On a problem of Sakai in unbounded derivations

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as a quantization of spaces, especially n-dimensional real lines, Sakai [7] posed the following interesting problem: are there simple C*-algebras A and a family \( \{ S_i \}_{i=1}^{\infty} \) of non approximately bounded pregenerators of A such that given a *-derivation \( \delta \) of A with \( D(\delta) = \bigcap_{n=1}^{\infty} D(S_n) \), there exist \( k_1, k_2 \in \mathbb{R} \) and an approximately bounded *-derivation \( \delta_0 \) of A with the property that \( \delta = \sum_{n=1}^{\infty} k_n S_n + \delta_0 \).

In this note, we show that there is at least one model for two-dimensional case. It is nothing but the irrational rotation algebra, namely the \( C^* \)-crossed product \( C(T) \times_\theta \mathbb{Z} \) of the C*-algebra \( C(T) \) of all continuous functions on the one-dimensional torus \( T \) by an irrational angle \( \theta \). More precisely, we have the following:

**Theorem 1.** Let \( A_0 \) be the irrational rotation algebra. Then there exist two non approximately bounded pregenerators \( S_1, S_2 \) of \( A_0 \) such that any
*-derivation \( \delta \) of \( B_0 \) with \( D(\delta) = D(\delta_1) \cap D(\delta_2) \) can be expressed as \( \delta = k_1 \delta_1 + k_2 \delta_2 + \delta_0 \) for some \( k_1, k_2 \in \mathbb{R} \) and an approximately bounded *-derivation \( \delta_0 \) of \( B_0 \).

**Remark 1.** Suppose \( D(\delta) = D(\delta_1) \) \( (i=1 \text{ or } 2) \), then one can show that \( \delta = k \delta_1 + \delta_0 \) for some \( k \in \mathbb{R} \).

We now state our main theorem as follows:

**Theorem 2.** Let \( (\mathcal{A}, G, \alpha) \) be a C*-dynamical system where \( \mathcal{A} \) is unital abelian, \( G \) is discrete abelian, and \( \alpha \) is effective. Suppose \( \beta = \exp t \delta_0 \) \( (t \in \mathbb{T}) \) commuting with \( \alpha \), and there exists an eigenunitary \( \nu \) for \( \beta \) which generates \( \mathcal{A} \). Then for any *-derivation \( \delta \) of \( \mathcal{A} \times_\alpha G \) with \( D(\delta) = D(\delta_0) = D(\delta_1) \otimes B \) there exist a \( k \in \mathbb{R} \), free-generator \( \delta_1 \) and an approximately bounded *-derivation \( \delta_2 \) of \( \mathcal{A} \times_\alpha G \) such that

1) \( D(\delta_2) = D(\delta) \) \( (2.1.2) \)
2) \( \delta|_\mathcal{A} = 0 \), \( \delta_1 \) commutes with \( \delta_0 \)
3) \( \delta = k \delta_1 + \delta_0 + \delta_2 \),

where \( D(\delta) \otimes B \) is the set of all \( D(\delta) \)-valued function of \( G \) with finite supports, and \( \delta_0(x)(g) = \delta_0(x)(g) \) \( (x \in D(\delta) \otimes B) \).

**Remark 2.** If \( G = \mathbb{Z} \), \( \delta_1 = l \delta_1' \) for some \( l \in \mathbb{R} \) where \( \delta_1' \) is independent of \( \delta \).
Let \((\mathbb{R}, G, \alpha)\) and \((\mathbb{R}, H, \beta)\) be two C\(^*\)-dynamical systems where \(\alpha, \beta\) commute. Then there is a C\(^*\)-dynamical system \((\mathbb{R} \times G, H, \hat{\beta})\) such that \(\beta_\alpha(x)(\xi) = \beta_\alpha(x) \xi\) (\(\xi \in L(G; \mathbb{R})\)). Then we have the following proposition of fixed point type:

**Proposition 3.** \((\mathbb{R} \times G)^{\hat{\beta}} = \mathbb{R}^g \times_{\alpha} G\)

**Proof.** By definition, \(\mathbb{R}^g \times_{\alpha} G \subset (\mathbb{R} \times G)^{\hat{\beta}}\).

Suppose the inclusion is proper, then \((\mathbb{R} \times G)^{\hat{\beta}} \times_{\alpha} G \not\subset (\mathbb{R} \times G)^{\hat{\beta}} \times_{\alpha} G\) since \(\beta\) commutes with \(\alpha\).

