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<td>引用</td>
<td>数理解析研究所講究録 398: 31-44</td>
</tr>
<tr>
<td>発行日</td>
<td>1980-10</td>
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<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/105057">http://hdl.handle.net/2433/105057</a></td>
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<tr>
<td>タイプ</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>版本</td>
<td>publisher</td>
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<tr>
<td>発行機関</td>
<td>Kyoto University</td>
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On a problem of Sakai in unbounded derivations

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As a quantization of spaces, especially n-dimensional real lines, Sakai [7] posed the following interesting problem: are there simple $C^*$-algebras $\mathfrak{A}$ and two families of non-approximately bounded pregenerators of $\mathfrak{A}$ such that given a $*$-derivation $\mathfrak{D}$ of $\mathfrak{A}$ with $\mathfrak{D}(\mathfrak{D}) = \sum_{n=1}^{\infty} D(D_n)$, there exist $k_1, k_2 \in \mathbb{R}$ and an approximately bounded $*$-derivation $\mathfrak{D}_0$ of $\mathfrak{A}$ with the property that $\mathfrak{D} = \sum_{n=1}^{\infty} k_n \mathfrak{D}_0 + \mathfrak{D}_0$.

In this note, we show that there is at least one model for two-dimensional case. It is nothing but the irrational rotation algebra, namely the $C^*$-crossed product $C(T) \times_\theta \mathbb{Z}$ of the $C^*$-algebra $C(T)$ of all continuous functions on the one-dimensional torus $T$ by an irrational angle $\theta$. More precisely, we have the following:

Theorem 1. Let $\mathfrak{A}_0$ be the irrational rotation algebra. Then there exist two non-approximately bounded pregenerators $\mathfrak{D}_1, \mathfrak{D}_2$ of $\mathfrak{A}_0$ such that any
\textit{*-derivation} $\delta$ of $\mathcal{A}_0$ with $D(\delta) = D(\delta_1) \cap D(\delta_2)$ can be expressed as $\delta = k\delta_1 + k_2\delta_2 + \delta_0$ for some $k, k_2 \in \mathbb{R}$ and an approximately bounded \textit{*-derivation} $\delta_0$ of $\mathcal{A}_0$.

\textbf{Remark 1.} Suppose $D(\delta) = D(\delta_1)$ (i = 1, 2), then one can show that $\delta = k\delta_1 + \delta_0$ for some $k \in \mathbb{R}$.

We now state our main theorem as follows:

\textbf{Theorem 2.} Let $(\mathcal{A}, G, \alpha)$ be a C\textsuperscript{*}-dynamical system where $\mathcal{A}$ is unital abelian, $G$ is discrete abelian, and $\alpha$ is effective. Suppose $\beta = \exp t\delta_0$ ($t \in \mathbb{T}$) commuting with $\alpha$, and there exists an eigenunitary $u$ for $\beta$ which generates $\mathcal{A}_0$. Then for any \textit{*-derivation} $\delta$ of $\mathcal{A} \times_\alpha G$ with $D(\delta) = D(\delta_0) = D(\delta_1) \cap \mathcal{O}_0 \hat{G}$ there exist a $k \in \mathbb{R}$, free-generator $\delta_1$ and an approximately bounded \textit{*-derivation} $\delta_2$ of $\mathcal{A} \times_\alpha G$ such that

1) $D(\delta_2) = D(\delta_1)$ \hspace{1cm} (2.1)

2) $\delta_j|_{\mathcal{A}_0} = 0$, $\delta_j$ commutes with $\delta_0$, \hspace{1cm} (2.2) $\delta = k\delta_1 + \delta_2 + \delta_3$,

where $D(\delta_1) \cap \mathcal{O}_0 \hat{G}$ is the set of all $D(\delta_0)$-valued function of $G$ with finite support, and $\delta_0(x)(g) = \delta_0[x(g)] \hspace{1cm} (x \in D(\delta_1) \cap \mathcal{O}_0 \hat{G})$.

\textbf{Remark 2.} If $G = \mathbb{Z}$, $\delta_1 = h\alpha$, for some $h \in \mathbb{R}$ where $\delta_1$ is independent of $\delta$. 

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Let \((\mathcal{A}, G, \alpha)\) and \((\mathcal{A}, H, \beta)\) be two C*-dynamical systems where \(\alpha, \beta\) commute. Then there is a C*-dynamical system \((\mathcal{A}\times G, H, \beta)\) such that 
\[
\beta_\b(x)(\gamma) = \beta_\b(x(\gamma)) \quad (\gamma \in \mathcal{L}(G; \mathcal{A})).
\]
Then we have the following proposition of fixed point type:

**Proposition 3.** \((\mathcal{A}\times G)^\b = \mathcal{A} \times_\alpha G\)

**Proof.** By definition, \((\mathcal{A}\times G)^\b \subseteq (\mathcal{A}\times G)^\b\).

Suppose the inclusion is proper, then \((\mathcal{A}\times G)^\b \times_\alpha G \nsubseteq (\mathcal{A}\times G)^\b \times_\alpha G\) since \(\beta\) commutes with \(\alpha\).

