数理解析研究所講究録
204巻 1974年 100-109

The motion of a particle in a central field.

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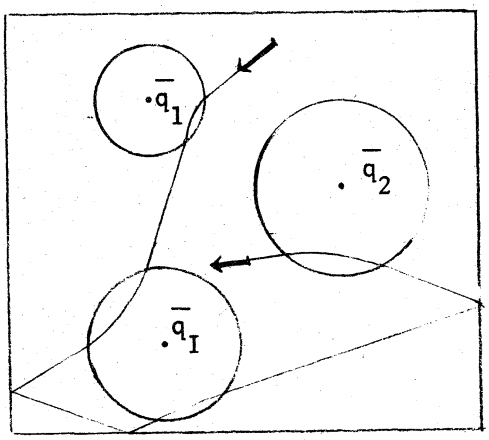
§1. Introduction

Let us consider a 2-dimensional rectangle box, in which there exist several fixed particles with non overlapped potentials of central force (see Fig. 1). Such field will be called *a compound central field*. Observe the motion of another free particle in the compound central field. The purpose of this report is to announce that *the motion is ergodic if the central potentials are "bell-shaped" and the energy level of the moving particle is low*. In this report, we say that a central potential U is *bell-shaped* if

(1) $U(s)$ is continuous for $s > 0$, and $U(s) = 0$ $s \geq R$,

(2) $U(s)$ belongs to $C^{(3)}$-class for $0 < s < R$ and there exist left derivatives $U'(R-0)$, $U''(R-0)$, $U'''(R-0)$,

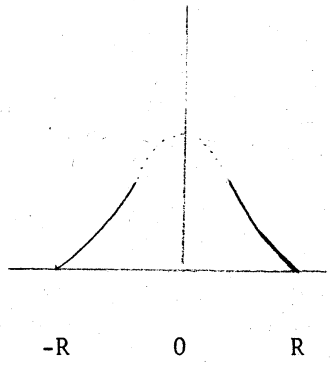
(3) $-sU'(s)$ is monotone decreasing and $U'(R-0) < 0$.



$\bar{q}_1, \bar{q}_2, \cdots \bar{q}_I$ are centers of particles.

$U_\iota(|\bar{q}(\iota)-q|)$ is the central potential corresponding to $\bar{q}_\iota$.

Fig. 1



a figure of "bell-shaped" potential.

Fig. 2

The method of my proof is based on the fact that our system is a perturbation of a Sinai's billiard system. Let T be the basic automorphism of the natural Kakutani-Ambrose representation of our dynamical system, whose basic space is the set of all incident vectors at the boundaries of the potentials. You can image of this representation by Fig. 3. The automorphism T can be resolved into two automorphisms T' and T'', such as

$$T = T''T',$$

where T'' is an automorphism found in Sinai's billiard system and T' is Anzai's automorphism, as you can see it.

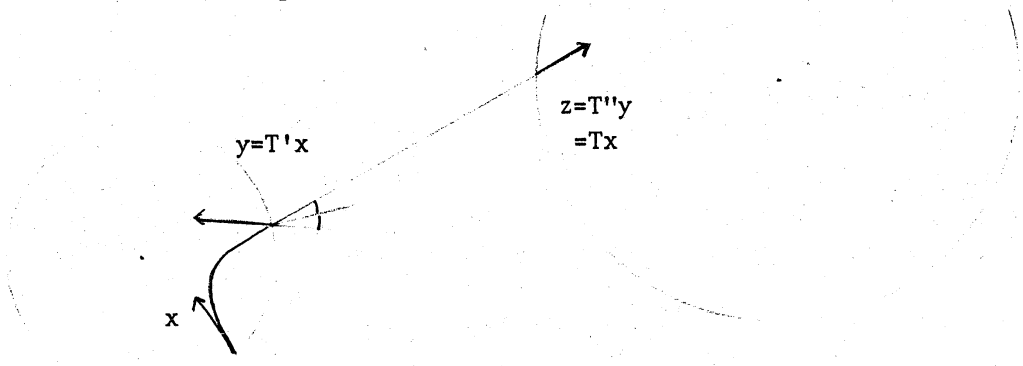


Fig. 3

§2. Perturbed billiard system

Let L be a 2-dimensional torus[1)] and let $\bar{Q}_\iota$ be a strictly convex open domain in L with boundary ∂Q_ι of C^3-class, $\iota = 1, 2, \cdots I$. Put $Q = L - \bigcup_{\iota=1}^{I} Q_\iota$. The motion of a particle in the domain Q with ellastic collision at the boundary $\partial Q = \bigcup_{\iota=1}^{I} \partial Q_\iota$ is called *Sinai's billiard system*. Then, the energy surface M is the product of Q and a one dimensional sphere $S^{(1)}$. We denote by $\{S''_t\}$ the dynamical system on M. Let π be the natural projection of M onto Q. Let X be the set of all incident vectors, that is,