Since \((\mathbb{R} \times G)^{\hat{\beta}} \times_{\alpha} G \subset (\mathbb{R} \times G)^{\hat{\beta}} \times_{\alpha} G\), it follows from duality \[\psi\] that \(\alpha^g \psi \in (\mathbb{R} \times G)^{\hat{\beta}} \times_{\alpha} G\), which is a contradiction. Q.E.D.

**Comment.** We only consider locally compact abelian groups throughout this note.

In what follows, let \(D\) be a \(*\)-derivation of \(\mathbb{R} \times G\) such that \(D(\delta) = D(\delta)\) where \(\delta\) is a generator of \(G\) commuting with \(\alpha\). Suppose \(\delta\) commutes with \(\hat{\alpha}\), and \(G\) is discrete. Then \(D(\alpha) = C\) for \(a \in D(\delta)\). Let \((x_n) \subset D(\delta)\) with \(x_n \to 0\) if \(D(x_n) \to \gamma \in \mathbb{R} \times G\). Since \(x_n = \sum_k \alpha_k^m \lambda_k\) \((\alpha_k^m \in D(\delta))\), using the conditional expectation \(\mathbb{E}\) of \(\mathbb{R} \times G\) onto \(\mathbb{R}\).
one has \( E(x, \lambda(\gamma)*) \to 0 \) and \( E[\langle \delta(x, z) - y, \lambda(\gamma) \rangle] \to 0 \)

for each \( \varphi \) in \( \mathcal{F} \). Thus \( a^m \to 0 \) and \( E[\sum_k (\delta(a^m_k) \lambda(k) + a^m_k \delta(\lambda(k)) \lambda(\gamma)^* \to \gamma_k \lambda(\eta) \lambda(k) \rangle] \to 0 \) where \( \gamma = \sum_k \gamma_k \lambda(k) \) be the Fourier expansion of \( \gamma \) in \( \mathbb{R} \times \mathcal{F} \) (where \( \mathcal{F} \)). Then \( a^m \to 0 \) and \( \delta(a^m_k) \to \gamma_k \) for all \( k \) in \( \mathcal{F} \). Since \( \mathcal{D}(\delta) = \mathcal{D}(\delta) \), it follows from Batty's result \( [2] \) that \( \mathcal{D}(\delta) \) is closable. So \( \gamma_k = 0 \) for all \( k \in \mathcal{F} \). Consequently we have the following:

**Lemma 4.** If \( \mathcal{F} \) is discrete, any \( * \)-derivative \( \delta \) of \( \mathbb{R} \times \mathcal{F} \) such that \( \delta(\delta) = \mathcal{D}(\delta) \) and \( \delta \) commutes with \( \delta \) is closable.

**Remark 3.** In the above lemma, the conclusion is unclear unless the condition \( (\Omega) \) is added.

Now let \( \delta \) be a \( * \)-derivative of \( \mathbb{R} \times \mathcal{F} \) with \( \mathcal{D}(\delta) = \mathcal{D}(\delta) \). Define \( \sigma = \{ x \in \mathcal{D}(\delta) : \alpha \to \delta(\alpha x) \} \) is continuous from \( \mathcal{D}(\delta) \) into \( \mathbb{R} \times \mathcal{F} \}. \) Since \( \delta(\alpha \lambda(\varphi)) = \delta((\lambda(\varphi) \alpha) x) + \lambda(\varphi) \delta(\alpha(\lambda(\varphi) x)) \) and \( \delta \) commutes with \( \delta \), we have \( x \lambda(\varphi) = 0 \) for all \( \varphi \in \mathcal{F} \) and \( x \in \sigma \) if \( \alpha x \in \mathcal{D}(\delta) \to 0 \) and \( \delta(\alpha x) \to x \in \mathbb{R} \times \mathcal{F} \}. \) Then \( \mathcal{E}(x \lambda(\varphi)) = 0 \) where \( \mathcal{E} \) is the projection of norm one from \( \mathbb{R} \times \mathcal{F} \) onto \( \mathbb{R} \). So \( \mathcal{E}(x \lambda(\varphi)) \in \mathcal{L}(\delta) \), the left annihilator.
of $I$. Since $I$ is a two-sided ideal of $D(b)$, it follows from the same way as Longo [4] that $L(I) = 0$. Thus $E(x \lambda(t)) = 0$ for all $f \in G$. Let $x = \sum x_t \lambda(t)$ be the Fourier expansion of $x$. Then $x_t = 0$. So $x = 0$. Then $S$ is closable from $(D(b), \|\cdot\|_S)$ into $A \times G$. Therefore we have the following:

**Lemma 5.** Let $S$ be a $*$-derivation of $A \times G$ with $D(b) = D(b)$. Then $S$ is relatively bounded on $D(b)$ with respect to $D_0$, namely \( \|S(a)\| = K(\|a\| + \|S(a)\|) \) for all $a \in D(b)$, with some positive constant $K$.