Since \((\mathcal{A}\times G)^\b \times_\alpha G \subseteq ((\mathcal{A}\times G)^\b \times_\alpha G)^\b\), it follows from duality \([8,9]\) that \(\mathcal{A} \triangleleft \mathcal{L}(G; \mathcal{A}) \subseteq (\mathcal{A} \triangleleft \mathcal{L}(G; \mathcal{A}))^{\text{op}}\),

which is a contradiction. Q.E.D.

**Comment 1.** We only consider locally compact abelian groups throughout this note.

In what follows, let \(\delta\) be a *-derivation of \(\mathcal{A}\times G\) such that \(\delta(D) = D(\delta_0)\) where \(\delta_0\) is a generator of \(\mathcal{A}\) commuting with \(\alpha\). Suppose \(\delta\) commutes with \(\alpha\), and \(G\) is discrete. Then \(\delta(\alpha) \in D(\delta_0)\) for \(a \in D(\delta_0)\). Let \((x_n)_n \subseteq D(\delta)\) with \(x_n \to 0\), \(\delta(x_n) \to \gamma \in \mathcal{A}\times G\). Since \(x_n = \sum_k q_k(n)\chi_k\) (\(q_k \in D(\delta_0)\)), using the conditional expectation \(\mathbb{E}\) of \(\mathcal{A}\times G\) onto \(\mathcal{A}\).
one has $E(\lambda^i k^* \gamma^i) \to 0$ and $E(\delta(x, y) \lambda(x^i) \gamma^i) \to 0$ for each $\gamma$ in $G$. Thus $a_{\gamma}^{(m)} \to 0$ and $E(\xi_{x, y} \delta(a_{\gamma}^{(m)}) \lambda(x^i) \gamma^i) \to 0$ where $\xi = \xi_{x, y} \lambda(x^i) \gamma^i$ be the Fourier expansion of $\gamma$ in $\mathbb{R} \times G$. Then $a_{\gamma}^{(m)} \to 0$ and $\delta(a_{\gamma}^{(m)}) \to y_\gamma$ for all $\gamma$ in $G$. Since $B(\delta|_{\mathbb{R}}) = B(y_\gamma)$, it follows from Batty's result [2] that $\delta|_{\mathbb{R}}$ is closable. So $y_\gamma = 0$ for all $\gamma$ in $G$. Consequently we have the following:

**Lemma 4.** If $G$ is discrete, any $*$-derivative $\delta$ of $\mathbb{R} \times G$ such that 0 $B(\delta) = B(\delta_0)$ and 0 $\delta$ commutes with $\hat{a}$ is closable.

**Remark 3.** In the above lemma, the conclusion is unclear unless the condition (R) is added.

Now let $\delta$ be a $*$-derivative of $\mathbb{R} \times G$ with $B(\delta) = B(\delta_0)$. Define $\mathcal{S} = \{x \in B(\delta_0) | a \to \delta(ax)\}$ is continuous from $B(\delta) \to \mathbb{R} \times G$. Since $\delta(\lambda(x^i) \gamma^i a_{\lambda^i}^{(m)} + \lambda(x^i) \delta(a_{\lambda^i}^{(m)}) a_{\lambda^i}^{(m)})$ and $\delta$ commutes with $\hat{a}$, we have $x \lambda(x^i) a_{\lambda^i}^{(m)} = 0$ for all $x \in G$ and $a \in \mathcal{S}$ if $a_{\lambda^i}^{(m)} \to 0$ and $\delta(ax) \to x \in \mathbb{R} \times G$. Then $E(x \lambda(x^i)) \to 0$ where $E$ is the projection of norm one from $\mathbb{R} \times G$ onto $\mathbb{R}$. So $E(x \lambda(x^i)) \in L(\mathcal{S})$, the left annihilator.
of $I$. Since $I$ is a two-sided ideal of $D(D)$, it follows from the same way as [4, 4] that $L(I) = 0$. Thus $E(x \lambda(g)) = 0$ for all $g \in G$. Let $X = \sum \lambda(g) X_g$ be the Fourier expansion of $X$. Then $X_g = 0$. So $X = 0$. Then $\delta \pi$ is closable from $(D(D), \| \cdot \|_D)$ into $AX(G)$.

