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Ergodic properties of the equilibrium processes associated with infinitely many Markovian particles

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Consider a system of independent identically distributed Markov processes which have an invariant measure λ. It is known that if each process starts from each point of a λ-Poisson point process at time zero, these particles are λ-Poisson distributed at every later time t>0. In this paper we are concerned with the ergodic properties of the stationary process obtained from such a system of particles, which is called the equilibrium process. Sinai's ideal gas model is a special example of the equilibrium processes.

Let $(X\ \mathcal{B}_X\ \lambda)$ be a σ-finite measure space, and denote by $\mathcal{K}(X)$ a family of all the counting measures on X, i.e. each element of $\mathcal{K}(X)$ is an integer-valued measure with a countable set as its support. $\mathcal{K}(X)$ is equipped with a σ-field $\mathcal{G}$ which is generated by $\{\xi\in\mathcal{K}(X)\ ;\ \xi(A)=n\}$, $n\geq 0$, $A\in\mathcal{B}_X$. An element ξ of $\mathcal{K}(X)$ is represented by $\xi=\sum_i\delta_{x_i}$ where $\delta_x(A)=1$ if $x\in A$ and $\delta_x(A)=0$ if $x\notin A$. Let Π_λ be a probability measure on $(\mathcal{K}(X)\ \mathcal{G})$.

Π_λ is λ-Poisson point process if it satisfies the following ; for any disjoint system $A_1,\ldots,A_n$ of $\mathcal{B}_X$ such that $\lambda(A_i)<+\infty_{i=1\ldots n}$ $\xi(A_1),\ldots,\xi(A_n)$ are independent random variables on $(\mathcal{K}(X),\mathcal{G},\Pi_\lambda)$, and

$$\Pi_\lambda\{\xi\ ;\ \xi(A_i)=n\}=\frac{[\lambda(A_i)]^n}{n}\exp[-\lambda(A_i)],\ i=1,\ldots,n.$$

Next, we define the equilibrium processes associated with Markovian particles.

Let X be a locally compact separable Hausdorff space and $\mathcal{B}_X$ be the topological Borel field of X. Denote by W the path space of X, that is, each element of W is a X-valued right continuous function with left limit defined on $(-\infty,\infty)$, and define the shift operators $\{\theta_t\}_{-\infty<t<\infty}$ of W as usual ; $(\theta_t f)_s = f_{t+s}$ for each f of W.

Put $S=\mathcal{M}(X)$ and $\Omega=\mathcal{M}(W)$. Denote by $\{\Theta_t\}_{-\infty<t<\infty}$ the shift operators on Ω induced by the shift operators $\{\theta_t\}_{-\infty<t<\infty}$ on W, i.e.

$$\Theta_t\omega=\sum_i \delta_{\theta_t f^i} \quad \text{if } \omega=\sum_i \delta_{f^i}$$

Define S-valued process $\{\xi_t(\omega)\}_{-\infty<t<\infty}$ on Ω as follows ;

$$\xi_t(\omega)=\sum_i f_t^i \quad \text{if } \omega=\sum_i \delta_{f^i}$$

Then $\xi_t(\omega)$ is right continuous in t in a natural topology.

In our situation a motion of one particle is given as a Markov process on X and denote by $\{P_t(x,dy)\}$ its transition probabilities.

<u>Assumption</u>

$P_t(x,dy)$ is a conservative Feller Markov process and have a Radon invariant measure λ, that is, $\{P_t(x,dy)\}$ induces a semi-group of contraction operators $\{T_t\}$ on $C_\infty(X)$, and $\int T_t f(x)\,\lambda(dx)=\int f(x)\,\lambda(dx)$ for every f of $C_0(X)$.

Under this assumption $\{T_t\}$ is, also, a semi-group of contraction operators on $L^2(X\ \mathcal{B}_X\ \lambda)$.

<u>Lemma</u> There is only one σ-finite measure Q on $(W,\mathcal{B}_W)$ such that for $-\infty<t_1<t_2<\cdots<t_n<+\infty$ and $\{A_i\}_{i=1,2,..,n}$

$$Q[f;\ f_{t_1}\in A_1, f_{t_2}\in A_2,..,f_{t_n}\in A_n]$$

$$=\int_{A_1}\lambda(dx_1)\int_{A_2}P_{t_2-t_1}(x_1,dx_2)\cdots\int_{A_n}P_{t_n-t_{n-1}}(x_{n-1},dx_n)$$

In Particular Q is $\{\theta_t\}$-invariant.

Denote by $\mathbb{B}$ the σ-field generated by $\{\omega \in \Omega;\ \omega(A)=n\}$ $n \geq 0$, $A \in \mathcal{B}_X$ and put $\mathbb{P} = \Pi_Q$ (Q-Poisson point process). We consider $(\Omega, \mathbb{B}, \mathbb{P})$ as our basic probability space.

Proposition 1. $\{\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}\}_{-\infty<t<\infty}$ is a right-continuous Markov stationary process with Π_λ as its absolute law.

The Markov process defined above is called the equilibrium process associated with $[\{T_t\}, \lambda]$. Our purpose is to investigate the ergodic properties.

Proposition 2. The following (i), (ii), and (iii) are equivalent.

(i) $(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\})$ is metrically transitive.

(ii) $\lim_{t\to\infty} \frac{1}{t} \int_0^t \int_K P_s(x,K)\lambda(dx)ds = 0$ for every compact subset K of X.

(iii) $\lim_{t\to\infty} \frac{1}{t} \int_0^t (T_s f, g)_{L^2(\lambda)} ds = 0$ for all f and g of $L^2(X,\lambda)$.

Proposition 3. The following three statements are equivalent.