**Remark 4.** Since $D_0$ is a pre-generator, one can not directly apply Longo's result. However, the crucial part of the above proof is due to his idea [4].

By the above lemma, let $F_t = \exp t D_0 (t \in \mathbb{R})$. Then there exist derivations $F_g (g \in L^1(G))$ of $A \times G$ such that \( D(F_g) = D(D_0) \) and \( F_g = \int_G f(t) D_0 \delta * F_0 \).

In fact, since \( \|S(a)\| \leq M (\|a\| + \|S(a)\|) \) for $a \in D(b)$, \( \|S \circ \beta (a) - S \circ \beta (a)\| \leq M \{ \|\beta (a) - \beta (a)\| + \|\beta (a) - \beta (a)\| \} \).

So $t \mapsto S \circ \beta (a)$ is continuous for each $a \in D(b)$. Thus $t \mapsto S \circ \beta (x)$ is also continuous for $x \in D(b)$ which gives
derivations $\hat{\mathcal{F}}_\delta$ for $\delta \in \mathcal{D}(\mathbb{R})$ of $\mathbb{R} \times G$ satisfying (i) and (ii). Similarly, for each $g \in G$ one has a derivation $\hat{\mathcal{F}}_\delta$ of $\mathbb{R} \times G$ such that $\mathcal{D}(\hat{\mathcal{F}}_\delta) = \mathcal{D}(\hat{\mathcal{F}}_g)$ and (i) $\hat{\mathcal{F}}_\delta = \int_G \hat{\mathcal{F}}_g \, dp \cdot \hat{\mathcal{F}}_g \cdot \hat{\mathcal{F}}_g \, dp$.

Moreover, suppose $\hat{\mathcal{F}}_g = \exp(\mathcal{F}_\delta)$ is periodic, then we have that $(\hat{\mathcal{F}}_g)^* = (\hat{\mathcal{F}}_{g})^*$ commutes with $\hat{\mathcal{F}}_\delta$, which follows from Lemma 4 that it is closable. Hence one may assume that it is closed. Let $x \in C^*(G)$, and $(x_t) \in \mathcal{D}^*(\mathcal{F})$ which converge to $x$. But $\mathcal{F}_x = \int_G \hat{x}(t) \, dp \in \mathbb{R} \times G$.

Since $\mathcal{F}$ commutes with $\hat{\mathcal{F}}_\delta$ and $\mathcal{F}$ is closed, $\mathcal{F}_x \in \mathcal{D}^*(\mathcal{F})$ and $(\mathbb{R} \times G)^* \mathcal{F}$ and $\mathcal{F}_x \rightarrow x$ since $(\mathbb{R} \times G)^* \mathcal{F} = C^*(G)$ by Proposition 1. So $\delta|_{C^*(G)}$ is a closed $*$-derivation of $C^*(G)$ since $\delta(\mathcal{F}_x) \in C^*(G)$.

