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Kyoto University
On a problem of Sakai in unbounded derivations

H. TAKAI (都立大理)

as a quantization of spaces, especially $n$-dimensional real lines, Sakai [7] posed the following interesting problem: are there simple $C^*$-algebras $A$ and non-approximately bounded pregenerators of $A$ such that given a $*$-derivation $\delta$ of $A$ with $D(\delta) = \bigcap_{n=1}^{\infty} D(\delta_n)$, there exist $k_1, k_2 \in \mathbb{R}$ and an approximately bounded $*$-derivation $\delta_0$ of $A$ with the property that $\delta = \sum_{n=1}^{\infty} k_n \delta_n + \delta_0$.

In this note, we show that there is at least one model for the two-dimensional case. It is nothing but the irrational rotation algebra, namely the crossed product $C(T) \times_{\theta} \mathbb{Z}$ of the $C^*$-algebra $C(T)$ of all continuous functions on the one-dimensional torus $T$ by an irrational angle $\theta$. More precisely, we have the following:

Theorem 1. Let $A_0$ be the irrational rotation algebra. Then there exist two non-approximately bounded pregenerators $\delta_1, \delta_2$ of $A_0$ such that any
\(-\) derivation \(\delta\) of \(\mathfrak{A}\) with \(D(\delta) = D(\delta_1) \cap D(\delta_2)\) can be expressed as \(\delta = k_1 \delta_1 + k_2 \delta_2 + \delta_0\) for some \(k_1, k_2 \in \mathbb{R}\) and an approximately bounded \(-\) derivation \(\delta_0\) of \(\mathfrak{A}\).

Remark 1. Suppose \(D(\delta) = D(\delta_2) (\delta = 1 + 2)\), then one can show that \(\delta = k \delta_2 + \delta_0\) for some \(k \in \mathbb{R}\).

We now state our main theorem as follows:

**Theorem 2.** Let \((\mathfrak{A}, G, \alpha)\) be a \(C^*\)-dynamical system where \(\mathfrak{A}\) is unital abelian, \(G\) is discrete abelian, and \(\alpha\) is effective. Suppose \(\beta = \exp t \delta_0 (t \in T)\) commuting with \(\alpha\), and there exists an eigenunitary \(u\) for \(\beta\) which generates \(\mathfrak{A}\). Then for any \(-\) derivation \(\delta\) of \(\mathfrak{A} \times_\alpha G\) with \(D(\delta) = D(\delta_0) = D(\delta) \cap \mathfrak{A}\), there exist a \(k \in \mathbb{R}\), free generator \(\delta_1\) and an approximately bounded \(-\) derivation \(\delta_2\) of \(\mathfrak{A} \times_\alpha G\) such that

1. \(D(\delta_2) = D(\delta)\)
2. \(\delta_1|_\mathfrak{A} = 0\), \(\delta_1\) commutes with \(\delta_0\)
3. \(\delta = k \delta_2 + \delta_1 + \delta_0\)

where \(D(\delta) \cap \mathfrak{A}\) is the set of all \(D(\delta)\)-valued function of \(G\) with finite support, and \(\tilde{\delta}_0(x)(\tilde{\delta}) = \delta_0[x(\delta)] (x \in D(\delta) \cap \mathfrak{A})\).

Remark 2. If \(G = \mathbb{Z}\), \(\delta_1 = l \delta_1\) for some \(l \in \mathbb{R}\) where \(\delta_1\) is independent of \(\delta\).
Let \((R, G, \alpha)\) and \((R, H, \beta)\) be two C*-dynamical systems where \(\alpha, \beta\) commute. Then there is a C*-dynamical system \((R \times_\alpha G, H, \beta)\) such that 
\[\beta_a(x)(f) = \beta_a(x(f)) \quad (x \in L^1(G; R)).\] Then we have the following proposition of fixed point type:

**Proposition 3.** \((R \times_\alpha G)^\beta = R \times_\alpha G\)

**Proof.** By definition, \(R \times_\alpha G \subset (R \times_\alpha G)^\beta\). Suppose the inclusion is proper, then \((R \times_\alpha G)^\beta \times_\beta G \not\subset (R \times_\alpha G)^\beta \times_\beta G\) since \(\beta\) commutes with \(\hat{G}\).