Therefore we have the following:

**Lemma 5.** Let $\delta$ be a $*$-derivation of $AX(G)$ with $D(D) = D(D)$. Then $\delta$ is relatively bounded on $D(D)$ with respect to $\delta_0$, namely $\| \delta(a) \| = K (\| a \| + \| \delta_0(a) \|)$ for all $a \in D(D)$, with some positive constant $K$.

**Remark 4.** Since $\delta_0$ is a pregenerator, one can not directly apply [4, 4]'s result. However the crucial part of the above proof is due to his idea [4, 4].

By the above lemma, let $\beta_0 = \exp \delta_0 \cdot (t \in \mathbb{R})$. Then there exist derivations $\beta_0^\delta (\in \mathcal{L}(\mathcal{L}(R)))$ of $AX(G)$ such that $\delta \beta_0 = \delta (\bar{\beta})$ and (i) $\beta_0^\delta = \int_0^\infty H(t) \beta_0^\delta \circ \beta_0^\delta \ dt$ and (ii) $\beta_0^\delta = \int_0^\infty H(t) \beta_0^\delta \circ \beta_0^\delta \ dt$.

In fact, since $\| \delta(a) \| = M (\| a \| + \| \delta_0(a) \|)$ for $a \in D(D)$, $\| \delta \beta_0 (a) - \delta \beta_0 (a) \| = M \{ \| \beta_0 (a) - \beta_0 (a) \| + \| \beta_0 \circ \delta_0 (a) - \beta_0 \circ \delta_0 (a) \| \}$. So $t \mapsto \delta \beta_0 (a)$ is continuous for each $a \in D(D)$. Thus $t \mapsto \delta \beta_0 (X)$ is also continuous for $X = D(D)$ which gives
derivations $\mathcal{D}_f$ for $f \in L^1(\mathbb{R})$ of $\mathcal{R}_xG$ satisfying (i) and (ii). Similarly, for each $g \in G$ one has a derivation $\mathcal{D}_g$ of $\mathcal{R}_xG$ such that $\mathcal{D}(g) = \mathcal{D}(g)$ and (ii) $\mathcal{D}_g = \int G \langle \phi, \rho \rangle \hat{\omega}_g \circ \hat{\rho} \circ \hat{\omega}_g \hat{d}p$. Moreover, suppose $g \in \exp \cdot S_0$ is periodic, then we have that $\mathcal{D}_g = (\hat{g})_1$ commutes with $\hat{\omega}$, $\hat{\rho}$. In what follows we treat $*-$derivations of $\mathcal{R}_xG$ with the same domain as $\mathcal{D}(g)$ commuting with $\hat{\omega}$ and $\hat{\rho}$, which are denoted by $\mathcal{D}$. Since it commutes with $\hat{\omega}$, it follows from Lemma 4 that it is closable. Hence one may assume that it is closed. Let $x \in C^*(G)$, and $(x_n) \in \mathcal{D}(\mathcal{D})$ which converge to $x$. But $y_n = \int F (x_n) d\tau \in \mathcal{R}_xG$. Since $\mathcal{D}$ commutes with $\hat{\rho}$ and $\mathcal{D}$ is closed, $y_n \to x$ since $(\mathcal{R}_xG)^\mathcal{D} = C^*(G)$ by Proposition 1. So $\mathcal{D}|_{C^*(G)}$ is a closed $*-$derivative of $C^*(G)$ since $\mathcal{D}(x_n) \in C^*(G)$. Since $\hat{\omega}_p \circ \hat{\omega}_p = \mathcal{D}$ for $p \in \mathcal{G}$ and $\mathcal{D} \circ \hat{\omega}_p \circ \hat{\omega}_p = \hat{\omega}_p$ on $C(\mathcal{G})$, $\hat{\mathcal{D}} = \mathcal{F} \circ \hat{\omega} \circ \hat{\mathcal{F}}^*$ commutes with $\hat{\omega}$ on $C(\mathcal{G})$ where $\hat{\mathcal{F}}$ is the Fourier isomorphism of $C^*(G)$ onto $C(\mathcal{G})$, and $\hat{\omega}$ is the shift action of $\hat{\mathcal{F}}$ on $C(\mathcal{G})$. It follows from Goodman-Nakagato [3, 5] that there exists a one parameter subgroup $(\rho_x)$ of $\mathcal{G}$ such that $\hat{\omega}(\rho_x)(p) = \lim_{t \to 0} \mathcal{F}(\mathcal{F}(F)(p))$
for all \( f \in D(\hat{\mathcal{S}}) \). Since \( \langle \mathcal{S}, \cdot \rangle \in D(\hat{\mathcal{S}}) \), one has \( \mathcal{S}(\lambda(f)) = \mathcal{S}(f^* \lambda(f)) \) for all \( f \in \mathcal{G} \) where \( \mathcal{D}(\mathcal{G}) = \lim_{t \to 0} (\mathcal{G}, P_t) - 1 \). Let \( d_{t} (a \lambda(f)) = \mathcal{D}(\mathcal{G}) a \lambda(f) \) for all \( a \in D(\hat{\mathcal{G}}) \) and \( f \in \mathcal{G} \). Then it is a pregenerator of \( \mathcal{A} \times \mathcal{G} \) such that \( \mathcal{D}(\mathcal{G}) = \hat{\mathcal{G}} \) and \( \mathcal{D}^* \mathcal{G} = 0 \), \( \mathcal{D}_{t} \) commutes with \( \mathcal{D} \). Since \( \mathcal{S} \) is a closed \(*\)-derivation of \( \mathcal{A} \times \mathcal{G} \) and \( \mathcal{D}^* \mathcal{G} \) commutes with \( \mathcal{D} = \exp t \mathcal{D}_{t} \), it follows from Batty [1] that \( \mathcal{D}^* \mathcal{G} = \mathcal{A} \mathcal{D}_{t} \mathcal{G} \) for some \( k \in \mathbb{R} \). Therefore we have that \( \mathcal{D}^* (a \lambda(f)) = k \mathcal{D}_{t} (a \lambda(f)) + a \lambda(f) \).