$$X = \{x = (q,p) \ ; \ x \in \pi^{-1}(\partial Q), \ (p, n(q)) \leq 0\},$$

where $n(q)$ is the inward normal at q. Since ∂Q_ι has a natural arclength coordinate r, we can introduce a natural coordinates (ι, r, φ) for the space X, where r is the arclength coordinate of $\pi(x) \in \partial Q_\iota$ and φ is the angle of incidence. An automorphism T'' of X is defined by

$$(4) \quad T''x = S''_{\sigma(x)-0}\, x,$$

where $\sigma(x) = \inf\{t > 0, \ S''_t\, x \in X\}$, that is, $\sigma(x)$ is the next incident time. Then $\{S''_t\}$ is the Kakutani-Ambrose flow with basic automorphism T'' and ceiling function $\sigma(x)$.

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1) For the case of rectangle box, we can reduce it to the case of torus.

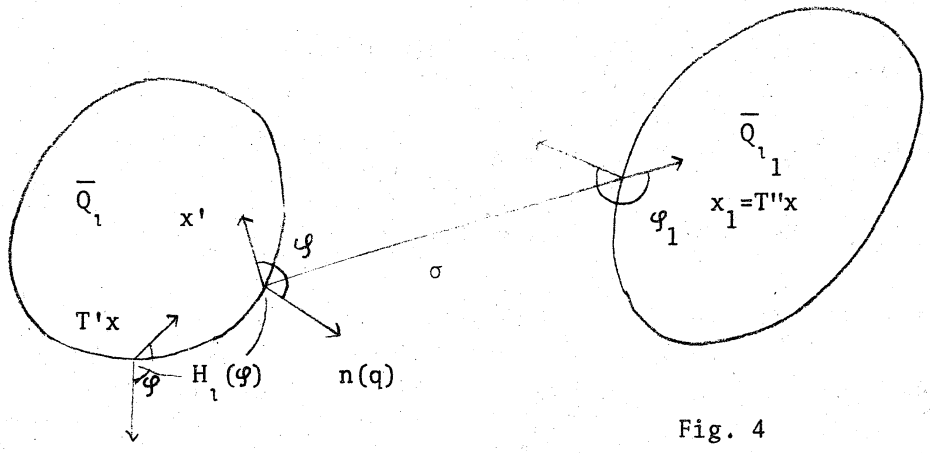

Fig. 4

We now introduce Anzai's transformation of X by

(5) $T' : (\iota, r, \varphi) \to (\iota, r+H_\iota(\varphi), \varphi)$

with functions $H_\iota(\varphi)$, $1 \le \iota \le I$, of C^3-class.

Definition : We say that the Kakutani-Ambrose flow $\{S_t\}$ with the basic automorphism $T = T''T'$ and the ceiling function $\sigma(x)$ is a *perturbed billiard system.*

Lemma 1. Let us suppose that $(\iota, r', \varphi) = T'(\iota, r, \varphi)$ and $(\iota_1, r_1, \varphi_1) = T''(\iota, r', \varphi)$. Then the Jacobian matrix of T is given by

$$(6)\quad \begin{pmatrix} \dfrac{\partial r_1}{\partial r} & \dfrac{\partial r_1}{\partial \varphi} \\ \dfrac{\partial \varphi_1}{\partial r} & \dfrac{\partial \varphi_1}{\partial \varphi} \end{pmatrix} = \begin{pmatrix} -\dfrac{\cos\varphi + k'\tau_1}{\cos\varphi_1}, & -\dfrac{\tau_1 + (\cos\varphi + k'\tau_1)h}{\cos\varphi_1} \\ -\dfrac{k_1\cos\varphi + k\cos\varphi_1 + k'k_1\tau_1}{\cos\varphi_1}, & -1-\dfrac{k_1\tau_1 + (k_1\cos\varphi + k'\cos\varphi_1 + k'k_1\tau_1)h}{\cos\varphi_1} \end{pmatrix},$$

where $k' = k(\iota, r')$ and $k_1 = k(\iota_1, r_1)$ are curvature of ∂Q at (ι, r') and (ι_1, r_1), respectively, and where $h = \dfrac{dH_\iota}{d\varphi}$ and $\tau_1 = -\sigma(\iota, r', \varphi)$.