Since \((R \times_\alpha G)^\beta \times_\beta G \subset ((R \times_\alpha G)^\beta \times_\beta G)^\beta\), it follows from duality \([G,G]\) that \(G^\beta \subseteq (L^1(G))^\beta\) which is a contradiction. Q.E.D.

**Comment 1.** We only consider locally compact abelian groups throughout this note.

In what follows, let \(S\) be a *-derivation of \(R \times G\) such that \(D(S) = D(\hat{G})\) where \(\hat{G}\) is a generator of \(Z\) commuting with \(\alpha\). Suppose \(S\) commutes with \(\hat{G}\), and \(G\) is discrete. Then \(S(a) \in R\) for \(a \in D(\hat{G})\). Let \((x_n)_n \subset D(S)\) with \(x_n \to 0\) \(\forall S(x_n) \to 0\). Let \(x \in R \times_\alpha G\). Since \(x = \sum_k q_k \lambda_k^{(n)} (q_k^{(n)} \in D(\hat{G}))\), using the conditional expectation \(\hat{G}\) of \(R \times G\) onto \(R\).
one has $E(x, \lambda(x)^*) \to 0$ and $E[x, (S(x) - y)\lambda(k)^*] \to 0$ for each $y$ in $G$. Thus $a_0^{(m)} \to 0$ and $E[\sum_k (S(a_0^{(m)})\lambda(k)^* + a_0^{(m)}\xi^*(x(k))\lambda(k)^* - \xi^*(y(x(k))))] \to 0$ where $y = \sum_k \xi(x(k))$ be the Fourier expansion of $y$ in $\mathbb{R} \times \mathbb{R}$ ($y \in \mathbb{R}$). Then $a_0^{(m)} \to 0$ and $S(a_0^{(m)}) \to Y_y$ for all $y$ in $G$. Since $D(D) = D(D)$, it follows from Batty's result [2] that $D(D)$ is closable. So $Y_y = 0$ for all $y \in G$. Consequently we have the following:

**Lemma 4.** If $G$ is discrete, any *-derivative $S$ of $\mathbb{R} \times \mathbb{R}$ such that $D(D) = D(D)$ and $S$ commutes with $\lambda$ is closable.

**Remark 3.** In the above lemma, the conclusion is unclear unless the condition (iv) is added.

Now let $S$ be a *-derivative of $\mathbb{R} \times \mathbb{R}$ with $D(D) = D(D)$. Define $S = \{ x \in D(D) \mid a \mapsto S(ax) \}$ is continuous from $D(D)$ into $\mathbb{R} \times \mathbb{R}$.

Since $S(\lambda(x)^*a) = S(\lambda(x)^*a + \lambda(x)^*S(\lambda(x)^*a) = S(\lambda(x)^*a)$ and $S$ commutes with $\lambda$, we have $x\lambda(y)^*a = 0$ for all $y \in G$ and $a \in S$ if $ax \in D(D) \to 0$ and $S(ax) \to x \in \mathbb{R} \times \mathbb{R}$. Then $E(x, \lambda(y)) = 0$ where $E$ is the projection of norm one from $\mathbb{R} \times \mathbb{R}$ onto $\mathbb{R}$. So $E(x, \lambda(y)) \in L(S)$, the left annihilator.
of $I$. Since $I$ is a two-sided ideal of $D(\omega)$, it follows from the same way as Longo [4] that $L(I) = 0$. Thus $E(\chi_\lambda(\omega)) = 0$ for all $\lambda \in \mathbb{G}$. Let $x = \sum x_\lambda \chi_\lambda(\omega)$ be the Fourier expansion of $x$. Then $x_\lambda = 0$. So $x = 0$. Therefore $d_x$ is closable from $(D(\omega), \| \cdot \|_{\mathcal{D}})$ into $\mathcal{A} \times \mathbb{G}$.