Therefore, we have that \( \mathcal{D}^* (a \lambda(f)) = k \mathcal{D}_{t} (a \lambda(f)) + a \lambda(f) \), which implies the following lemma:

**Lemma 6.** Let \((\mathcal{A}, \mathcal{G}, \mathcal{x})\) be a \(C^*\)-dynamical system where \( \mathcal{A} \) is unital abelian and \( \mathcal{G} \) is discrete abelian. Let \( \mathcal{S} = \exp t \mathcal{D}_{t} \) be a periodic action of \( \mathcal{G} \) on \( \mathcal{A} \). Suppose \( \mathcal{S} \) is ergodic, then given a \(*\)-derivation \( \mathcal{D} \) of \( \mathcal{A} \times \mathcal{G} \) with the property that \( \mathcal{D}(\mathcal{G}) = \mathcal{D}(\hat{\mathcal{G}}) \) and \( \mathcal{D} \) commutes with \( \mathcal{S} \), \( \mathcal{S} \), there exist a \( k \in \mathbb{R} \) and a pregenerator \( \mathcal{D} \) of \( \mathcal{A} \times \mathcal{G} \) such that

- \( \mathcal{D}(\mathcal{G}) = \mathcal{D}(\hat{\mathcal{G}}) \), \( \mathcal{D}^* \mathcal{G} = 0 \), \( \mathcal{D} \) commutes with \( \mathcal{D} \), and
- \( \mathcal{D} = k \mathcal{D}_{t} + \mathcal{S} \) on \( D(\hat{\mathcal{G}}) \).

**Remark 5.** The pregenerator \( \mathcal{D} \) defined above would

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be written as $\delta = r \delta'$ for some $r \in \mathbb{R}$ where $\delta'$ is not depending on $\delta$. Actually if $G = \mathbb{Z}$, we have $\delta'(a \lambda(m)) = l(m) \lambda(a)$ for $a \in D(\delta)$ and $n \in \mathbb{Z}$.

Let $\delta'$ be a linear mapping from a $*$-subalgebra $D(\delta)$ of $\mathcal{A}$ into $\mathbb{R}$ such that $\delta'(a \lambda) = \delta(a) \lambda(a) + \lambda(a)$ for all $a, \lambda \in D(\delta)$ where $\lambda \in G$ is a fixed element.

Suppose there is a unitary $U$ of $D(\delta)$ such that $1 \neq \text{sp}(\lambda(U) U^*)$, then we have by direct computation that $\delta(U^n) = \sum_{k=0}^{n-1} \lambda(U^k) U^k \delta(U) U^{-k}$. Since $1 \neq \text{sp}(\lambda(U) U^*)$, one has that $\sum_{k=0}^{n-1} \lambda(U^k) U^k = (\lambda(U) U^n - 1)(\lambda(U) U^{-1})$.

