-projections satisfying $1 = J^+ + J^-$. We set $L^+ = L^+(*)$, $L^- = L^-(*)$. $L^+$ is the elements $x$ of $L$ satisfying $\|x\|^2 = (Jx\ x)$, and is identical to the range of $J^+$. Similarly, $L^-$ is identical to the range of $J^-$, these are the eigenspaces for the eigenvalues 1 and -1 of $J$, respectively, and $L = L^+ + L^-$ is a canonical partition of $L$.

Next, we return to a general unitary norm $p$, and let $K$ be the *-positive #-unitary operator in $L$ which induces $p$. Then

$$J_p = JK^2 = K^{\#}JK.$$

An element $x$ of $L$ belongs to $L^+(p)$ iff

$$\|Kx\|^2 = \langle x,x\rangle = (JKx,Kx),$$

ie., $Kx = JKx$ or $x = J_p x$ holds. In other words $x$ is an eigenvector of $J_p$ for the eigenvalue 1. Then $L^+(p)$ is the eigenspace of $J_p$ for the eigenvalue 1. Similarly, $L^-(p)$ is the eigenspace of $J_p$ for the eigenvalue -1. Now we set

$$J_p^+ = \frac{1}{2}(1 + J_p) = K^\# J^+ K,$$

$$J_p^- = \frac{1}{2}(1 - J_p) = K^\# J^- K.$$

Then $L^+(p)$ and $L^-(p)$ are the ranges of $J_p^+$ and $J_p^-$, and

$$\mathcal{D}(J_p) = \mathcal{D}(K^2) = L^+(p) + L^-(p).$$

$L^+(p)$ and $L^-(p)$ are mutually #-orthogonal to each other. Then it determines a canonical quasi-partition of $L$.

2.4.4. (Theorem). Let $L = [L_1 + L_2]$ be a canonical quasipartition of $L$. Then it is identical to a certain quasi-partition of $L$ defined by a certain unitary norm of $L$.

Proof. Define an operator $S$ with the domain $\mathcal{D}$ in $L$ by

$$S : x_1 + x_2 \to x_1 - x_2 \quad \text{for} \quad x_1 \in L_1, \quad x_2 \in L_2.$$

We show that $S$ is a #-positive #-unitary operator in $L$. $S$ satisfies $S = S^{-1}$, and

$$\langle Sx, x\rangle = \langle x_1, x_1\rangle - \langle x_2, x_2\rangle \geqq 0$$

for $x = x_1 + x_2 \in \mathcal{D}$. Then $S^\# \supseteq S$ holds. We show $S^\# \subseteq S$. Let $y$ and $z$ be elements of $L$ satisfying

$$\langle Sx,y\rangle = \langle x,z\rangle \quad \text{for} \quad x \in \mathcal{D}.$$

Let

$$u = \tfrac{1}{2}(y+z), \qquad v = \tfrac{1}{2}(y-z).$$

Then

$$\langle x_1,v\rangle = \langle x_2,u\rangle \qquad \text{for} \quad x_1 \in L_1, \quad x_2 \in L_2.$$

Since $L_1$ and $L_2$ contains $0$, we have

$$\langle x_1,v\rangle = \langle x_2,u\rangle = 0 \qquad \text{for} \quad x_1 \in L_1, \quad x_2 \in L_2,$$

and $v$ belongs to the #-orthogonal complement $L_2$ of $L$. Similarly, $u$ belongs to $L_1$. Then remarking

$$y = u+v, \quad z = u-v,$$

we find that $z = Sy$, and $S^{\#} = S$ holds. Hence $S$ is a #-positive #-unitary operator in $L$ and identical to the defining operator $J_p$ of a certain unitary norm. Hence $L_1 = L^{+}(p)$ and $L_2 = L^{-}(p)$ holds, and the quasipartition is defined by $p$.

§2.5. Reduction by #-selforthogonal subspace.

2.5.1. (Definition). Let $\mathfrak{M}$ be a certain closed #-selforthogonal subspace of $L$. Then setting $L_1 = \mathfrak{M}$, $L_3 = J\mathfrak{M}$, and $L_2$ the *-orthogonal complement of $L_1+L_3$ in $L$, $L$ is represented as a linear sum $L = L_1+L_2+L_3$ of mutually *-orthogonal spaces $L_1$, $L_2$ and $L_3$. We call it the reduction of $L$ by the space $\mathfrak{M} = L_1$. We immediately obtain that :