Therefore we have the following:

**Lemma 5.** Let $\delta$ be a $*$-derivation of $\mathcal{A} \times \mathbb{G}$ with $D(\omega) = D(\overline{\omega})$. Then $\delta$ is relatively bounded on $D(\omega)$ with respect to $D_0$, namely $\| \delta(a) \| = \| a \| + \| \delta(a) \|$ for all $a \in D(\omega)$, with some positive constant $K$.

**Remark 4.** Since $D_0$ is a preradical, one can not directly apply Longo's result. However, the crucial part of the above proof is due to his idea [4].

By the above lemma, let $\beta_\omega = \exp \delta_0$ ($\omega \in \mathbb{R}$). Then there exist derivations $\delta_\omega$ ($\omega \in L'(\mathbb{R})$) of $\mathcal{A} \times \mathbb{G}$ such that $\delta D(\delta_\omega) = D(\delta_\omega)$ and $\delta_\omega = \sum \int_{\mathbb{R}} f(t) \beta_\omega \circ \beta_\omega \circ dt$.

In fact, since $\| \delta(a) \| \leq M (\| a \| + \| \delta(a) \|)$ for $a \in D(\omega)$,

$\| \delta \circ \beta_\omega (a) - \delta \circ \beta_\omega (a) \| \leq M \{ \| \beta_\omega (a) - \beta_\omega (a) \| + \| \beta_\omega \circ \delta_\omega (a) - \beta_\omega \circ \delta_\omega (a) \| \}$.

So $t \mapsto \delta \circ \beta_\omega (x)$ is continuous for each $a \in D(\delta_\omega)$. Thus $t \mapsto \delta \circ \beta_\omega (x)$ is also continuous for $x \in D(\delta_\omega)$ which gives
derivations $\tilde{S}_f$ for $f \in L^1(\mathbb{R})$ of $\mathbb{R} \times G$ satisfying (1) and (2). Similarly, for each $g \in G$ one has a derivation $\hat{S}_g$ of $\mathbb{R} \times G$ such that (1) $\hat{S}_g(D(\hat{g})) = D(\hat{g})$ and (2) $\hat{S}_g = \int_{\mathbb{R}} \hat{g}(t) \hat{S}_t \circ \hat{g}$.

Moreover, suppose $f_0 = \exp \cdot S_0$ is periodic, then we have that $(\hat{S}_{f_0}^\infty) = (\hat{S}_0)_{\infty}$ commutes with $\hat{S}$, which follows from what we treat *-derivations of $\mathbb{R} \times G$ with the same domain as $D(\hat{g})$ commuting with $\hat{S}$ and $\hat{S}$, which are denoted by $\hat{S}$. Since it commutes with $\hat{S}$, it follows from Lemma 4 that it is closed. Hence one may assume that it is closed. Let $x \in C^*(G)$, and $(x_n) \subset D(D)$ which converges to $x$. But $x_n = \int_{\mathbb{R}} \hat{g}_n(x_n) \, dt \in \mathbb{R} \times G$.