We introduce some constants :

$$g_{min}(\iota) = ((\min_{(r,\varphi)}|\tau(\iota,r,\varphi)|)^{-1} + \max_{r} k(\iota,r))^{-1} ,$$

$$k_{min} = \min_{\iota,r} k(\iota,r) , \quad |\tau|_{min} = \min_{(\iota,r,\varphi)}|\tau(\iota,r,\varphi)| ,$$

$$k_{max} = \max_{\iota,r} k(\iota,r) \quad \text{and} \quad \eta = k_{min}|\tau|_{min}.$$

Now we assume the following assumption (H) ;

$$\text{(H)} \quad \min_{\iota,\varphi} h_{\iota}(\varphi) + \frac{1}{k_{max} + \frac{1}{|\tau|_{min}}} = K > 0 ,$$

where $h_{\iota}(\varphi) = dH_{\iota}(\varphi)/d\varphi$.

Under the assumption, we have inequalities

$$(7) \quad \tau_1 + (\cos\varphi + k'\tau_1)h \le -K\eta$$

and

$$(8) \quad \cos\varphi + k'\tau_1 + (k_1\cos\varphi + k'\cos\varphi_1 + k'k_1\tau_1)h \le -K\eta k_{min}.$$

These inequalities and Lemma 1 garantee the following properties.

Lemma 2. Let γ be a smooth curve in $M_{\iota} = \pi^{-1}(\partial Q_{\iota})$, which is defined by an equation $r = r(\varphi)$.

(i) If $\frac{dr}{d\varphi} \ge g_{min}(\iota)$, then

$$g_{min}(\iota_1) \le \frac{dr_1}{d\varphi_1} \le \frac{1}{k_{min}} \qquad \text{and}$$

$$-\frac{d\varphi_1}{d\varphi} = (1+\frac{k_1\tau_1}{\cos\varphi_1}) + \frac{k'\cos\varphi_1+k_1\cos\varphi+k'k_1\tau_1}{\cos\varphi_1}(\frac{dr}{d\varphi} + h) \ge 1+\eta.$$

(ii) If $-\frac{dr_1}{d\varphi_1} \ge 0$, then

$$k \le -\frac{dr}{d\varphi} \le \frac{1}{k_{min}} + \max_{(\iota,\varphi')} h_{\iota}(\varphi') \qquad \text{and}$$

$$-\frac{d\varphi}{d\varphi_1} = (1+\frac{k'\tau_1}{\cos\varphi}) + \frac{k'\cos\varphi_1+k_1\cos\varphi+k'k_1\tau_1}{\cos\varphi}\frac{dr_1}{d\varphi_1} \ge 1+\eta.$$

Using these properties, we can prove the following theorem by the similar proof of the ergodicity of Sinai's billiard system.

Theorem 1. Under the assumption (H), T is a K-system, and the perturbed billiard system is ergodic.

§3. The motion of a particle in a compound central field.

Let L be a two dimensional torus.[2] We suppose that $\bar{q}_1, \cdots, \bar{q}_I$ are centers of bell-shaped central potentials $U_1, \cdots, U_I$, respectively. Let us observe the motion of a particle with energy E. The motion is described by the cannonical equations

$$(8) \qquad \frac{dq_i}{dt} = \frac{\partial}{\partial p_i} H(q,p), \qquad \frac{dp_i}{dt} = -\frac{\partial}{\partial q_i} H(q,p)$$

with Hamiltonian

$$(9) \qquad H(q,p) = \frac{1}{2m}(p_1^2 + p_2^2) + \sum_{\iota=1}^{I} U_\iota(|q - \bar{q}_\iota|).$$

Our purpose of this section is to show the following theorem. Let $\bar{Q}_\iota$ be the range of the potential U_ι, R_ι be the radius of $\bar{Q}_\iota$ and $L_{\iota,\iota'}$ be the distance between the ranges $\bar{Q}_\iota$ and $\bar{Q}_{\iota'}$. Put $R_{min} = \min_\iota R_\iota$ and $L_{min} = \min_{\iota,\iota'} L_{\iota,\iota'}$.