2.5.2. (Lemma). Let $L = L_1 + L_2 + L_3$ be a reduction of $L$ by a certain closed #-selforthogonal subspace $:= L_1$ of $L$. Then $L_2$ is a Krein space whose #-innerproduct and *-norm are the restrictions of those of $L$ on $L_2$. Each element $x$ of $L$ has a unique representation

$$x = x_1 + x_2 + Jx_3,$$

where $x_1,\ x_3 \in \mathfrak{M},\ \ x_2 \in L_2$. The #-innerproduct and the *-norm of $L$ has the following representation :

$$\langle x,y\rangle = (x_1|y_3) + \langle x_2,y_2\rangle + (x_3|y_1),$$

$$\|x\|^2 = \|x_1\|^2 + \|x_2\|^2 + \|x_3\|^2.$$

If $A$ is a continuous linear operator in $L$, $A$ and $A^{\#}$ have the following matrix representation.

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{pmatrix}, \quad A^{\#} = \begin{pmatrix} A_{33}^{*} & A_{23}^{\#} & A_{13}^{*} \\ A_{32}^{\#} & A_{22}^{\#} & A_{12}^{\#} \\ A_{31}^{*} & A_{21}^{\#} & A_{11}^{*} \end{pmatrix}$$

## Chapter 3. Spectral analysis in Pontrjagin space.

### §3.1. Pontrjagin space.

3.1.1. (Definition). A Krein space $L$ is called a Pontrjagin space if $C$ has a certain canonical partition $L = L_1 + L_2$ such that either $L_1$ or $L_2$ is of finite dimensional.

In this chapter we always denote by $L$ a Pontrjagin space with a canonical partition $L = L_1 + L_2$ whose $L_2$ is of dimension $n < +\infty$. The partition determines a certain finite unitary norm $\| \ \|$ which we call the *-norm of $L$. The operator $J$ which defines the *-norm of $L$ is represented as $J = J^+ - J^-$. $L_2$ is the range of $J^-$, and hence $\dim J =$ holds.

3.1.2. (Lemma). If $\mathfrak{M}$ is a #-selforthogonal linear subspace of $L$, then the dimension of $\mathfrak{M}$ is $\leqq n$.

Proof. Let $E$ be the *-projection in $L$ whose range is the closure $[\mathfrak{M}]$ of the range of $\mathfrak{M}$. $[\mathfrak{M}]$ is also #-selforthogonal. Then

$$(JEx|Ey) = \langle Ex, Ey\rangle = 0$$

and $EJE = 0$. Remarking $J = I - 2J^-$, we have

$$E = 2EJ^-E \quad \text{and} \quad \dim E \quad \dim J^- = \quad .$$

3.1.3. (Theorem). Every #-unitary operator in $L$ is continuous.

Proof. Let $U$ be a #-unitary operator, and take its Cartan decomposition $U = KV$. $V$ is biunitary, and continuous on $L$. Hence it is sufficient to see that $K$ is continuous on $L$. Consider the spectral decomposition $K = \int_0^{+} \lambda dE(\lambda)$ of $K$. Since $JKJ = K^{-1}$, we have $J\psi(K)J = \psi(K^{-1})$ for any bounded Borel function $\psi$ on $0 \leqq \lambda \leqq +\infty$. In particular, we consider the projections $E_1$, $E_2$, $E_3$ which are spectral measures on the sets $(0,1)$, $\{1\}$ and $(1,+\infty)$, respectively, since the mapping $\lambda \to \lambda^{-1}$ carries $(0,1)$ onto $(1,+\infty)$, we find that $JE_1J = E_3$, $JE_2J = E_2$, and the ranges of $E_1$ and $E_3$ are #-selforthogonal. By Lemma 3.1.2 we have $\dim E_3 = \dim E_1 < n$. Then the spectrum of $K$ consists of at most $2n+1$ points, and $K$ ~~and~~ become continuous in $L$ and we have the Theorem.

3.1.4. (Corollary). Every unitary norm of $L$ is finite.

Proof. Every unitary norm is induced from a certain *-positive #-unitary operator in $L$, and by Theorem 4.1.2 it is finite-valued on $L$.

3.1.5. (Theorem). Let $\mathscr{Q} = M_1+M_2$ be a canonical quasi-partition of $L$. Then it is a canonical partition of $L$ and $\dim M_2 = n$ holds.

Proof. The partition determines a unitary norm $P$ of $L$, and $\mathscr{Q}$ is identical to $\mathscr{Q}(K^2)$, where $K$ is a *-positive #-unitary operator which induces the norm $P$. Since $K$ is continuous, we have $L = \mathscr{Q}$ and $L= M_1+M_2$ is a partition of $L$. $M_2$ is the range of the operator $J_p$. Since $J_p^- = K^\# J^- K$, $\dim M_2 = n$ holds, and the Theorem is proved.