Since $S$ commutes with $\hat{S}$ and $\hat{S}$ is closed, $\hat{x}_n \in D(D)$ and $(\mathbb{R} \times G)_{\hat{S}}$ and $\hat{x}_n \to x$ since $(\mathbb{R} \times G)_{\hat{S}} = C^*(G)$ by Proposition 1. So $\hat{S} \mid_{C^*(G)}$ is a closed *-derivative of $C^*(G)$ since $\hat{S}(x_n) \in C^*(G)$. Since $T_{\hat{g}} \circ \hat{S} \circ \hat{g}_n = \hat{g}_n$ for $g \in \hat{G}$ and $\hat{S} \circ \hat{g}_n \circ \hat{g}_n = T_{\hat{g}_n}$ on $C(\hat{G})$, $\hat{S} = T_{\hat{g}} \circ \hat{g}$ commutes with $T$ on $C(\hat{G})$ where $\hat{g}$ is the Fourier isomorphism of $C^*(G)$ onto $C(\hat{G})$, and $T$ is the shift action of $\hat{G}$ on $C(\hat{G})$. Let follows from Goodman-Nakagato [3, 5] that there exists a one parameter subgroup $(P_t)$ of $\hat{G}$ such that $\hat{S}(\hat{f})(P_t) = \lim_{t \to t} \hat{f}(T_t(P_t))$.
- $f(\mathcal{G})$ for all $f \in D(\mathcal{G})$. Since $\langle \mathcal{G}, \cdot \rangle \in D(\mathcal{G})$, one has $\mathcal{E}(\mathcal{G}(\xi)) = D(\mathcal{G}) \mathcal{G}(\xi)$ for all $\xi \in \mathcal{G}$ where $D(\mathcal{G}) = D(\mathcal{G}(\xi)) = L^{\infty}_{\mathcal{G}} \langle \mathcal{G}, \mathcal{P} \rangle - I$. Let $D(\lambda(\mathcal{G})) = D(\mathcal{G}) \lambda(\mathcal{G})$ for all $\lambda \in D(\mathcal{G})$ and $\mathcal{G} \in \mathcal{G}$. Then it is a pre-generator of $\alpha_{\mathcal{G}} \mathcal{G}$ such that $D(\mathcal{G}) = D(\mathcal{G}(\xi))$ and $D(\mathcal{G}) = 0$, $D$ commutes with $\mathcal{G}$. Since $D$ is a closed $\ast$-derivation of $\alpha_{\mathcal{G}} \mathcal{G}$ and $D_{\mathcal{G}}$ commutes with $\beta = \exp \tau \mathcal{G}$, it follows from Batty [1] that $D_{\mathcal{G}} = \lambda k \mathcal{G}$ for some $k \in \mathbb{R}$. Therefore we have that $\mathcal{E}(D(\mathcal{G})) = \mathcal{E}(\mathcal{G}(\xi)) = D(\mathcal{G}) \mathcal{G}(\xi) + \mathcal{G}(\xi) \mathcal{G}(\xi) = (\lambda k \mathcal{G} + \mathcal{G}) (D(\mathcal{G}))$, which implies the following lemma:

**Lemma 6.** Let $(\mathcal{G}, \mathcal{G}, \mathcal{G})$ be a $\ast$-dynamical system where $\mathcal{G}$ is unital abelian and $\mathcal{G}$ is discrete abelian. Let $\beta = \exp \tau \mathcal{G}$ be a periodic action of $\mathcal{G}$ on $\mathcal{G}$. Suppose $\beta$ is ergodic, then given a $\ast$-derivation $D$ of $\alpha_{\mathcal{G}} \mathcal{G}$ with the property that

1. $D(\mathcal{G}) = D(\mathcal{G}(\xi))$ and
2. $D$ commutes with $\mathcal{G}$,

there exist a $\lambda \in \mathbb{R}$ and a pre-generator $D_\xi$ of $\alpha_{\mathcal{G}} \mathcal{G}$ such that

1. $D(D_\xi) = D(\mathcal{G})$, $D_{\mathcal{G}} = 0$, $D$ commutes with $D_\xi$, and
2. $D = \lambda \mathcal{G} + D_\xi$ on $D(\mathcal{G})$.

**Remark 5.** The pre-generator $D_\xi$ defined above would

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be written as $\delta = \tau \delta'$ for some $\tau \in \mathbb{R}$ where $\delta'$ is not depending on $\delta$. Actually if $\mathcal{G} = \mathbb{Z}$, we have $\delta'(a \lambda(n)) = i \pi a \lambda(n)$ for $a \in D(\mathcal{G})$ and $n \in \mathbb{Z}$.

Let $\delta$ be a linear mapping from a $*$-subalgebra $D(\mathcal{G})$ of $\mathcal{A}$ into $\mathbb{R}$ such that $\delta(ab) = \delta(a) \delta(b) + \delta(b) \delta(a)$ for all $a, b \in D(\mathcal{G})$ where $\tau \in \mathcal{G}$ is a fixed element.

Suppose there is a unitary $u$ of $D(\mathcal{G})$ such that $1 \neq \text{sp}(\delta_u(u) u^*)$, then we have by direct computation that $\delta(u^n) = \sum_{k=0}^{n-1} \delta_u(u^k) u^{k+1} \delta_u(u) u^{n-k}$. Since $1 \neq \text{sp}(\delta_u(u) u^*)$, one has that $\sum_{k=0}^{n-1} \delta_u(u^k) u^{k+1} = (\delta_u(u^n) u^{n-k} - 1) (\delta_u(u) u^{k+1} - 1)$.