Since $\hat{\mathcal{F}}_g \cdot \hat{\mathcal{F}}_\delta \cdot \hat{\mathcal{F}}_g = \delta$ for $p \in \hat{G}$ and $\mathcal{F} \cdot \hat{\mathcal{F}}_\delta \cdot \mathcal{F} = \hat{\mathcal{F}}$ on $C(\hat{G})$, $\hat{\mathcal{F}} = \mathcal{F} \cdot \mathcal{F} \cdot \mathcal{F}$ commutes with $T$ on $C(\hat{G})$ where $\mathcal{F}$ is the Fourier isomorphism of $C^*(G)$ onto $C(\hat{G})$, and $T$ is the shift action of $\hat{G}$ on $C(\hat{G})$. It follows from Goodman-Nakagato [3, 5] that there exists a one parameter subgroup $(\hat{P}_\phi)$ of $\hat{G}$ such that $\hat{\mathcal{F}}(\phi) = \lim_{\delta \to 0} \hat{\mathcal{F}}(\mathcal{F}(\phi))$.
\[ f(\lambda f) \text{ for all } f \in D(\mathcal{D}). \] Since \( \langle f, \cdot \rangle \in D(\mathcal{D}) \), one has \( \sigma(\lambda f) = \mathcal{D}(\lambda f) \) for all \( f \in \mathcal{G} \) where \( \mathcal{D}(\lambda f) = \lim_{t \to 0} \langle f, Pe^{-t} \rangle - 1 \). Let \( \mathcal{D}(a \lambda f) = \mathcal{D}(\lambda f) a \lambda f \) for all \( a \in D(\mathcal{D}) \) and \( f \in \mathcal{G} \). Then it is a pregenerator of \( \mathcal{A} \times \mathcal{G} \) such that \( \mathcal{D}(\mathcal{D}) = \mathcal{D}(\mathcal{D}) \) and \( \mathcal{D}/\mathcal{G} = 0 \), \( \mathcal{D} \) commutes with \( \mathcal{D} \). Since \( \mathcal{D} \) is a closed \(*\)-derivation of \( \mathcal{A} \times \mathcal{G} \) and \( \mathcal{D} \) commutes with \( \mathcal{D} = \text{exp}\, t\mathcal{D} \), it follows from Batty [1] that \( \mathcal{D}/\mathcal{G} = k\mathcal{G} \) for some \( k \in \mathbb{R} \). Therefore we have that \( \mathcal{D}(a \lambda f) = k \mathcal{D}(a \lambda f) + a \mathcal{D}(\lambda f) \frac{d}{dt}(a \lambda f) = (k\mathcal{G} + \mathcal{D})(a \lambda f) \), which implies the following lemma:

**Lemma 6.** Let \( (\mathcal{A}, \mathcal{G}, \lambda) \) be a \(*\)-dynamical system where \( \mathcal{A} \) is unital abelian and \( \mathcal{G} \) is discrete abelian. Let \( \mathcal{D} = \text{exp}\, t\mathcal{D} \) be a periodic action of \( \mathcal{A} \) on \( \mathcal{G} \). Suppose \( \mathcal{D} \) is ergodic, then given a \(*\)-derivation \( \mathcal{D} \) of \( \mathcal{A} \times \mathcal{G} \) with the property that \( \mathcal{D}(\mathcal{D}) = \mathcal{D}(\mathcal{D}) \) and \( \mathcal{D} \) commutes with \( \mathcal{D}, \mathcal{G} \), there exist a \( k \in \mathbb{R} \) and a pregenerator \( \mathcal{D} \) of \( \mathcal{A} \times \mathcal{G} \) such that

1. \( \mathcal{D}(\mathcal{D}) = \mathcal{D}(\mathcal{D}), \mathcal{D}/\mathcal{G} = 0 \), \( \mathcal{D} \) commutes with \( \mathcal{D} \), and
2. \( \mathcal{D} = k\mathcal{G} + \mathcal{D} \) on \( D(\mathcal{D}) \).

**Remark 5.** The pregenerator \( \mathcal{D} \) defined above would
be written as \( \delta = r \delta' \) for some \( r \in \mathbb{R} \) where \( \delta' \) is not depending on \( \delta \). Actually if \( G = \mathbb{Z} \), we have \( \delta'(a \lambda(n)) = i n a \lambda(n) \) for \( a \in D(\delta) \) and \( n \in \mathbb{Z} \).

Let \( \delta \) be a linear mapping from a \( * \)-subalgebra \( D(\delta) \) of \( A \) into \( A \) such that \( \delta(ab) = \delta(a)\delta(b) + \delta(a) \) for all \( a, b \in D(\delta) \) where \( \delta \in \mathbb{R} \) is a fixed element.

Suppose there is a unitary \( u \) of \( D(\delta) \) such that \( \delta = 1 + \delta(u)u^* \), then we have by direct computation that
\[
\delta(u^n) = \sum_{k=0}^{n+1} \delta(u)^k u^k \delta(u) u^{n-k}.
\]
Since \( 1 + \Delta(u)(u^*) \), one has that
\[
\sum_{k=0}^{n+1} \delta(u)^k u^k = (\delta(u)u^n - 1)(\delta(u)u^n - 1)\delta(u).
\]
So \( \delta(u^n) = \delta(u)u^n(\delta(u)u^n - 1)\delta(\text{id})(u^n) = \delta(u)(\delta(u) - u)(\delta(u) - u)\delta(u) \) for all \( n \in \mathbb{Z} \) since \( \delta(1) = 0 \). But
\[
\delta = \delta(u)(\delta(u) - u)\delta(\text{id}) \in A.
\]
Since \( \delta(\delta - \text{id}) \) is bounded on \( A \), the conclusion follows. Namely we have the following:

**Lemma 7.** Suppose \( \mathbb{R} \) is unital abelian and \( G \) is discrete. Let \( \delta \) be a linear mapping of a \( * \)-subalgebra \( D(\delta) \) of \( A \) into \( A \) such that \( \delta(ab) = \delta(a)\delta(b) + \delta(a) \) for all \( a, b \in D(\delta) \) for a fixed \( \delta \in \mathbb{R} \).

