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Limit Theorems for Continued-Fractions

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O. Introduction: In this paper we shall discuss several limit theorems for continued-fractions. In the first section we define the continued-fraction expansion with notations in Billingsley's "Ergodic theory and information" and mention its ergodic property with applications.

Next, under the mixing condition of exponential order we shall state limit theorems which are sufficient for applying to continued-fractions and to related algorithms. These theorems are the generalizations of the central limit theorem and the law of the iterated logarithm to non-independent cases.

Finally, applying these limit theorems for continued-fractions, we get results in theorems 1, 2, and 3.

1. Definitions and preliminary remarks.

Any number ω in the unit interval has a simple continued-fraction expansion

(1) $$\omega = \frac{1}{|a_1(\omega)} + \frac{1}{|a_2(\omega)} + \cdots ,$$

where the partial quotients $a_n(\cdot)$ are positive integers. The expansion terminates after finitely many steps if and only if ω is rational. If ω has the expansion (1), then

(2) $$\omega = \frac{1}{a_1(\omega) + \omega'},$$

where

(3) $$\omega' = \frac{1}{|a_2(\omega)} + \frac{1}{|a_3(\omega)} + \cdots,$$

which is also a number in the unit interval whose partial quotients are the same as the translates of those of ω. Since $a_1(\omega)$ is the integer part $[1/\omega]$ of $1/\omega$, and ω' is its fractional part $\{1/\omega\}$, we are led to study the transformation T that carries ω to $\{1/\omega\}$.

For easier comprehension, we introduce the usual set-up: Let $\Omega = [0, 1)$, $\mathcal{F}$ the Borel σ-field over Ω and define T by

(4) $$T\omega = \begin{cases} \{1/\omega\} & \text{if } \omega \neq 0 \\ 0 & \text{if } \omega = 0. \end{cases}$$

If we define

(5) $$a(\omega) = \begin{cases} [1/\omega] & \text{if } \omega \neq 0 \\ \infty & \text{if } \omega = 0 \end{cases}$$

and

$$a_n(\omega) = a(T^{n-1}\omega), \qquad n = 1, 2, \cdots,$$

then $a_1(\omega)$, $a_2(\omega)$, --- are just the partial quotients in the continued-fraction expansion of ω. The n-th approximant of ω is

(6) $$\frac{1}{|a_1(\omega)} + \frac{1}{|a_2(\omega)} + \cdots + \frac{1}{|a_n(\omega)} = \frac{p_n(\omega)}{q_n(\omega)},$$

and ω is then represented by

$$\omega = \frac{p_n(\omega) + (T^n\omega)\, p_{n-1}(\omega)}{q_n(\omega) + (T^n\omega)\, q_{n-1}(\omega)}, \tag{7}$$

where these integral-valued functions $p_n(\omega)$, $q_n(\omega)$ are defined by the following recursion formulae;

$$\left\{ \begin{array}{l} p_{-1}(\omega) = 1,\ p_o(\omega) = 0,\ p_n(\omega) = a_n(\omega) p_{n-1}(\omega) + p_{n-2}(\omega), \quad n \geq 1, \\ q_{-1}(\omega) = 0,\ q_o(\omega) = 1,\ q_n(\omega) = a_n(\omega) q_{n-1}(\omega) + q_{n-2}(\omega), \quad n \geq 1, \end{array} \right. \tag{8}$$

, which satisfy

$$p_{n-1}(\omega)\, q_n(\omega) - p_n(\omega)\, q_{n-1}(\omega) = (-1)^n, \quad n \geq 0. \tag{9}$$

The transformation T does not preserve Lebesgue measure λ, but does a useful measure on $\mathfrak{F}$, namely Gauss's measure

$$P(A) = \frac{1}{\log 2} \int_A \frac{dx}{1+x}, \qquad A \in \mathfrak{F}. \tag{10}$$

This measure is not only T-invariant but also equivalent to Lebesgue measure λ, i.e. P and λ are absolutely continuous with respect to each other. But if we omit the second condition, one could find another T-invariant measure, which will be discussed somewhere else.

Khinchine proved many measure-theoretic results about continued-fractions, but his proofs are complicated by the fact that he made no use of the ergodic theorem.

The ergodicity of this transformation T was first proved by Knopp in 1926 and with a different proof by Ryll-Nardzewski. According to the ergodic theorem, if f is an integrable function on the unit interval Ω, then it holds

(11) $$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^n\omega) = \frac{1}{\log 2} \int_0^1 \frac{f(x)}{1+x}\, dx$$

for almost all ω with respect to λ (or P).