3.2. #-spectral and #-prespectral operators.

We now introduce operators in a Krein space with certain nice properties.

3.2.1. (Definition). An operator $A$ in a Krein space $L_0$ is called #-spectral if there is a continuous #-unitary operator $X$ and a biselfadjoint operator $B$ in $L_0$ satisfying $A = X^\# BX$.

3.2.2. We return again to the Pontrjagin space L. The discussion of spectral analysis of #-selfadjoint operator would finish if every #-selfadjoint operator in L would have been #-spectral, because such an operator A have the spectral representations $A = \int \lambda dE(\lambda)$ by continuous #-projections $\{E(\lambda) : -\infty < \lambda < +\infty\}$. But even in Minkowsky spaces, #-self-adjoint operators generally do not have such a wishful property.

3.2.3. So we introduce the concept of #-prespectrality, which is a property of operators in L slightly weaker than the #-spectrality. On the other hand it preserves the following two conditions.

3.2.4. Every #-selfadjoint operator in Minkowsky space is #-prespectral.

3.2.5. #-prespectrality of operators in L is preserved under the strong convergences in bounded nets of operators in L.

3.2.6. We start from modifying the representation of a #-spectral operator A in L. A is written as $A = X^{\#}CX$, by a (continuous) #-unitary operator X and a biselfadjoint operator C in L. Consider the Cartan decomposition $X = UK$, where U is biunitary and K is *-positive #-unitary in L. We set $B = U^{\#}CU$, and represent A again by $A = K^{\#}BK$. Then B is also biselfadjoint. We define a *-positive operator T in L by

$$T = K^2(I+K^2)^{-1}.$$

From $JKJ = K^{-1}$, we have $JTJ = I-T$, and

$$TA(I-T) = (TA(I-T))^*.$$

Indeed,

$$TA(I-T) = (I+K^2)^{-1}KBK(I+K^2)^{-1}.$$

Summarizing these properties of A we define :

3.2.7. (Definition). An operator A in L is called #-prespectral if A is #-selfadjoint and has a certain *-positive operator T in L satisfying $JTJ = I-T$, and

$$TA(I-T) = (TA(I-T))^*.$$

3.2.8. (Lemma). A continuous operator A in L is #-spectral iff A is #-selfadjoint, and satisfies the condition of Definition 3.2.7 whose operator T in L is injective.

Proof. Indeed, if there is such an injective T, we can take a *-positive operator K satisfying $T = K^2(I+K^2)^{-1}$. K is #-unitary, and $B: = KAK^{\#}$ is biselfadjoint in L. Then $A: = K^{\#}BK$ is #-spectral.

§3.3. Canonical represntations of #-selfadjoint operators.

Now we generalize the Jordan's canonical representation of matrices to the cases of #-selfadjoint operators in L.

3.3.1. (Theorem). Let $A$ be a continuous #-selfadjoint operator in a Pontrjagin space $L$ with $\dim L^{-} = n$. Then $A$ has the following matrix representation :

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix},$$

where $A_{11}^{*} = A_{33}$, $A_{23}^{*} = A_{12}$, $A_{13}^{*} = A_{13}$. The representation is determined by the reduction $L = L_1 + L_2 + L_3$ of $L$ by a certain #-selforthogonal closed subspace $L_1$ of $L$ satisfying $\dim L_1 = \dim L_3$ $n$, and $A_{22}$ becomes a #-spectral operator in $L_2$.

3.3.2. (Definition). The matrix representation of an operator $A$ in $L$ satisfying the requirement of Theorem 3.3.1 is called the canonical representation of $A$.

3.3.3. (Lemma). A continuous operator $A$ in $L$ is #-prespectral iff $A$ has the canonical representation.

Indeed, let $A$ be a continuous operator with the canonical representation. Then $A$ is #-selfadjoint. $A_{22}$ is #-spectral and has a representation $A_{22} = K_2^{\#} B_2 K_2$, where $B_2$ is biselfadjoint and $K$ is #-unitary, and further we can suppose that $K$ is *-positive in $L_2$. Let $I_K$ denote the identity in $L_K$, and define the operators $T_2$ in $L_2$ and $T$ in $L$ by

$$T_2 = K_2^2 (I_2 + K_2^2)^{-1}, \quad T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0^2 & I_3 \end{pmatrix}$$

Then clearly, $T$ is *-positive and satisfies $JTJ = I-T$ and $TA(I-T) = (TA(I-T))^*$. Hence $A$ is #-prespectral.