Suppose there exists a unitary \( u \in D(\delta) \) such that \( 1 + \delta(u)u^* \), then \( \delta = \delta(u)(\delta - \text{id}) \) on \( D(\delta) \cap C^*(\mathbb{R}) \).
for some $a_3 \in A$.

**Remark 6.** By the above lemma, there is no unbounded $\gamma$-cocycle closed $\ast$-derivation of $a_3$ has an eigenunitary generating $\Omega$.

Now let $\hat{S}_\gamma (\gamma \in \Gamma)$ be a derivation of $A \times \mathbb{F}$ as in the previous way (following to Remark 4). Then it implies that $\hat{S} = \sum_\gamma \hat{S}_\gamma$ on $D(\gamma)$. In fact, let $S(\lambda)(\gamma) = \sum_\gamma S(\lambda)(\gamma) \cdot \lambda(\gamma) \cdot \gamma(\gamma)$ and $S(\lambda)(\gamma) = \sum_\gamma S(\lambda)(\gamma) \cdot \gamma(\gamma)$ be the Fourier expansion of $S(\lambda)$ and $S(\lambda)(\gamma)$ respectively.

Then $\hat{S}_\gamma (\lambda)(\gamma) = \sum_\gamma S(\lambda)(\gamma) \cdot \gamma(\gamma)$ and $\hat{S}_\gamma (\lambda)(\gamma) = \sum_\gamma S(\lambda)(\gamma) \cdot \gamma(\gamma)$.

**Suppose** $S$ commutes with $\beta$, it follows from Lemma 6 that $\hat{S}_\gamma = \lambda_\gamma + \hat{S}_\gamma$ on $D(\gamma)$ where $\lambda, \gamma$ are as in Lemma 6. Let $\hat{S}_\gamma (\lambda)(\gamma) = \sum_\gamma S(\lambda)(\gamma) \cdot \gamma(\gamma)$ for $\gamma \in D(\gamma) (\gamma \in \Gamma)$.

Then $S$ satisfies the condition of Lemma 7. Suppose there exists a unitary $u \in D(\gamma)$ such that (a) $1 \ast S(\gamma(u)u^*) = (\gamma \in \Gamma)$ and (a) $\Omega = C^* (u)$. Since $S$ commutes with $\beta$, and $\gamma$ commutes with $\beta = e^{\gamma}$, which is ergodic, we have $a_3 \in A_1$. Then $\hat{S}_\gamma (\lambda)(\gamma) = \sum_\gamma S(\lambda)(\gamma) \cdot \gamma(\gamma)$ is $[a_3 \lambda(\gamma), \gamma(\gamma)]$. Hence $\hat{S}_\gamma (\lambda)(\gamma) = \hat{S}_\gamma (\lambda)(\gamma) + a_3 \lambda(\gamma)(\gamma) = [a_3 \lambda(\gamma), \gamma(\gamma)] + a_3 \lambda(\gamma)(\gamma)$. Since $\hat{S}_\gamma - \text{ad} (a_3 \lambda(\gamma))$ is a derivation on $D(\gamma)$, one has $\hat{S}_\gamma (\lambda)(\gamma) = 0$ for $\gamma \in \Gamma$.
In fact, since $\delta^*(\lambda(k)) = \delta^*(\lambda(h))(h+k)\lambda(h+k)$, we have that $\delta^*(\lambda(h+k))(h+k)\lambda(h+k)U = \delta^*(\lambda(h))(h+k)U_2 U_2 + \delta^*(\lambda(h))(h+k)U$ for all $h, k \in \mathbb{Q}$. Since $1 \in D(\delta)$, we have $\delta^*(\lambda(h))U = 0.

