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Generalized Unique Continuation Property for Hyperfunctions with Real Analytic Parameters

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In these notes we give some elementary lemmas on hyperfunctions, especially on those with real analytic parameters. These are expressed in terms of local operators with constant coefficients. As for the local operators we refer the reader to [1], of which some results are cited here.

Lemma 1 Let $u \in B_*(R^n)$ be a hyperfunction with compact support. Then $\operatorname{supp} u \not\ni 0$ if and only if for every local operator $J(D)$ with constant coefficients we have

$$\lim_{\varepsilon \downarrow 0} \int_{R^n} J(\xi)\tilde{u}(\xi)e^{-\varepsilon|\xi|}d\xi = 0.$$

Here $\tilde{u}(\xi) = F[u] = \int e^{ix\cdot\xi}u(x)dx$ denotes the Fourier transform, and $|\xi| = (\xi_1^2+\dots+\xi_n^2)^{1/2}$, $x\cdot\xi = x_1\xi_1+\dots+x_n\xi_n$.

Proof Necessity If $\operatorname{supp} u \not\ni 0$, then $\operatorname{supp} J(D)u \not\ni 0$ for every $J(D)$. Recall that $E(x,\varepsilon) = F^{-1}[e^{-\varepsilon|\xi|}]$ is the Poisson kernel for the boundary value problem:

$$\begin{cases} (\Delta_x + \partial^2/\partial\varepsilon^2)E(x,\varepsilon) = 0, \\ E(x,0) = \delta(x). \end{cases}$$

When $\varepsilon \downarrow 0$, $E(x,\varepsilon)$ converges to zero uniformly for x on some complex neighborhood of any real compact set K which does not contain the origin. Taking $\operatorname{supp} J(D)u$ as the set K, we have

$$\lim_{\varepsilon \downarrow 0} \langle E(x,\varepsilon),\ J(D)u(x)\rangle = 0,$$

where $\langle\ ,\ \rangle$ denotes the pairing between $A(K)$ and $B[K]$, and also the one between $A(R^n) \cap Q$ and $B_*(R^n)$ for $\varepsilon > 0$, Q denoting the space of Fourier hyperfunctions. Hence by the Parseval formula we conclude that

$$\lim_{\varepsilon \downarrow 0} \int_{R^n} e^{-\varepsilon|\xi|} J(\xi)\tilde{u}(\xi)d\xi = 0.$$

<u>Sufficiency</u> By the Parseval formula we have

$$\int_{R^n} J(\xi)\tilde{u}(\xi)e^{-\varepsilon|\xi|}d\xi = F^{-1}[J(\xi)\tilde{u}(\xi)e^{-\varepsilon|\xi|}](0)$$

$$= J(D)(u(x) * E(x,\varepsilon))\big|_{x=0}.$$

Put $u_k(x) = u(x) * E(x,1/k)$. Our assumption implies that $\{u_k(x)\}$ is a converging sequence in $A_{\underline{J}}(\{0\})$. Here $A_{\underline{J}}(K)$ denotes the space of real analytic functions f on K endowed with the seminorms $\|f\|_J = \sup_{x \in K} |J(D)f(x)|$, J running over all the local operators with constant coefficients (see [1], Definition 2.1). Thus it converges uniformly on some complex neighborhood of the origin ([1], Proposition 2.4). Thus the limit function is real analytic in a neighborhood of the origin. On the other hand, $u(x)$ is the boundary value of $u(x,\varepsilon) = u(x) * E(x,\varepsilon)$ with respect to the operator $\Delta_x + \partial^2/\partial\varepsilon^2$. Since the operation of taking the boundary value to a non-characteristic surface is of local character (see [3]), we conclude that $u(x)$ agrees with the above limit in that neighborhood of the origin. Thus $u(x)$ itself is analytic there. Finally letting $J(D)$ run all the finite order derivatives we conclude that $(\partial/\partial x)^{\alpha}u(0) = 0$ for any α, hence $\operatorname{supp} u \not\ni 0$. q.e.d.

We can slightly rewrite the result:

<u>Lemma 2</u> Let $u \in B_*(R^n)$. Then $\operatorname{supp} u \cap \{x_n = 0\} = \emptyset$ if and only if for every local operator $J(D)$ with constant coefficients and for any fixed $x' \in R^{n-1}$, the finite limit

$$\lim_{\varepsilon \downarrow 0} \int_{R^n} J(\xi)\tilde{u}(\xi)e^{-\varepsilon|\xi|-ix'\xi'}d\xi$$

exists.

<u>Proof</u> From the proof of Lemma 1, we see that $u(x)$ is real analytic in some neighborhood of every point $(x',0) \in R^n$. Since $u(x)$ has compact support, we conclude that $\operatorname{supp} u \cap \{x_n = 0\} = \emptyset$ due to the uniqueness of analytic continuation.

q.e.d.