So $\delta(U^n) = \delta(U) U^*(\lambda(U) U^n - 1)^{-1} (\lambda(U) U^{-1}) U^n = \delta(U) (\lambda(U) - U^{-1})(\lambda(U) U^{-1})$ for all $n \in \mathbb{Z}$ since $\delta(1) = 0$. But $\delta(U^n) = \delta(U) (\lambda(U) - U^{-1})^n \in \mathcal{A}$. Since $\lambda(U) (\lambda(U) - U^{-1})$ is bounded on $\mathcal{A}$, the conclusion follows. Namely we have the following:

**Lemma 7.** Suppose $\mathcal{A}$ is unital abelian and $G$ is discrete. Let $\delta$ be a linear mapping of a $*$-subalgebra $D(\delta)$ of $\mathcal{A}$ into $\mathbb{R}$ such that $\delta(ab) = \delta(a) \lambda(b) + \lambda(a)$ for $a, b \in D(\delta)$ for a fixed $\lambda \in G$.

Suppose there exists a unitary $U \in D(\delta)$ such that $1 \neq \text{sp}(\lambda(U) U^*)$, then $\delta = \lambda(U) (\lambda(U) - U^{-1})^n$ on $D(\delta) \cap C^*(\mathcal{A})$. 

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for some $a_g \in \mathcal{A}$.

Remark 6. By the above lemma, there is no unbounded $\delta_g$-cocycle closed $\ast$-derivation of $\mathcal{A}_g$ has an eigenunitary generating $\mathcal{A}$.

Now let $\hat{f}_g (\lambda \theta)$ be a derivation of $\mathcal{A}_g$, as in the previous way (following to Remark 4). Then it implies that $\hat{f} = \sum \hat{f}_g$ on $\mathcal{D}(\mathcal{F})$. In fact, let $\hat{f}(a) = \sum \hat{f}_g (a)(\lambda \theta) \lambda (\theta)$ and $\hat{f}(\lambda (\theta)) = \sum \hat{f}_g (\lambda (\theta))(\lambda \theta)(\lambda \theta)$ be the Fourier expansion of $\hat{f}(a)$ and $\hat{f}(\lambda (\theta))$ respectively.

Then $\hat{f}_g (a) = \hat{f}(a)(\lambda \theta) \lambda (\theta)$ and $\hat{f}_g (\lambda (\theta)) = \hat{f}(\lambda (\theta))(\lambda \theta)(\lambda \theta)$.

Suppose $\hat{f}$ commutes with $\hat{\beta}$, it follows from Lemma 6 that $\hat{f}_g = k \hat{f}_g + \hat{f}_g \lambda (\theta)$ on $\mathcal{D}(\mathcal{F})$ where $k, \delta$ are as in Lemma 6. Let $\hat{f}_g (a) = \hat{f}_g (a)(\lambda \theta) \lambda (\theta)$ for $a \in \mathcal{D}(\mathcal{F}) (\# e)$.

Then $\hat{f}_g$ satisfy the condition of Lemma 7. Suppose there exists a unitary $u \in \mathcal{D}(\mathcal{F})$ such that $\# e 1 \# e \hat{f} (\lambda (u)(u)^*) = \# e \hat{f}(u)(u)^*$ and $\overline{\mathcal{A}} = \mathcal{C}^*(u)$. Since $\hat{f}$ commutes with $\hat{\beta}$, and $a$ commutes with $\hat{\beta} = \exp t \delta$, which is ergodic, we have $a_g \in \mathcal{A}_g$. Then $\hat{f}_g (a) = a_g (a_g^{-1} a_g)(a)(\lambda (\theta) = [a_g \lambda (\theta), a_g]$. Hence $\hat{f}_g (a \lambda (\theta)) = \hat{f}_g (a) \lambda (\theta) + a_g \hat{f}_g (\lambda (\theta)) = [a_g \lambda (\theta), a_g \lambda (\theta)] + a_g \hat{f}_g (\lambda (\theta))$. Since $\hat{f}_g - \text{ad} (a_g \lambda (\theta))$ is a derivation on $\mathcal{D}(\mathcal{F})$, one has $\hat{f}_g (\lambda (\theta)) = 0$ for $k \in \mathcal{F}$.
In fact, since \( \tilde{S}_y(\lambda h) = \tilde{S}(\lambda h)(h+k) \lambda (h+k) \), we have that \( \delta (\lambda (h+k)) (h+k) \chi = \delta (\lambda h) h + \delta (\lambda k) (h+k) \chi \) for all \( h, k \in \mathcal{G} \). Since \( 1 \notin D(\delta) \), we have \( \delta (\lambda(1)) = 0 \).

So \( \delta (\lambda h) (h+k) = 0 \) for all \( h, k \in \mathcal{G} \) or \( \delta (\lambda h) = 0 \). Since \( 1 \notin Sp (y h(i) h^*) \), we have \( \delta (\lambda h) (h+k) = 0 \) for all \( h, k \in \mathcal{G} \).