So $\delta^*(\lambda(h))(h+k)U = 0$ for all $h \in \mathbb{Q}$ or $U_2 U_2 = U$. Since $1 \neq sp(\lambda(h)(h+k))$, we have $\delta^*(\lambda(h))(h+k) = 0$ for all $h \in \mathbb{Q}$. Consequently $\delta = k\delta_0 + \delta_1 + \sum_{\gamma \in \mathbb{F}} ad(\lambda(\gamma))$ on $D(\delta)$. Let $\delta_\mathbb{F} = ad(\sum_{\gamma \in \mathbb{F}} \lambda(\gamma))$ for a finite set $\mathbb{F}$ of $\mathbb{Q}$-rational with $\mathbb{F} = -\mathbb{F}$. Then $\delta_\mathbb{F}$ are bounded $*$-derivations of $\mathfrak{A} \times \mathbb{Q}$ such that $\delta_\mathbb{F}(\lambda(h)) = 0$ and $\delta_\mathbb{F}$ converges to $\delta_2$ pointwisely on $D(\delta)$ where $\delta_2(\lambda(h)) = \sum_{\gamma \in \mathbb{F}} \lambda(\gamma)$, $\lambda(h) = \sum_{\gamma \in \mathbb{F}} \lambda(\gamma)$. Then $\delta = k\delta_0 + \delta_1 + \delta_2$ on $D(\delta)$ and $\delta_2(\lambda(h)) = 0$ for all $h \in \mathbb{Q}$, which implies the following proposition:

**Proposition 8.** Let $(\mathfrak{A}, G, \alpha)$ be a $C^*$-dynamical system where $\mathfrak{A}$ is unital abelian and $G$ is discrete. Let $\beta = \exp t\alpha$ be an ergodic action of $T$ on $\mathfrak{A}$ commuting with $\alpha$. Suppose there exists a unitary $U \in D(\delta)$ such that (i) $1 \neq sp(\lambda(\alpha)(U^2 U^*) (\beta U_* U^*))$, (ii) $\mathfrak{A} = C^*(U)$, then given a $*$-derivation $\delta$ of $\mathfrak{A} \times \mathbb{Q}$ such that (i) $D(\delta) = D(\delta_0)$ and (ii) $\delta$ commutes with $\beta$, there exist $A \in \mathfrak{A}$, a generator $\delta_\mathbb{F}$, and an approximately bounded $*$-derivation $\delta_2$ of $\mathfrak{A} \times \mathbb{Q}$ such that (i) $D(\delta_2) = D(\delta)$, $\delta_2|_{\mathfrak{A}} = 0$, $\delta_2$ commutes with $\delta_0$,
1) \( D(\beta) = D(\delta_1), \delta_1(\lambda(\phi)) = 0 \) for all \( \phi \in \Phi \), and \( \Pi \delta = \delta_1 + \delta_2 \).

**Remark 7.** In the case of discrete abelian groups, the Fourier expansion of any element of \( AX_\Phi \) can be taken in the uniform sense. In fact, taking a net \( \{ \phi_i \} \) of positive definite functions on \( \Phi \) with finite support converging to 1, one can show that \( \Sigma \phi_i(\phi) \lambda(\phi) \) converges to \( \Sigma \phi \lambda(\phi) \in AX_\Phi \) uniformly.

**Proof of Theorem 2:** Since \( \beta \) commutes with \( \delta \) and \( \beta \) is ergodic, we have \( \phi_\beta(u)u^* \in C^1 \). Since \( AX = C^1(u) \) and \( \alpha \) is effective, there are \( Q^1 + 1 \) (Prop. C) such that \( q(\phi) = Q^1 u \). Let \( 1 \neq \phi_\beta(u) = \phi(u) \) (Prop. C). Let \( \delta_0 = \int_t e^{int} \delta t \) for \( t \in D(\beta) \) for \( n \in \mathbb{Z} \). Since \( \delta_0 \) commutes with \( \beta \), it follows from Proposition 8 that \( \delta_0 = \delta_1 + \delta_2 \delta_1 \) where \( \delta_1 \) is as in Proposition 8. Since \( \delta_0 \delta_1 + \delta_2 = e^{int} \delta_0 (n \in \mathbb{Z}) \), \( \beta_\delta \delta_0 (\lambda(\phi)) = e^{int} \delta_0 (\lambda(\phi)) \). Since \( \beta_\delta (u^*) = e^{int} u^* \), we have that \( u^* \delta_0 (\lambda(\phi)) \in (AX_\Phi \hat{\phi} = C^1(\hat{\phi}) \) so there are \( \delta_\lambda \delta_0 (\lambda(\phi)) \in C^1(\Phi) \) such that \( \delta_0 (\lambda(\phi)) = u^* \beta_\delta (\lambda(\phi)) \). Let \( \delta(\lambda(\phi)) = \sum_\phi \delta(\lambda(\phi)) (\phi) \lambda(\phi) \) and \( \delta(\lambda(\phi)) = \sum_\phi \delta(\lambda(\phi)) (\phi) \lambda(\phi) \) be the Fourier expansion of \( \delta(\lambda(\phi)) \) and \( \delta(\lambda(\phi)) \) respectively.
Since \( \mathcal{A} = C^*(\mathcal{A}) \) and \( \beta_\varepsilon (\mathcal{A}) = e^{\varepsilon \mathcal{A}} \), we have that \( \delta (\lambda (\beta))(\mathcal{A}) = \alpha (\mathcal{A}) + \sum_{n \neq 0} \delta (n, \beta)(\mathcal{A}) \mathcal{A}^n \) where \( \alpha (\mathcal{A}) \) is the 0-component of the expansion of \( \delta (\lambda (\beta))(\mathcal{A}) \) in \( \mathcal{A} \). Since \( \delta \beta_{\varepsilon} = \alpha \delta_{\varepsilon} + \delta_{\varepsilon} \delta \), one has \( \delta \beta_{\varepsilon} (\lambda (\beta)) = \delta (\lambda (\beta)) \lambda (\beta) \). By uniqueness, \( \int \beta \delta \beta_{\varepsilon} (\lambda (\beta))(\mathcal{A}) \, dt = \lambda (\beta) \lambda (\beta) \) (\( \mathcal{A} \neq \{0\} \), = 0 (otherwise), which is nothing but \( \alpha (\mathcal{A}) \).