A more refined form of this lemma will be effectively employed in the forthcoming paper for the study of linear exceptional sets of real analytic solutions of partial differential equations with constant coefficients.

The last one concerns the unique continuation property for the real analytic parameters. We say that $u(x)$ contains x_n as a real analytic parameter if $S.S.u$ (the singular spectrum of $u(x)$) does not contain $\pm idx_n$ on each fibre of $iS^*_\infty R^n$. In this case we can restrict the hyperfunction $u(x)$ to the hyperplane $\{x_n = 0\}$. For the details see [4] or [5].

<u>Lemma 3</u> Let u be a hyperfunction defined on a cylindrical domain $U \times I$, where $U \subset R^{n-1}_{x'}$ is open and $I \subset R^1_{x_n}$ is an open interval containing zero. Assume that u contains x_n as a real analytic parameter and for every local operator $J(D)$ with constant coefficients it satisfies

$$J(D)u(x)\big|_{x_n=0} = 0.$$

Then $u \equiv 0$ on a neighborhood of $U \times \{0\}$.

<u>Sketch of Proof</u> Without loss of generality we can assume that U contains the origin of $R^{n-1}_{x'}$. Let $K \subset U$ be a compact set whose interior $\overset{\circ}{K}$ contains the origin of $R^{n-1}_{x'}$. By the flabbiness of the sheaves C and Q, and by the vanishing of cohomology groups of the sheaf P_* of rapidly decreasing real

analytic functions, we can find a Fourier hyperfunction $v(x)$ on $D^{n-1} \times I$, which contains x_n as a real analytic parameter, is real analytic and rapidly decreasing outside $K \times I$, and on $\overset{\circ}{K} \times I$ differs from $u(x)$ by a real analytic function $f(x)$. (For such argument see [2].) We are going to prove that v is, hence u is real analytic in a neighborhood of the origin. Since v contains x_n as a real analytic parameter, the functions

$$v_\varepsilon(x) = v(x) *' F^{-1}[e^{-\varepsilon|\xi'|}]$$

are real analytic in $R^{n-1}_{x'} \times I$ in the whole variables, where $*'$ denotes the convolution with respect to x'. Now for any local operator $J(D)$ with constant coefficients we have

$$J(D)v_\varepsilon(0) = \{(J(D)v(x)|_{x_n=0}) *' F^{-1}[e^{-\varepsilon|\xi'|}]\}|_{x'=0}.$$

Here by the assumption on $u(x)$, $J(D)v(x)|_{x_n=0}$ is real analytic outside K, and rapidly decreasing at infinity. Recalling that $F^{-1}[e^{-\varepsilon|\xi'|}]$ converges uniformly to zero on a complex neighborhood of K and converges to $\delta(x')$ in D', we conclude, as in the proof of Lemma 1 but with more delicacy, that $J(D)v_\varepsilon(0)$ converges to a finite value when $\varepsilon \longrightarrow 0$. We proceed in the same way as in Lemma 1. Again employing [1], Proposition 2.4 (after replacing ε by $1/k$), and considering that the limit takes place in the local sense, we conclude that $v(x)$ is, hence $u(x)$ is analytic in some neighborhood of the origin, hence $u(x)$ is identically equal to zero there.

Since the origin can be replaced by any other points of $U \times \{0\}$, we have proved $u \equiv 0$ in a neighborhood of $U \times \{0\}$.

q.e.d.

<u>Remark</u> If $\operatorname{supp} u \subset K \times I$, where $K \subset U$ is compact, then we can easily prove that $u \equiv 0$ in $U \times I$ if and only if $(\partial/\partial x_n)^k u|_{x_n=0} = 0$ for $k = 0,1,2,\ldots$. On the other hand, for a general hyperfunction the latter condition does not imply $u \equiv 0$. The following is a famous counter-example by M. Sato (unpublished): Let $P_n(z)$ be the polynomials in one variable which approximate $1/z$ locally uniformly outside the nsgative real axis; namely there exists a sequence of compact subsets $K_1 \subset K_2 \subset \ldots$, $\bigcup K_n = C \setminus]-\infty, 0]$, and a decreasing sequence of positive numbers ε_n such that

$$\left| \frac{1}{z} - P_n(z) \right| < \varepsilon_n; \quad \text{if} \quad z \in K_n.$$

Further writing $\delta_n = \operatorname{dist}(0, K_n)$ we can assume that $\sqrt[n]{\delta_n} \longrightarrow 0$ if $n \longrightarrow \infty$. Then $F(z,t) = \sum_{n=0}^{\infty} P_n(z)t^n$ defines a holomorphic function in $(C \setminus]-\infty, 0]) \times \{t \in C;\ |t|<1\}$. The associated hyperfunction $f(x,t) = F(x+i0,\ t+i0) - F(x-i0,\ t+i0)$ contains t as a holomorphic parameter; every finite derivative vanishes when $t = 0$; but supp $f(x,t)$ contains the origin.

References

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