Consequently \( \delta = k \tilde{S}_0 + \delta_1 + \sum_{y \in \mathcal{G}} a_1 \mathcal{D}(\lambda y) \) in \( D(\delta) \). Let \( \delta_n = \mathcal{D}(\sum_{y \in \mathcal{F}} a_1 \lambda y) \) for a finite set \( \mathcal{F} \) of \( \mathcal{G} - \{ e \} \) with \( \mathcal{F} = -\mathcal{F} \). Then \( \delta_n \) are bounded \(*\)-derivations of \( \mathcal{A} \times \mathcal{G} \) such that \( \delta_n (\lambda h) = 0 \) and \( \delta_n \) converges to \( \delta_2 \) pointwise on \( D(\delta) \) where \( \delta_2 (a \lambda (h)) = \sum_{y \in \mathcal{G}} a_1 \mathcal{D}(\lambda y) a \lambda (h) \lambda \mathcal{D}(\lambda y) \).

Then \( \delta = k \tilde{S}_0 + \delta_1 + \delta_2 \) in \( D(\delta) \) and \( \delta_2 (\lambda h) = 0 \) for all \( h \in \mathcal{G} \), which implies the following proposition:

**Proposition 8.** Let \( (\mathcal{A}, \mathcal{G}, \alpha) \) be a \( C^* \)-dynamical system where \( \mathcal{A} \) is unital abelian and \( \mathcal{G} \) is discrete. Let \( \beta = \exp ik \alpha \) be an ergodic action of \( T \) on \( \mathcal{A} \) commuting with \( \alpha \). Suppose there exists a unitary \( U \in D(\tilde{S}) \) such that

1. \( 1 \notin \text{Sp}(y(i) U U^*) \) (\( y \in \mathcal{G} \)),
2. \( \mathcal{A} = C^*(U) \),
then given a \(*\)-derivation \( \delta \) of \( \mathcal{A} \times \mathcal{G} \) such that

a. \( D(\delta) = D(\tilde{S}) \) and \( \delta \) commutes with \( \beta \),

b. there exists a \( k \in \mathcal{A} \), a generator \( \delta_1 \),

c. an approximately bounded \(*\)-derivation \( \delta_2 \) of \( \mathcal{A} \times \mathcal{G} \) such that

i. \( D(\delta_2) = D(\delta) \),
ii. \( \delta_1 |_{\mathcal{A}} = 0 \),
iii. \( \delta_2 \) commutes with \( \tilde{S}_0 \),

\[ \tag{10} \]
\( D(\mathbb{Z}) = D(\mathbb{Z}) \),\( \mathcal{L}(\lambda(x)) = 0 \) for all \( x \in \mathcal{G} \), and \( \mathcal{F} = k_{i_0} + \delta_1 + \delta_2 \).

**Remark 7.** In the case of discrete abelian groups, the Fourier expansion of any element of \( \mathbb{Z} \times \mathbb{G} \) can be taken in the uniform sense. In fact, taking a net \( \{ \delta_i \} \) of positive definite functions on \( \mathcal{G} \) with finite support converging to 1, one can show that \( \sum \delta_i(x) \varphi(x) \lambda(x) \) converges to \( \sum \varphi(x) \lambda(x) \) in \( \mathbb{Z} \times \mathcal{G} \) uniformly.