Therefore we deduce that \( \delta (\lambda (\beta)) = \delta (\lambda (\beta)) \lambda (\beta) + \sum_{n \neq 0} \delta (n, \beta)(\mathcal{A}) \mathcal{A}^n \lambda (\beta) = \delta (\lambda (\beta)) \lambda (\beta) + \sum_{n \neq 0} \delta (n, \beta)(\mathcal{A}) \mathcal{A}^n \).

Moreover \( \delta (\varepsilon) = \delta \beta_{\varepsilon} \) for all \( \varepsilon \in D(\mathcal{A}) \). It follows from Lemma 7 that \( \beta_{\varepsilon} \lambda (\beta) = \beta_{\varepsilon} (\delta_{\varepsilon} - \mathcal{A}) \lambda (\beta) \) for some \( \beta_{\varepsilon} \in \mathcal{A} \). So \( \beta_{\varepsilon} (\lambda (\beta)) = [\beta_{\varepsilon} \lambda (\beta), \lambda (\beta)] \) for all \( \varepsilon \in D(\mathcal{A}) \).

Since \( \delta \beta_{\varepsilon} \) commutes with \( \mathcal{A} \), we have \( \delta \beta_{\varepsilon} (\varepsilon) \in \mathcal{A} \) for all \( \varepsilon \in D(\mathcal{A}) \). Since \( \delta \beta_{\varepsilon} \) commutes with \( \mathcal{A} \) and \( \beta \), it means that \( \delta \beta_{\varepsilon} \) = \( \delta \beta_{\varepsilon} \delta_{\varepsilon} + \delta_{\varepsilon} \delta \), where \( \delta_{\varepsilon} \) are as in Lemma 6.

Then \( \int e^{\varepsilon \mathcal{A}} \beta_{\varepsilon} \delta \beta_{\varepsilon} (\mathcal{A}) \, dt = \delta \beta_{\varepsilon} (\mathcal{A}) \). Since \( \beta_{\varepsilon} (\mathcal{A}) = e^{\varepsilon \mathcal{A}} \mathcal{A} \), we have \( \delta (\varepsilon) = \delta \mathcal{A} \). Let \( \delta \beta_{\varepsilon} (\mathcal{A}) = \sum a_n \mathcal{A}^n \mathcal{A} \). Then \( a_1 = \delta \mathcal{A} \). Therefore \( \delta \beta_{\varepsilon} (\mathcal{A}) = \delta \mathcal{A} \mathcal{A} + \sum a_n \mathcal{A}^n \). Since \( \delta \beta_{\varepsilon} \) is a *-derivation, we deduce that \( \delta \beta_{\varepsilon} (\mathcal{A}) = \mathcal{A} \delta \beta_{\varepsilon} (\mathcal{A}) + \sum a_n \mathcal{A}^n \mathcal{A} \).

Hence \( \delta \beta_{\varepsilon} (\mathcal{A}) \lambda (\beta) = \delta \beta_{\varepsilon} (\mathcal{A}) \lambda (\beta) + \sum a_n \mathcal{A}^n \mathcal{A} \lambda (\beta) \).