Taking f to be the indicator of the set $\{\omega ; a_1(\omega) = k\}$, we see that the asymptotic relative frequency of k among the partial quotients $a_1(\omega)$, $a_2(\omega)$, --- is equal to

(12) $$\frac{1}{\log 2} \int_{\frac{1}{k+1}}^{\frac{1}{k}} \frac{dx}{1+x} = \frac{1}{\log 2} \log \frac{(k+1)^2}{k(k+2)}$$

$$= \lim_{N\to\infty} \frac{1}{N} \# \{ i \le N ; a_i(\omega) = k \}, \quad \text{a. e.},$$

where $\#\{ i \le N ; \quad \}$ indicates the number of integers $i \le N$ satisfying the condition in the bracket. In particular, it is seen that the partial quotients are a. e. unbounded.

Taking $f(\omega) = \log a_1(\omega)$, we see that

(13) $$\lim_{n\to\infty} \sqrt[n]{a_1(\omega) \cdots a_n(\omega)} = \prod_{k=1}^{\infty} \left(1 + \frac{1}{k^2 + 2k}\right)^{\log k/\log 2} \quad \text{a. e.}$$

Also by virtue of the ergodic theorem, we see that

(14) $$\lim_{n\to\infty} \frac{1}{n} \log q_n(\omega) = \frac{\pi^2}{12 \log 2} \quad \text{a. e.}$$

Another type of limiting properties of continued-fractions is the limit theorem, such as Gaussian law, the law of the iterated logarithm or Poisson's law, most of which were proved by Doeblin in 1940. These limiting laws are based on the mixing condition of T of exponential order, which is called Gauss-Kuz'min's theorem (or simply, Kuz'min's theorem). In the next section we give more precise limit theorems with remainder for mixing sequence of random variables.

2. The central limit theorem with remainder and the law of the iterated logarithm.

We shall state the central limit theorem with remainder and the law of the iterated logarithm according to W. Phillip by which he attempts to unify probabilistic number theory. First, we consider a sequence of probability spaces $\langle (\Omega_N, \mathfrak{F}_N, P_N),\ N = 1, 2, \cdots \rangle$ and for each $N = 1, 2, \cdots$ a sequence of random variables $\langle X_{N_n},\ n = 1, 2, \cdots, n_N \rangle$ defined on $(\Omega_N, \mathfrak{F}_N, P_N)$. By $\mathcal{M}^{(N)}_{ab}$ we mean the σ-algebra generated by the random variables X_{N_n}, $1 \leq a \leq n \leq b \leq n_N$. Under the mixing condition of exponential order,

$$P(AB) - P(A)P(B) \ll e^{-\lambda n} P(A)P(B) \quad (\lambda > 0) \tag{15}$$

for any events $A \in M^{(N)}_{1t}$, and $B \in M^{(N)}_{t+n\ n_N}$,

we have the next result.

<u>Theorem</u> A. Let $\langle X_{N_n},\ n = 1, 2, \cdots, n_N ;\ N = 1, 2, \cdots \rangle$ be a double sequence of random variables centered at expectations with $\sup_{n,N} \| X_{N_n} \|_\infty \leq 1$, and $s_N^2 = E(\sum_{n \leq n_N} X_{N_n})^2 \to \infty$ as $N \to \infty$.

Suppose the condition (15) and moreover

(16) $$\sum_{n=M+1}^{M+H} E\,|X_{N_n}| \ll E\left(\sum_{n=M+1}^{M+H} X_{N_n}\right)^2 \quad (H \to \infty)$$

uniformly in M = 0, 1, 2, ---. Then

(17) $$P(s_N^{-1} \sum_{n \le n_N} X_{N_n} < \alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} \exp(-t^2/2)\,dt + O(s_N^{-1/2} \log^3 s_N).$$

We can relax the condition (15) by replacing $e^{-\lambda n}$ by $\chi(n)$ with $\sum \chi^{1/3}(n) < \infty$, but with some additional conditions (say, the estimate for the sum of forth moments) required. The remainder term worsens to $O(s_N^{-1/10})$.