**Proof of Theorem 2:** Since \( \beta \) commutes with \( \alpha \) and \( \beta \) is ergodic, we have \( \beta_\alpha(x) = \beta(x) \). Since \( \mathbb{Z}^* = C^*(\mathbb{Z}) \) and \( \alpha \) is effective, there are \( q \in \mathbb{G} \) such that \( \alpha(q) = q \cdot \mathbb{Z} \). Let \( f_n = \int_{\mathbb{Z}} e^{i \pi n t} \beta_\alpha(x) \), \( \delta_1 \) and \( \delta_2 \) be in \( D(\mathcal{G}) \). Since \( \beta_\alpha \) commutes with \( \beta \), it follows from Proposition 8 that \( \beta_\alpha = k_{i_0} + \delta_1 + \delta_2 \) where \( \delta_1 \) is as in Proposition 8. Since \( \beta_\alpha \cdot \beta_\alpha = \beta_\alpha \cdot \beta_\alpha = e^{i \pi n \beta_\alpha(x)} \), \( \beta_\alpha \cdot \beta_\alpha(x) = e^{i \pi n \beta_\alpha(x)} \). Since \( \beta_\alpha(x) = e^{i \pi n \beta_\alpha(x)} \), we have that \( \beta_\alpha(x) = \beta(x) \). So there are \( \delta(x) \in C^*(\mathcal{G}) \) such that \( \beta_\alpha(x) = \beta_\alpha(x) \). Let \( \beta_\alpha(x) = \sum \beta_\alpha(x) \lambda(x) \) and \( \delta(x, \lambda) = \sum \beta_\alpha(x) \lambda(x) \lambda(x) \) be the Fourier expansion of \( \beta_\alpha(x) \) and \( \delta(x, \lambda) \) respectively.
Since $\mathfrak{H} = C^{r}(\mathcal{U})$ and $\beta_{0}(\mathcal{U}) = e^{i\theta} \mathcal{U}$, we have that $\delta(\lambda(\theta))(h) = a(0) + \sum_{n=0}^{\infty} \delta_{n}(\lambda, \theta)(h) \mathcal{U}^{n}$ where $a(0)$ is the 0-component of the expansion of $\delta(\lambda(\theta))(h)$ in $\mathfrak{H}$. Since $\delta_{0} = k \delta_{0} + \delta_{1} + \delta_{2}$, one sees $\delta_{0}(\lambda(\theta)) = \delta(\theta) \lambda(\theta)$. By uniqueness, $\int_{\mathcal{H}} \beta_{0}(\mathcal{V}(\theta))(h) \, dt = \delta(\theta) \cdot 1$ (otherwise), which is nothing but $a(0)$. Therefore we deduce that $\delta(\lambda(\theta)) = \delta(\theta) \lambda(\theta) + \sum_{n=0}^{\infty} \delta_{n}(\lambda, \theta)(h) \mathcal{U}^{n} \lambda(\theta) = \delta(\theta) \lambda(\theta) + \sum_{n=0}^{\infty} \lambda(\theta)^{n} \lambda(\theta)$, i.e., $\delta(\theta) \lambda(\theta) + \sum_{n=0}^{\infty} \mathcal{U}^{n} \lambda(\theta) = \delta(\theta) \lambda(\theta) + \sum_{n=0}^{\infty} \delta_{n}(\theta) \lambda(\theta)$. Moreover $\delta(\theta) = \sum_{n=0}^{\infty} \delta_{n}(\theta)$ for all $\theta \in \mathbb{D}(\mathcal{H})$. It follows from Lemma 7 that $\hat{\delta}_{0}(\theta) = \hat{\delta}_{0}(\theta - i\theta)(\mathcal{U}) \lambda(\theta)$ for some $\hat{\delta}_{0} \in \mathfrak{H}$ (\text{\textit{p+e}}). So $\hat{\delta}_{0}(\theta) = [\hat{\delta}_{0}(\lambda(\theta)), \mathcal{U}]$ for all $\theta \in \mathbb{D}(\mathcal{H})$.

Since $\hat{\delta}_{0}$ commutes with $\mathcal{U}$, we have $\hat{\delta}_{0}(\mathcal{U}) \in \mathfrak{H}$ for all $\theta \in \mathbb{D}(\mathcal{H})$. Since $(\hat{\delta}_{0})^{*}$ commutes with $\mathcal{U}$ and $\nabla$, it means that $(\hat{\delta}_{0})^{*} = k \delta_{0} + \delta_{1}$ where $k, \delta_{1}$ are as in Lemma 6.

Then $\int_{\mathcal{H}} e^{i\theta} \beta_{0} \, \hat{\delta}_{0}(\mathcal{U}) \, dt = k \delta_{0}(\mathcal{U})$. Since $\beta_{0}(\mathcal{U}) = e^{i\theta} \mathcal{U}$, we have \( \delta_{0}(\mathcal{U}) = \mathcal{U} \mathcal{U} \). Let $\hat{\delta}_{0}(\mathcal{U}) = \sum_{n=0}^{\infty} \mathcal{U}^{n} \mathcal{U}$. Then $a_{2} = i k$. Therefore $\hat{\delta}_{0}(\mathcal{U}) = -k \delta_{0}(\mathcal{U}) + \sum_{n=0}^{\infty} \mathcal{U}^{n} \mathcal{U}$. Since $\hat{\delta}_{0}$ is a $*$-derivation, we deduce that $\hat{\delta}_{0}(\mathcal{U}) = k \delta_{0}(\mathcal{U}) \lambda(\theta) + \sum_{n=0}^{\infty} \mathcal{U}^{n+1} \lambda(\theta)$. Hence $\hat{\delta}_{0}(\mathcal{U}) = -k \delta_{0}(\mathcal{U}) \lambda(\theta) + \sum_{n=0}^{\infty} \mathcal{U}^{n+1} \lambda(\theta)$.