Consequently, we have that \( \delta (\mathcal{A} \lambda (\beta)) = \delta (\mathcal{A} \lambda (\beta)) + \mathcal{A} \mathcal{A} \lambda (\beta) \).

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+ \sum_{m=0}^{n} n \alpha_{m+1} u^{n-m} \lambda(\beta).

Since \delta = \delta_0 - \delta_1 is a \ast-derivation,
so is \sum_{m=0}^{n} \delta_0 [\delta_0 \lambda(\beta)] + \sum_{m=0}^{n} \delta_1 [\delta_1 \lambda(\beta)] + \sum_{m=0}^{n} n \alpha_{m+1} u^{n-m} \lambda(\beta).

Since \text{ad} (\delta_0 \lambda(\beta)) (u^n \lambda(\beta)) + u^n \text{ad} (\delta_0 \lambda(\beta)) (\lambda(\beta)) = \text{ad} (\delta_0 \lambda(\beta)) (u^n \lambda(\beta)),
we deduce that \( u^n (\sum_{m=0}^{n} \delta_0 [\delta_0 \lambda(\beta)]) - \sum_{m=0}^{n} [\delta_0 \lambda(\beta), \lambda(\beta)] + \sum_{m=0}^{n} n \alpha_{m+1} u^{n-m} \lambda(\beta) \)
\( \lambda(\beta) \) is a \ast-derivation. Let \( a = \sum_{m=0}^{n} a_{m+1} u^m \in \mathfrak{g}. \) Conventionally put \( \sigma(\lambda(\beta)) = \sum_{m=0}^{n} \delta_0 [\delta_0 \lambda(\beta)] - \sum_{m=0}^{n} [\delta_0 \lambda(\beta), \lambda(\beta)] \).

Moreover, put \( \Delta (u^n \lambda(\beta)) = u^n \sigma(\lambda(\beta)) + n a u^n \lambda(\beta). \)
Since \( \delta_0 (u^n) = i^n u, \) we see \( n a u^n \lambda(\beta) = (-1)^n a \delta_0 (u^n \lambda(\beta)). \) Now since \( \Delta (u^n \lambda(\beta)) = u^n \lambda(\beta) + u^n \lambda(\beta) \Delta (u^n \lambda(\beta)), \) we can show that \( u^n (\sigma(\lambda(\beta)) \lambda(\beta) - \lambda(\beta) \sigma(\lambda(\beta))) = n (\delta_0 (u^n) - a) u^{n-1} \lambda(\beta). \) Put \( a = 0 \)
and \( n = 1. \) Then we have \( u^n \sigma(\lambda(\beta)) = (\delta_0 (u^n) - a) u^{n-1} \lambda(\beta) for all \( n \in \mathbb{N} \)
and \( a \in \mathfrak{g}. \) Therefore \( \Delta (u^n \lambda(\beta)) = (\delta_0 (a) - a) u^n \lambda(\beta) + n a u^n \lambda(\beta) = (\delta_0 (a) + (n-1) a) u^n \lambda(\beta). \)
Since \( \Delta \) is a derivation,
we get \( \delta_0 (a) = a \) for all \( a \in \mathfrak{g}. \) So \( \lambda = \mathfrak{z} \Gamma \) for some \( \beta \in \mathfrak{g}. \) Then \( \Delta (u^n \lambda(\beta)) = a u^n u^n \lambda(\beta) = a \delta_0 (u^n \lambda(\beta)). \) Finally,
we obtain that \( \delta (u^n \lambda(\beta)) = (c \delta_0 + \delta_1) (u^n \lambda(\beta)) + \sum_{m=0}^{n} [\delta_0 \lambda(\beta), \delta_1 \lambda(\beta)] \) for some \( c \in \mathbb{R}. \)
Let \( \delta_0 (\lambda(\beta)) = \sum_{a \in \mathcal{F}} [\delta_0 \lambda(\beta), a \lambda(\beta)] \) for \( a \in \mathcal{D}(\beta) \) and \( \beta \in \mathfrak{g} \) where \( \mathcal{F} \) is a finite set of \( \mathfrak{g} - \{0 \} \) with \( \mathcal{F} = - \mathcal{F}. \) Then \( \delta_0 \) is a bounded \ast-derivative
of \( \mathfrak{g}, \mathfrak{g} \) for all \( \mathcal{F} \) and \( \delta_0 \to \delta \) pointwisely. Hence \( \delta \) is approximately bounded. This completes the proof.
References