So $\delta(u^n) = \delta(u) u^n (\delta_u(u) u^n - 1) (\delta_u(u) - u^{-1}) (\delta_u(u) - u)$ for all $n \in \mathbb{Z}$ since $\delta(1) = 0$. Put $\delta_u = \delta(u) (\delta_u(u) - u)^{-1} \in \mathcal{A}$. Since $\delta_u (\delta_u - \text{id})$ is bounded on $\mathcal{A}$, the conclusion follows. Namely, we have the following:

**Lemma 7.** Suppose $\mathcal{A}$ is unital abelian and $\mathcal{G}$ is discrete. Let $\delta$ be a linear mapping of a $*$-subalgebra $D(\mathcal{G})$ of $\mathcal{A}$ into $\mathbb{R}$ such that $\delta(ab) = \delta(a) \delta(b) + \delta(b) \delta(a)$ for $a, b \in D(\mathcal{G})$ for a fixed $\tau \in \mathcal{G}$.

Suppose there exists a unitary $u \in D(\mathcal{G})$ such that $1 \neq \text{sp}(\delta_u(u) u^*)$, then $\delta = \delta_u (\delta_u - \text{id})$ on $D(\mathcal{G}) \cap C^*(\mathcal{A})$. 

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for some $a_\phi \in \mathcal{A}$.

**Remark 6.** By the above lemma, there is no unbounded $\phi$-cocycle closed $\ast$-derivation of $\mathcal{A}_\phi$ with an eigenunitary generating $\mathcal{K}$.

Now let $\hat{S}_\phi (\phi \in \Phi)$ be a derivation of $\mathcal{A}_\Phi \Phi$ as in the previous case (following to Remark 4). Then it implies that $\hat{S} = \Sigma_\phi \hat{S}_\phi$ on $D(\Phi)$. In fact, let $\hat{S}(x) = \Sigma_\phi \hat{S}(x)(\phi) \lambda(\phi)$ and $\hat{S}(\lambda(\phi)) = \Sigma_\phi \hat{S}(\lambda(\phi))(\phi) \lambda(\phi)$ be the Fourier expansion of $\hat{S}(x)$ and $\hat{S}(\lambda(\phi))$ respectively. Then $\hat{S}_\phi (x) = \hat{S}(x)(\phi) \lambda(\phi)$ and $\hat{S}_\phi (\lambda(\phi)) = \hat{S}(\lambda(\phi))(\phi) \lambda(\phi)$.

Suppose $\hat{S}$ commutes with $\check{\Phi}$, it follows from Lemma 6 that $\hat{S}_\phi = k \hat{S}_\phi + 1$ on $D(\phi)$ where $k, d$ are as in Lemma 6. Let $\hat{S}_\phi (x) = \hat{S}_\phi (x) \lambda(\phi) \ast$ for $x \in D(\phi)$ ($\phi \in \Phi$).

Then $\hat{S}_\phi$ satisfy the condition of Lemma 7. Suppose there exists a unitary $u \in D(\phi)$ such that
1) $1 \ast \hat{S}_\phi (x u) = \hat{S}_\phi (x u \ast)$ for $x \in D(\phi)$ ($\phi \in \Phi$).
2) $\mathcal{A} = C^*(u)$. Since $\hat{S}$ commutes with $\check{\Phi}$, and $\phi$ commutes with $\check{\Phi} = \exp t \phi$, which is ergodic, we have $a_\phi \in \mathcal{A}_1$. Then $\hat{S}_\phi (x) = a_\phi (a_\phi - id)(x) \lambda(\phi) = [a_\phi \lambda(\phi), a_\phi \lambda(\phi)]$. Hence $\hat{S}_\phi (a \lambda(\phi)) = \hat{S}_\phi (a \lambda(\phi)) + a \hat{S}_\phi (\lambda(\phi)) = [a_\phi \lambda(\phi), a \lambda(\phi)] + a \hat{S}_\phi (\lambda(\phi))$. Since $\hat{S}_\phi - ad (a_\phi \lambda(\phi))$ is a derivation on $D(\phi)$, one has $\hat{S}_\phi (\lambda(\phi)) = 0$ for $\phi \in \Phi$. 