(i) $(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}_{-\infty<t<\infty})$ has the strong mixing property.

(ii) $\lim_{t\to\infty} \int_K \lambda(dx) P_t(x,K) = 0$ for all f and g of $L^2(X,\lambda)$.

(iii) $\lim (T_t f, g)_{L^2(\ )} = 0$ for all f and g of $L^2(X,\ )$.

Proposition 4. The following three statements are equivalent.

(i) $(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}_{-\infty<t<\infty})$ is purely non-deterministic.

(ii) $\lim_{t\to\infty} \int_X \lambda(dx) [P_t(x,K)]^2 = 0$ for every compact subset K of X.

(iii) $\lim_{t\to\infty} \| T_t f \|_{L^2(\lambda)} = 0$ for every f of $L^2(X,\lambda)$.

Proposition 5. $(\Omega, \mathbb{B}, \mathbb{P}; \{\xi_t\}_{-\infty<t<\infty})$ is purely non-deterministic if and only if $\mathbb{E}[\xi_t | \xi_0]$ converges to λ vaguely in probability.

Next we study the Bernoulli property of the shift flow $\{\Theta_t\}_{-\infty<t<\infty}$. It is easy to see that $\{\Theta_t\}_{-\infty<t<\infty}$ is a flow on the probability space $(\Omega = \mathcal{K}(W), \mathbb{B}, \mathbb{P} = \Pi_Q)$.

So, we define the Bernoulli property in the strong sense.

$(\Omega, \mathbb{B}, \mathbb{P}; \{\Theta_t\}_{-\infty<t<\infty})$ is called a Bernoulli flow if there exists a σ-subfield ζ_0 of $\mathbb{B}$ and $\zeta_t = \Theta_t \cdot \zeta_0$ satisfies the following conditions ;

(i) $\zeta_t \subset \zeta_s$ for any $t < s$

(ii) $\bigcap_t \zeta_t = \{\phi, \Omega\}$ (mod. $\mathbb{P}$)

(iii) $\bigvee_t \zeta_t = \mathbb{B}$ (mod. $\mathbb{P}$)

(iv) for any $t < s$ there exists a σ-subfield ζ_t^s of $\mathbb{B}$ such that $\zeta_s = \zeta_t \vee \zeta_t^s$ and $\zeta_t \perp\!\!\!\perp \zeta_t^s$.

The following lemma is a criterion of the Bernoulli property of our shift flow $\{\Theta_t\}_{-\infty<t<\infty}$.

<u>Lemma</u> Suppose that there exists a real measurable function $\tau(f)$ on the σ-finite measure space $(W, \mathcal{B}_W, Q)$ such that for almost all f w.r.t. Q (a) $-\infty < \tau(f) < +\infty$

(b) $\tau(f) = t + \tau(\theta_t f)$ for all t of R^1.

Then, $(\Omega, \mathbb{B}, \mathbb{P}; \{\Theta_t\}_{-\infty<t<\infty})$ is a Bernoulli flow.

We can show the following proposition by appealing to this lemma.

<u>Proposition 6.</u> Suppose that $\{T_t\}$ is transient in the sense that $\int_0^\infty (T_t\varphi, \varphi)_{L^2(\lambda)} dt < +\infty$ for every φ of $C_0^+(X)$.

Then, $(\Omega\ \mathbb{B}\ \mathbb{P}; \{\Theta_t\}_{-\infty<t<\infty})$ is a Bernoulli flow.

The equilibrium process $\{\xi_t\}$ induces a factor flow of $\{\Theta_t\}$. Since a Bernoulli flow $\{\Theta_t\}$ in our sense is a Bernoulli flow in the weak sense (i.e. the automorphism $\{\Theta_t\}$ is Bernoulli for each $t \neq 0$), the shift flow induced by $\{\xi_t\}$ is also a Bernoulli flow in the weak sense by the theorem of Ornstein.

Finally we can prove a central limit theorem related to the equilibrium process. Denote by $G\varphi(x) = \int_0^\infty T_t\varphi(x)dt$ if the integral is well-defined.

Proposition 7. Consider any function $\varphi \in L^2(X,\lambda)$ which satisfies $(G|\varphi|,|\varphi|)_{L^2(\lambda)} < +\infty$ and $(G(|\varphi| G|\varphi|),|\varphi|)_{L^2(\lambda)} < +\infty$. Then, we have

$$\lim_{t\to\infty} \mathbb{P}[\omega ; \alpha < \frac{\int_0^t \langle \varphi, \xi_s \rangle ds - t \cdot \langle \varphi, \lambda \rangle}{\sqrt{2(\varphi, G\varphi)_{L^2(\lambda)} \times t}} < \beta] = \frac{1}{\sqrt{2\pi}} \int_\alpha^\beta \exp(-\frac{x^2}{2}) dx \quad \text{for } \forall\alpha < \forall\beta .$$

Let $\{Q_t(\xi, d\eta)\}$ the transition probability of the equilibrium process defined in Proposition 1.

In general, $\{Q_t(\xi, d\eta)\}$ has many invariant measures besides λ-Poisson point processes. In this paper we treated only the equilibrium processes with λ-Poisson point processes Π_λ as its absolute law. But this is reasonable because of the following proposition.

Proposition 8. Suppose that

$$\lim_{t\to\infty} \sup_{x\in X} P_t(x,K) = 0 \quad \text{for any compact set } K \subset X.$$

Let $\Pi(d\xi)$ an invariant probability measure with respect to $\{Q_t(\xi, d\eta)\}$. If the stationary process generated by $[\{Q_t(\xi, d\eta)\}, \Pi(d\xi)]$ is metrically transitive, $\Pi = \Pi_\lambda$ for some P_t-invariant measure λ.

Referrences

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