Conversely, suppose that $A$ is #-prespectral. Then there is a *-positive operator $T$ in $L$ such that $JTJ = I-T$ and $TA(I-T)$ is *-selfadjoint. Let $L_1$ be the kernel of $T$. Then $L_1$ is a closed subspace of $L$. $L_3 := JL_1$ is the kernel of $JTJ$ $(= I-T)$, and hence $L_1$ is *-orthogonal to $L_3$, and $L_1$ and $L_3$ are #-selforthogonal. Let $A := (A_{ij})$ be the matrix representation of $A$ defined by the reduction $L = L_1 + L_2 + L_3$. We show that $L_1$ is invariant under $A$. Indeed, if $x$ belongs to $L_1$, then $Tx = 0$, and hence $(I-T)x = x$ holds. Now

$$TAx = TA(I-T)x$$

$$= (TA(I-T))^*x = (I-T)A^*Tx = 0.$$

Therefore $Ax$ belongs to $L_1$, and $L_1$ is invariant under $A$. From this we find that $A_{21} = A_{31} = 0$, and $A_{12} = (A_{21})^\#$ $= 0$. $T$ has the matrix representation

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & T_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} .$$

$T_2$ is *-positive in $L_2$, and satisfies $J_2T_2J_2 = I_2 - T_2$, and $T_2A_{22}(I_2-T_2)$ is *-selfadjoint. Further, $T_2$ is injective on $L_2$. Then by Lemma 3.2. $A_{22}$ becomes #-spectral. Thus has the canonical representation.

## §3.4. Proof of the canonical representation theorem.

The assertion 3.2.4 is now equivalent to the following 3.4.1.

3.4.1. (Lemma). If $A$ is a #-selfadjoint operator in a Minkowsky space $L$, then $A$ has the canonical representation.

To prove 3.4.1, let $L_1$ be a maximal subspace of $L$ which is invariant under $A$, and consider the reduction $L = L_1+L_2+L_3$ of $L$ by the space $L_1$. Then $L$ has the representation

$$A = \begin{pmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{pmatrix} .$$

Suffice it to show that $A_{22}$ is #-spectral in $L_2$. Notice that $L_2$ does not contain any nontrivial #-selforthogonal subspace $\mathfrak{M}$ which is invariant under $A_{22}$. Indeed, if $\mathfrak{M}$ is such a space, then $L_2+\mathfrak{M}$ becomes a #-selforthogonal subspace of $L$ which is invariant under $A$.

Then 3.4.1 is verified by showing the next lemma.

3.4.2. (Lemma). If a #-selfadjoint operator $A$ in a Minkowsky space $L$ does not have any nontrivial invariant #-selforthogonal subspace, then $A$ is #-spectral.

Proof. Let $S$ be the set of eigenvalues of $A$. For each $\alpha \in S$ consider the operator

$$E(\alpha) := (2\pi i)^{-1}\int_C (A-z)^{-1}dz,$$

where $C$ is a small circular surrounding $\alpha$ to the positive direction. It is very well-known that $I = \Sigma_{\alpha \in S} E(\alpha)$, $E(\alpha)^2 = E(\alpha)$ and $E(\alpha)E(\beta) = 0$ for $\alpha \neq \beta$. The range of $E(\alpha)$ is the eigenspace of $A$ associated to the eigenvalue $\alpha$. Remarking that $A^{\#} = A$, if $\alpha$ belongs to $S$, then $\bar{\alpha}$ also belongs to $S$. $E(\alpha)$ commutes to $A$, and

$$E(\alpha)^{\#} = E(\bar{\alpha}) \neq 0.$$

$S$ is contained in the real line. Indeed, if $\alpha$ is a non-real element of $S$, then $E(\alpha)^{\#}E(\alpha) = 0$, and $E(\alpha)$ is #-selforthogonal. Then the range of $E(\alpha)$ is a nontrivial #-selforthogonal, invariant subspace of $A$, and it leads a contradiction. If $\alpha$ is real, then $AE(\alpha) = \alpha E(\alpha)$ holds. Indeed, let $n$ be the least number such that $(A-\alpha)^n E(\alpha) = 0$. If $n \geqq 2$, we can take a number $m$ such that $2m \geqq n$ and $m \leqq n-1$. The operator $(A-\alpha)^m E(\alpha)$ is $\neq 0$ and #-selforthogonal. Indeed,

$$((A-\alpha)^m E(\alpha))^{\#}((A-\alpha)^m E(\alpha))$$

$$= (A-\alpha)^{2m} E(\alpha) = 0.$$

This is a contradiciton.

The signs of $\langle x,x\rangle$ for elements $x \neq 0$ of the range of $E(\alpha)$ is constant, and we denote it by sign $E(\alpha)$.