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In fact, since \( \delta_{2}(\lambda(k)) = \delta(\lambda(k))(h+k) \lambda(h+k) \), we have that \( \delta(\lambda(k+h))(h+k+\xi) U = \delta(\lambda(k))(h+\xi) a_{g}(U) + \delta(\lambda(k))(h+\xi) U \) for all \( k, h \in G \). Since \( 1 \in D(\delta) \), we have \( \delta(1)(U) = 0 \).

So \( \delta(\lambda(k))(h+\xi) = 0 \) for all \( k \in G \) or \( a_{g}(U) = U \). Since

\( 1 \notin \text{sp}(a_{g}(U)N) \), we have \( \delta(\lambda(k))(h+\xi) = 0 \) for all \( k \in G \).

Consequently \( \delta = \delta_{0} + \delta_{1} + \sum_{g \in G} \text{ad}(a_{g}(\lambda(h))) \) on \( D(\delta) \). Let \( \delta_{0} = \text{ad}(\sum_{g \in G_{\lambda}} a_{g}(\lambda(h))) \) for a finite set \( \lambda \) of \( G - \{ e \} \) with \( \lambda^{-1} = -\lambda \). Then \( \delta_{1} \) is bounded \( \ast \)-derivations of \( A \times_{\lambda} G \) such that \( \delta_{1}(\lambda(h)) = 0 \) and \( \delta_{1} \) converges to \( \delta_{2} \) pointwisely on \( D(\delta) \) where \( \delta_{2}(a\lambda(h)) = \sum_{g \in G} [ag(\lambda(h)), a\lambda(h)] = (\delta_{2}([a\lambda(h)]) \).

Then \( \delta = \delta_{0} + \delta_{1} + \delta_{2} \) on \( D(\delta) \) and \( \delta_{2}(\lambda(h)) = 0 \) for all \( g \in G \), which implies the following proposition:

Proposition 8. Let \((A, G, \alpha)\) be a \( C^{*}\) dynamical system where \( A \) is unital abelian and \( G \) is discrete. Let \( \beta = \exp(t\alpha) \) be an ergodic action of \( T \) on \( A \) commuting with \( \alpha \). Suppose there exists a unitary \( U \in D(\delta) \) such that \( \delta(1 \ast \text{sp}(a_{g}(U)U^{*})(g \ast e), (ii) \ R = C^{*}(U) \), then given a \( \ast \)-derivation \( \delta \) of \( A \times_{\lambda} G \) such that \( (i) \ D(\delta) = D(\delta_{0}) \) and \( (ii) \delta \) commutes with \( \beta \), there exists a \( k \in R \), a generator \( \delta_{0} \), and an approximately bounded \( \ast \)-derivation \( \delta_{2} \) of \( A \times_{\lambda} G \) such that \( (i) \ D(\delta_{2}) = D(\delta_{0}), \delta_{2}|_{R} = 0, \delta \) commutes with \( \delta_{0} \).
\( D(\mathbb{G}) = D(\mathbb{B}), \hat{\xi}(\lambda(\mathbb{G})) = 0 \) for all \( \mathbb{B} \in \mathbb{G} \), and \( \mathfrak{m} = \mathfrak{a} + \mathfrak{b} + \mathfrak{c} \).

**Remark 7.** In the case of discrete abelian groups, the Fourier expansion of any element of \( \mathbb{R} \times \mathbb{G} \) can be taken in the uniform sense. In fact, taking a net \( \{ x_i \} \) of positive definite functions on \( \mathbb{G} \) with finite support converging to 1, one can show that \( \Sigma_x \hat{f}(x) \hat{a}(\lambda(\mathbb{G})) \) converges to \( \Sigma_x a(x) \lambda(\mathbb{G}) \in \mathbb{R} \times \mathbb{G} \) uniformly.