Consequently, we have that $\delta(\mathcal{U} \lambda(\theta)) = \delta(\mathcal{U}) \lambda(\theta) + \mathcal{U} \delta(\lambda(\theta)) = (k \delta_{0} + \delta_{1})(\mathcal{U} \lambda(\theta)) + \sum_{n=0}^{\infty} [\delta_{n}(\lambda(\theta)), \mathcal{U}] \lambda(\theta) + \sum_{n=0}^{\infty} \mathcal{U}^{n+1} \lambda(\theta)$.
+ \Sigma_{m=0}^{\infty} n a_m n \mu^m \mu^{\nu-1} \lambda(\theta)$. Since $\delta - k \delta_0 \delta_1$ is a $*$-derivation, so is $\Sigma_{n=0}^{\infty} \left[ \frac{f_n(\lambda(\theta))}{\lambda(\theta)} \right] \mu^n \mu^m \mu^{\nu-1} \lambda(\theta) + \Sigma_{m=0}^{\infty} \left[ \frac{f_m(\lambda(\theta))}{\lambda(\theta)} \right] \mu^n \mu^m \mu^{\nu-1} \lambda(\theta)$. Since $\text{ad} (f_n(\lambda(\theta))) (\mu^n \lambda(\theta)) + \mu^n \text{ad} (f_n(\lambda(\theta))) (\lambda(\theta)) = \text{ad} (f_n(\lambda(\theta))) (\mu^n \lambda(\theta))$, we deduce that $\mu^n (\Sigma_{m=0}^{\infty} \left[ \frac{f_m(\lambda(\theta))}{\lambda(\theta)} \right] \mu^m \mu^{\nu} \lambda(\theta)) = \Sigma_{m=0}^{\infty} \left[ \frac{f_m(\lambda(\theta))}{\lambda(\theta)} \right] \mu^n \mu^m \mu^{\nu} \lambda(\theta)$. Let $a = \Sigma_{m=0}^{\infty} a_m n \mu^m \in N$. Conventionally put $\delta (\lambda(\theta)) = \Sigma_{m=0}^{\infty} \left[ \frac{f_m(\lambda(\theta))}{\lambda(\theta)} \right] \mu^n \mu^m \mu^{\nu} \lambda(\theta)$. Moreover, put $\Delta (\mu^n \lambda(\theta)) = \mu^n \delta (\lambda(\theta)) + \nu a \mu^n \lambda(\theta)$. Since $\delta (\mu^n) = \nu \mu n$, we see $\nu a \mu^n \lambda(\theta) = (\nu) a \delta_1 (\mu^n \lambda(\theta))$. Now since $\Delta (\mu^n \lambda(\theta)) \mu^n \lambda(\theta)) = \Delta (\mu^n \lambda(\theta)) \mu^n \lambda(\theta)) + \mu^n \lambda(\theta)) \Delta (\mu^n \lambda(\theta))$, we can show that $\mu^n (\delta (\lambda(\theta)) \lambda(\theta) - \lambda(\theta) \delta (\lambda(\theta))) = \nu (\delta_0 (\lambda(\theta)) - \lambda(\theta)) \mu^n \lambda(\theta)$. For $\theta = e$ and $\nu = 1$, then we have $\mu^n \delta (\lambda(\theta)) = (\delta_0 (\lambda(\theta)) - \lambda(\theta)) \mu^n \lambda(\theta)$ for all $n \in \mathbb{N}$ and $\theta \in G$. Therefore $\Delta (\mu^n \lambda(\theta)) = (\delta_0 (\lambda(\theta)) - \lambda(\theta)) \mu^n \lambda(\theta)$ + $\nu a \mu^n \lambda(\theta) = (\delta_0 (\lambda(\theta)) - (\nu-1) a) \mu^n \lambda(\theta)$. Since $\delta$ is a derivation, we get $\delta_0 (\lambda(\theta)) = a \frac{\nu}{\nu-1}$ for all $\theta \in G$. So $a = \mathbb{R}^1$ for some $\theta \in G$. Then $\Delta (\mu^n \lambda(\theta)) = \beta_0 \mu^n \lambda(\theta) = c \delta_0 (\mu^n \lambda(\theta))$. Finally, we obtain that $\delta (\mu^n \lambda(\theta)) = (c \delta_0 + \delta_1) (\mu^n \lambda(\theta)) + \Sigma_{n=0}^{\infty} \left[ \frac{f_n(\lambda(\theta))}{\lambda(\theta)} \right] \mu^n \mu^m \mu^{\nu} \lambda(\theta)$ for some $c \in \mathbb{R}$. Let $\delta_0 (\lambda(\theta)) = \Sigma_{n=0}^{\infty} \left[ \frac{f_n(\lambda(\theta))}{\lambda(\theta)} \right] \mu^n \mu^m \mu^{\nu} \lambda(\theta)$, then $\delta_0$ is a bounded $*$-derivative of $\mu^\infty G$ for all $\theta$ and $\delta_0 \rightarrow \delta_2$ pointwise. Hence $\delta_2$ is approximately bounded. This completes the proof.
References