Indeed, if there are elements $x \neq 0$ and $y \neq 0$ such that $\langle x,x\rangle \geqq 0$ and $\langle y,y) \leqq 0$, we can take a certain element $z = \alpha x+(1-\alpha)y \neq 0$ satisfying $\langle z,z\rangle = 0$, and $\{\alpha z : \alpha \in C\}$ is a nontrivial invariant subspace of $A$.

Now we set

$$E^{+}(A) = \Sigma_{\text{sign } E(\alpha)=+} E(\alpha),$$

$$E^{-}(A) = \Sigma_{\text{sign } E(\alpha)=-} E(\alpha).$$

and let $M_1$ and $M_2$ be the ranges of $E^{+}(A)$ and $E^{-}(A)$. Then $L = M_1+M_2$ is a canonical partition of $L$ and defines a certain finite unitary norm $p$ of $L$. Consider the innerproduct $(x|y)_p$ corresponding to $p$. Then $A$ is p-selfadjoint on $L$. Let $K$ be the *-positive #-unitary operator in $L$ which induces $p$. Then $K^{\#}AK$ is biselfadjoint, and $A$ is #-spectral in $L$. Thus 3.4.2 and hence 3.4.1 are verified.

We now show 3.2.5 as the next lemma 3.4.3.

3.4.3. (Lemma). If $\{A_\alpha\}$ is a bounded net of #-prespectral operators in $L$ which converges strongly to an operator $A$ in $L$, then $A$ is #-prespectral.

Proof. For each $A_\alpha$ we take a certain *-positive operator $T_\alpha$ satisfying $JT_\alpha J = I-T_\alpha$ and

$$T_\alpha A_\alpha(I-T_\alpha) = (T_\alpha A(I-T_\alpha))^{*}.$$

Then in the Hilbert space $L$ we have $0 \leqq T_\alpha \leqq I$, and we can take a certain subnet $\{T_\beta\}$ of $\{T_\alpha\}$ converging weakly to a certain operator $T$ in $L_1$. $T$ is also *-positive and satisfies $JTJ = I-T$.

We show that

$$TA(I-T) = (TA(I-T))^*.$$

The identity $T_\beta A_\beta(I-T_\beta) = (T_\beta A_\beta(I-T_\beta))^*$ is equivalent to

$$T_\beta A_\beta - A_\beta^* T_\beta = T_\beta(A_\beta - A_\beta^*)T_\beta,$$

Where $\{A_\beta\}$ is norm-bounded, and $T_\beta A_\beta - A_\beta^* T_\beta$ converges weakly to $TA-A^*T$. Since $A_\beta$ are #-selfadjoint we have

$$A_\beta^\# = JA_\beta J = (1-2J^-)A_\beta(1-2J^-),$$

and

$$A_\beta - A_\beta^* = 2J^-A_\beta + 2A_\beta J^- - 4J^-A_\beta J^-.$$

$J^-$ is of finite dimensional, and the right hand of the above identity converges uniformly to $A-A^*$.

Remark that

$$T_\beta(A_\beta - A^*)T_\beta = T_\beta(A-A^*)T_\beta + T_\beta R_\beta T_\beta,$$

where $|R_\beta| \to 0$. Then to see that $T_\beta(A_\beta - A_\beta^*)T_\beta$ converges weakly to $T(A-A^*)T$, it is sufficient to find that if $X$ is an operator of finite dimensional, $T_\beta X T_\beta$ converges weakly to $TXT$. This also turns to verify the case that $X$ is an operator of one-dimensional range, say $X = ab^*$, which is an operator in $L : ab^* : x \to (x|b)a$. Notice that

$$T_\beta ab^* T_\beta = (T_\beta a)(T_\beta b)^*.$$

Then $T_\beta ab^* T_\beta$ converges weakly to $Tab^*T$. Thus we obtain the identity

$$TA(I-T) = ((TA(I-T))^*,$$

and A is #-prespectral. The Lemma is proved.

3.4.4. (Proof of Theorem 3.3.1). The discussion of this section insures the validity of Theorem 3.3.1, but for completeness we summalize how our argument goes to this conclusion.
If L is a Minkowsky space, the Theorem is valid (3.4.1).
So, even if L is a general Pontrjagin space, every #-selfadjoint operator of finite rank has the canonical representation.

If A is any continuous #-selfadjoint operator in L, we can take a bounded net of #-selfadjoint operators of finite ranks converging strongly to L. Now operators in this net are #-prespectral (by Lemma 3.4.1). Then by Lemma 3.4.3 A is #-prespectral. Then by Lemma 3.4.1 A has the canonical representation.