**Proof of Theorem 2:** Since \( \beta \) commutes with \( \mathfrak{a} \) and \( \beta \) is ergodic, we have \( \beta(x) = e^{i\theta} x \). Since \( \mathbb{G} = C^*(\mathbb{B}) \) and \( \alpha(x) = e^{i\theta} G \), it follows from Proposition 8 that \( \hat{\xi}_e = \mathfrak{a} + \mathfrak{b} + \mathfrak{c} \) where \( \mathfrak{a}, \mathfrak{b}, \mathfrak{c} \) are as in Proposition 8. Since \( \beta \), \( \hat{\xi}_e \), and \( \hat{\xi}_c \) are all \( \mathbb{B} \)-valued, we have that \( \hat{\xi}_c(\lambda(\mathbb{G})) = e^{i\tau(x)} \hat{\xi}_c(\lambda(\mathbb{G})) \). Since \( \beta(x) = e^{i\theta} x \), we have that \( \mathbb{B} = \mathbb{B} \), \( \hat{\xi}_c(\lambda(\mathbb{G})) = \mathbb{B} \), \( \mathbb{B} \), \( \hat{\xi}_c(\lambda(\mathbb{G})) = \mathbb{B} \), \( \mathbb{B} \), \( \hat{\xi}_c(\lambda(\mathbb{G})) = \mathbb{B} \), \( \mathbb{B} \), \( \hat{\xi}_c(\lambda(\mathbb{G})) = \mathbb{B} \), \( \mathbb{B} \), \( \hat{\xi}_c(\lambda(\mathbb{G})) = \mathbb{B} \), \( \mathbb{B} \), \( \hat{\xi}_c(\lambda(\mathbb{G})) = \mathbb{B} \), \( \mathbb{B} \). So there are \( \lambda(x) \in \mathbb{C} \) such that \( \hat{\xi}_c(\lambda(\mathbb{G})) = \mathbb{B} \). Let \( \delta(\lambda(\mathbb{G})) = \Sigma_x \delta(\lambda(\mathbb{G})) \lambda(x) \) and \( \delta(\lambda(\mathbb{G})) = \Sigma_x \delta(\lambda(\mathbb{G})) \lambda(x) \) be the Fourier expansion of \( \delta(\lambda(\mathbb{G})) \) and \( \delta(\lambda(\mathbb{G})) \) respectively.
Since \( a_t = c_t(u) \) and \( b_t(u) = e^{it} u \), we have that \( \delta(\lambda(t)) \delta u = a_0 + \sum_{n=0} a_n u^n \), where \( a_0 \) is the \( 0 \)-component of the expansion of \( \delta(\lambda(t)) \delta u \) in \( a_t \). Since \( \delta_y \lambda(t) = \delta (\lambda(t)) \lambda(t) \). By unicity, \( \int a_t b_t (\delta (\lambda(t)) \delta u) \, dt = \delta (\lambda(0)) (\delta (\lambda(t)) \lambda(t)) = 0 \) (otherwise), which is nothing but \( a_0 \).

Therefore we deduce that \( \delta (\lambda(t)) = \delta (\lambda(t)) \lambda(t) + \sum_{n=0} \sum_{m=0} b_m (\lambda(t)) \lambda(t) x \).

Moreover, \( \delta (a) = \sum_{n=0} b_n (\lambda(t)) \lambda(t) \) for all \( \lambda \in D(a) \). It follows from Lemma 7 that \( \delta_y (a) = \delta_y (a_0 - \delta t) \lambda(t) \lambda(t) + \sum_{n=0} \sum_{m=0} b_m (\lambda(t)) \lambda(t) \lambda(t) \).

Since \( \delta_y \) commutes with \( \lambda(t) \), we have \( \delta_y (a) \in D(a) \) for all \( a \in D(a) \). Since \( \delta_y \) commutes with \( \delta \) and \( \delta^* \), it means that \( \delta_y \) is a \(*\)-derivation, we deduce that \( \delta_y (\lambda(t)) = \lambda(t) \lambda(t) \lambda(t) + \sum_{n=0} a_n u^n \).