Theorem 2. If energy E satisfies the inequality

$$(10) \qquad E \le \frac{R_{min} L_{min}}{4m(R_{min} + L_{min})} \min_\iota (-U_\iota'(R_\iota - 0))$$

Then the motion of the particle in the compound central field is ergodic.

We shall reduce our problem to the case of §2. The energy

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2) The case of rectangle box can be reduced to that of torus.

surface $M_E = \{(q,p)\ ;\ H(q,p) = E\}$ is a fibre bundle. Let π be the natural projection of M_E to $Q_E = \{q\ ;\ U_\iota(|\bar{q} - q|) \le E,\ \iota = 1, 2, \cdots, I\}$. Let ∂Q_ι be the boundary of the potential range $\bar{Q}_\iota$ and let $Q = L - \cup\bar{Q}_\iota$ $(= \bigcap_{E>0} Q_E)$. Then the induced flow onto $M_0 = \pi^{-1}(Q)$ (in the sense of Kakutani) of our system is obviously a perturbed billiard system. Hence, in order to prove Theorem 1, it is sufficient to show that $H_\iota(\varphi)$ satisfies the assumption (H).

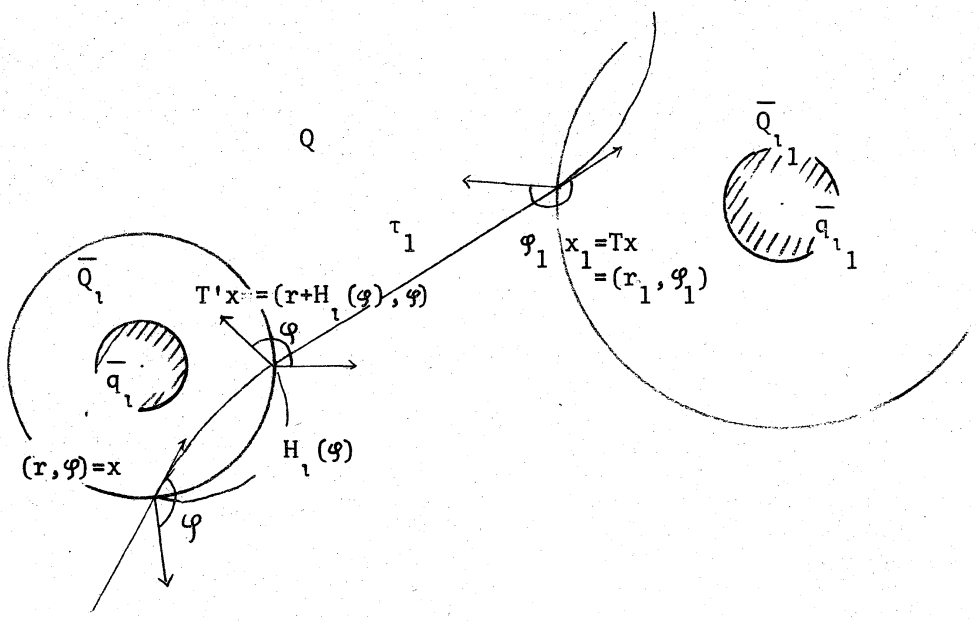

Fig. 5

Let us return back to a bell-shaped potential U. We introduce the polar coordinates (s, β). Then Hamitonian is given by

$$(11) \qquad H(s,\beta) = \frac{1}{2} m(\dot{s}^2 + s^2\dot{\beta}^2) + U(s).$$

It is well known that the angular momentum of the particle,

$$(12) \qquad A = ms^2\dot{\beta}.$$

is a first integral. Moreover, the equation of the motion is

given by

$$(13) \qquad m\ddot{s} - s\dot{\beta}^2 = -\frac{d}{ds}U(s).$$

Hence the equation of a path is written in the formula

$$(14) \qquad \beta = \int \frac{\pm mAs^{-2}}{\sqrt{2m(E-U(s))-A^2s^{-2}}}\,ds + \text{const.}$$

Especially, we shall observe a path which attains to the minimum value u of s-coordinate (see Fig. 7). Its angular momentum A is equal to $\sqrt{2m(E-U(u))}\cdot u$, by (11) and (12). We suppose that the path passes $(u,0)$. Let $(R,\alpha(u))$ be the point, at which the path goes out from the potential range. By (14), $\alpha(u)$ is given by

$$(15) \qquad \alpha(u) = m\int_u^R \sqrt{\frac{u^2(E-U(u))}{s^2(E-U(s))-u^2(E-U(u))}}\cdot\frac{ds}{s}.$$