<u>Theorem</u> B. Under the same hypothesis in the theorem A, we have

(18) $$P\left\{ \limsup_{N \to \infty} \frac{\left|\sum_{n=1}^{n_N} X_{N_n}\right|}{\sqrt{2 s_N^2 \log\log s_N^2}} = 1 \right\} = 1.$$

3. Application of limit theorems to continued-fractions.

We return to the ergodic transformation T in section 1. For a set of positive integers $(b_1, b_2, ---, b_n)$, we call $\Delta_{b_1 --- b_n}$ the fundamental interval of rank n with $(b_1, ---, b_n)$, where

(19) $$\Delta_{b_1 --- b_n} = \{ \omega \; ; \; a_1(\omega) = b_1, ---, a_n(\omega) = b_n \}.$$

Let $\varphi(n)$ be any sequence of positive integers and let E_n be the event $\{ \omega \; ; \; a_n(\omega) \ge \varphi(n) \}$. Since $P(E_n) = O(1/\varphi(n))$, it follows from Borel-Cantelli lemma that if $\sum 1/\varphi(n)$ converges, then $a_n(\omega) > \varphi(n)$ occurrs finitely often, except on a set of measure 0 (P or λ). If the events $\{E_n\}$, or the functions $a_n(\omega)$, were independent, the statement of Borel-Cantelli lemma in divergent case (i. e. $\sum 1/\varphi(n) = \infty$) would follow immediately. But by easy calculations, we see that the events E_n occur infinitely often for almost all ω if $\sum 1/\varphi(n) = \infty$. Our subject is to give a quantitative statement to the term "infinitely often" with the help of the preceding theorems.

In order to apply the theorem A and B, the following lemma is necessary. By M_{ab} we denote the σ-algebra generated by the fundamental intervals of rank k with $a \leq k \leq b$. Then we have

Lemma. For any sets $A \in M_{1t}$ and $B \in M_{t+n\ \infty}$ we have

(20) $$P(AB) - P(A)P(B) \ll P(A)P(B)\rho^n,$$

where $\rho < 1$ and the constants implied by $\ll$ are namerical.

We are now in a position to apply the theorem A and B to obtain

Theorem 1. Let A (N, ω) be the number of integers $n \leq N$ satisfying $a_n(\omega) \geq \varphi(n)$. Let $\varphi(n) \to \infty$ be a sequence of integers with $\sum 1/\varphi(n) = \infty$. Put

(21) $$\Phi(N) = \frac{1}{\log 2} \sum_{n \leq N} \log\left(1 + \frac{1}{\varphi(n)}\right).$$

Then for any probability measure ν, absolutely continuous with respect to the Lebesque measure λ, it holds

(22) $$\nu\left\{\omega\ ;\ \frac{A(N,\omega) - \Phi(N)}{\sqrt{\Phi(N)}} < \alpha\right\} \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} \exp(-t^2/2)\, dt$$

Moreover, for almost all

$$(23)\qquad \limsup_{N\to\infty} \frac{|A(N,\omega) - \Phi(N)|}{\sqrt{2\Phi(N)\log\log\Phi(N)}} = 1.$$

The order in their remainder terms depends on the measure ν .

By using almost the same procedure, we have

Theorem 2. For almost all ω,

$$(24)\qquad \limsup_{N\to\infty} \frac{\log q_N(\omega) - N\pi^2/(12\log 2)}{\sqrt{2\sigma^2 N \log\log N}} = 1$$

where

$$(25)\qquad \sigma^2 = \lim N^{-1}\int_0^1 (\log q_N(x) - N\pi^2/(12\log 2))^2 \frac{dx}{(1+x)\log 2} > 0.$$

Moreover

$$(26)\qquad P\left\{\frac{1}{\sigma\sqrt{N}}(\log q_N - N\pi^2/(12\log 2)) < \alpha\right\}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} \exp(-t^2/2)\,dt + O\left(\frac{\log\log N}{\log N}\right)^{1/2}.$$

A slight modification of theorem 1 is as follows

Theorem 3. Let f(k) be a function satisfying

$$(27)\qquad f(k) = O(k^{\frac{1}{2}-\varepsilon})$$

with some $\varepsilon > 0$. Put

$$(28)\qquad K = \frac{1}{\log 2}\sum f(k)\log\frac{(k+1)^2}{k(k+2)}.$$

Then we have ,

$$\limsup_{N \to \infty} \frac{\sum_{k=1}^{N} f(a_k(\omega)) - NK}{\sqrt{N \log \log N}} = \sigma \tag{29}$$

for almost all ω, σ being some constant.

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