Therefore, \( \delta_y (a) = \lambda(t) \lambda(t) \lambda(t) \lambda(t) \lambda(t) \). Since \( \delta_y \) is a \(*\)-derivation, we deduce that \( \delta_y (\lambda(t)) = \lambda(t) \lambda(t) \lambda(t) \lambda(t) \lambda(t) \).

Hence, \( \delta_y (\lambda(t)) \lambda(t) = \lambda(t) \lambda(t) \lambda(t) \lambda(t) \lambda(t) \).

Consequently, we have that \( \delta (\lambda(t)) \lambda(t) = \delta (\lambda(t)) \lambda(t) + \sum_{n=0} a_n u^n \).

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\[ + \sum_{m=0}^{n} a_m n^m u^{n+m} \lambda(t) \]. Since \( -A \delta_n - \delta_n \) is a \( * \)-derivation, so is \( \sum_{m=0}^{n} \left[ \delta_n \lambda(t), u^m \lambda(t) \right] + \sum_{m=0}^{n} a_m n^m u^{n+m} \lambda(t) \). Since \( ad(\delta_n \lambda(t)) (u^m \lambda(t)) + u^m ad(\delta_n \lambda(t)) (\lambda(t)) = ad(\delta_n \lambda(t)) (u^m \lambda(t)) \), we deduce that \( u^m \left( \sum_{m=0}^{n} \left[ \delta_n \lambda(t), \lambda(t) \right] \right) + \sum_{m=0}^{n} a_m n^m u^{n+m} \lambda(t) \) is a \( * \)-derivation. Let \( a = \sum_{m=0}^{n} a_m u^m \in \mathbb{R} \). Conventionally put \( \sigma(\lambda(t)) = \sum_{m=0}^{n} \left[ \delta_n \lambda(t), \lambda(t) \right] \). Moreover, put \( \Delta(u^m \lambda(t)) = u^m \sigma(\lambda(t)) + na u^m \lambda(t) \). Since \( \delta_n(u^m) = i_n u^m \), we see \( na u^m \lambda(t) = (-\sigma) a \delta_n(u^m \lambda(t)) \). Now since \( \Delta(u^m \lambda(t)) u^m \lambda(t) \) = \( \Delta(u^m \lambda(t)) u^m \lambda(t) + u^m \lambda(t) \Delta(u^m \lambda(t)) \), we can show that \( u^{n+m}(\sigma(\lambda(t)) \lambda(t) - \lambda(t) \sigma(\lambda(t))) = m (a_0(a) - A) u^{n+m} \lambda(t) \). Put \( A = \sigma \) and \( m = 1 \). Then we have \( u^m \sigma(\lambda(t)) = (a_0(a) - A) u^m \lambda(t) \) for all \( n \in \mathbb{N} \) and \( g \in G \). Therefore \( \Delta(u^m \lambda(t)) = (a_0(a) - A) u^m \lambda(t) \) + \( na u^m \lambda(t) = (a_0(a) + (n-1)A) u^m \lambda(t) \). Since \( \Delta \) is a derivation, we get \( a_0(a) = A \) for all \( g \in G \). So \( a = \mathbb{R} \) for some \( a \in \mathbb{C} \). Then \( \Delta(u^m \lambda(t)) = 2\pi u^m \lambda(t) = i \pi \delta_n(u^m \lambda(t)) \). Finally, we obtain that \( \delta(u^m \lambda(t)) = (c \delta_n + \delta_n) (u^m \lambda(t)) + \sum_{m=0}^{n} \left[ \delta_n \lambda(t), u^m \lambda(t) \right] \) for some \( c \in \mathbb{R} \). Let \( \delta_n(a \lambda(t)) = \sum_{n \in \mathbb{N}} \left[ \delta_n \lambda(t), a \lambda(t) \right] \) for \( a \in D(\delta) \) and \( g \in G \), where \( D(\delta) \) is a finite set of \( G \) with \( \mathbb{N} = \mathbb{N} \). Then \( \delta_n \) is a bounded \( * \)-derivative of \( \mathbb{R} \times G \) for all \( \mathbb{N} \) and \( \delta_n \to \delta \) pointwise. Hence \( \delta \) is approximately bounded. This completes the proof.
References