On the other hand, the angle $\psi(u)$ between the velocity and the radius vector at $(R,\alpha(u))$ is given by

$$(16) \qquad \psi(u) = \cos^{-1}\sqrt{\frac{R^2E-u^2(E-U(u))}{R^2E}}$$

A proof of the equality (16) is given in the following. The velocity of the path at (s,β) is given by $(\dot{s}\cos\beta-\dot{\beta}s\sin\beta,$ $\dot{s}\sin\beta+\dot{\beta}s\sin\beta)$. Hence

$$\cos\psi(u) = \left.\frac{\dot{s}}{\sqrt{\dot{s}^2+\dot{\beta}^2s^2}}\right|_{s=R}.$$

On the other hand, by (11) and (12), we have that

$$\left.\dot{s}^2 + s^2\dot{\beta}^2\right|_{s=R} = \left.\frac{2}{m}(E-U(s))\right|_{s=R} = \frac{2E}{m}$$

and

$$A = \left.ms^2\dot{\beta}\right|_{s=R}.$$

Hence we have (16) from the above formulae.

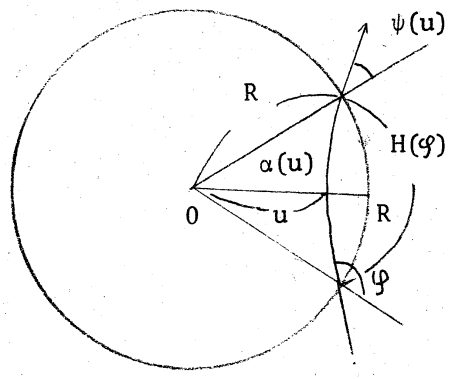

Fig. 7

Lemma 3. Let us consider a path passing into the range of the potential with an incident angle φ. Then the arclength distance between the position of going out from the range and the position of passing into the range, is given by

$$(17)\quad H(\varphi) = 2R\alpha(\psi^{-1}(|\pi - \varphi|))\cdot \operatorname{sign}(\varphi - \pi).$$

Moreover, for $u = \psi^{-1}(|\pi - \varphi|)$,

$$(18)\quad \frac{dH(\varphi)}{d\varphi} = \frac{-4mR(E-U(u))+2mR\sqrt{R^2E-u^2(E-U(u))}\cdot g(u)}{2(E-V(u))-uU'(u)}$$

holds with

$$g(u) = \int_1^{\log R/u} \frac{[-e^{2s}(E-U(e^s u))U'(u)+e^{3s}(E-U(u))U'(e^s u)]}{2\sqrt{E-U(u)}\,[e^{2s}(E-U(e^s u))-E+U(u)]^{3/2}}\,ds\ .$$

This Lemma is an immediate consequence of (15) and (16).

Proof of Theorem 2. From the property (3), of a bell-shaped potential, we can easily see that $g(u)$ is non negative. Hence we have inequalities

$$h(\varphi) = \frac{dH(\varphi)}{d\varphi} \geq \frac{-4mR(E-U(u))}{2(E-U(u))-uU'(u)}$$

$$\geq \frac{4mE}{U'(R+0)} \quad .$$

In our system, $k(\iota,r)$ is equal to $1/R_\iota$. Hence if the inequality

$$\max_\iota \frac{4mE}{-U'_\iota(R+0)} < \frac{1}{\frac{1}{R_{min}} + \frac{1}{L_{min}}}$$

holds, then assumption (H) is fulfilled. Q.E.D.

Examples of Bell-shaped Potentials :

(a) $$U^\alpha(s) = \begin{cases} \frac{c}{s^\alpha} - \frac{c}{R^\alpha} & 0 < s < R \\ 0 & R \leq s \end{cases} \qquad \alpha > 0.$$

(b) $$U^0(s) = \begin{cases} c \log \frac{R}{s} & 0 < s < R \\ 0 & R \leq s. \end{cases}$$

References

[1] Sinai, Ya. G. : On the foundations of the ergodic hypothesis for a dynamical system of statistical mechanics, Sov. Mat. Dokl. 4, No. 6 (1963), 1818-1822.

[2] Sinai, Ya. G. : Dynamical systems with elasitic reflections, Russian Math. Surveys 25, No. 2 (1970), 137-189.

[3] Kubo, I. : Billiard problem , Seminar on Probability vol.37, (1972), (in Japanese).