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Micro-hyperbolic pseudo-differential operators *

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Study of (fundamental solutions of) hyperbolic differential equations has a long history. See for example Riemann [1], Hadamard [1], Courant-Hilbert [1] and references cited in Courant-Hilbert [1], [2]. In such a long history the works of Petrowsky [1] and Gårding [1] are clearly outstanding milestones from the view point of the general theory of differential equations because of the generality of their results. Leray [1] has also influenced much the later development of the theory of hyperbolic differential equations by establishing the existence and uniqueness theorems of the solutions in an elegant and far-reaching way. See also Friedrichs-Lewy [1] and Friedrichs [1]. On the other hand Hörmander [1], [2] gave good existence and uniqueness theorems for real operators of principal type along these lines. One of the reasons for the success of Hörmander seems to us to be the fact that such operators are

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micro-hyperbolic. The notion of micro-hyperbolicity was introduced in Kashiwara-Kawai [1] in full generality. Hereafter Kashiwara-Kawai [1] will be often referred to as K-K for short. In the case of linear differential operators with constant coefficients the same notion was introduced by Andersson [1] influenced by the work of Atiyah-Bott-Gårding [1]. (See Kawai [2] and Gårding [2].) In most of the above quoted papers the fundamental solution plays its essential role. As for the construction and investigation of the fundamental solutions for hyperbolic operators or real operators of principal type, we refer to Courant-Lax [1], Lax [1], Leray [1], [2], Ludwig [1], Mizohata [1], Hörmander [3], Kawai [1] and Duistermaat-Hörmander [1]. Note that these works assume the regularity of characteristic variety of the operator. In order to treat the operators whose characteristic variety is not simple, the employment of hyperfunctions is very crucial as is shown by Bony-Schapira [1], [2]. (See Mizohata [2], where a necessary condition for hyperbolicity is discussed. See also Leray-Ohya [1], Mizohata-Ohya [1], Chazarain [1] and the references cited there about the operators with constant multiple characteristics.)

The purpose of this report is to explain in a sketchy way the idea of Kashiwara-Kawai [1], whose results cover all the above quoted (local) existence and uniqueness theorems as long as the operators under consideration have analytic coefficients. Note that the theory developed in Kashiwara-Kawai [1] is also related to Egorov [1], [2], Nirenberg-Treves [1] and Treves [2].

(Theorem 3 in the below.)

The topics of this report are completely restricted to the existence and uniqueness of the (fundamental) solutions of linear (pseudo-) differential equations and other topics of hyperbolic equations are not discussed here, though some of them are important and also expected to be closely related to the topics discussed here, e.g. hyperbolic mixed problems. (As for such problems we refer to the exposition of Chazarain [2] for example.)

Now we will sketch the idea of the proof of the existence of fundamental solutions for partially micro-hyperbolic pseudo-differential operator $P(x,D_x)$.

A pseudo-differential operator $P(x,D_x)$ is said to be partially micro-hyperbolic at $(x^0, i\xi^0)\in\sqrt{-1}S^*M$ with respect to the direction $\langle\vartheta,dx\rangle+\langle\rho,d\xi\rangle$ if $p_m(x+i\varepsilon\rho, i\xi+\varepsilon\rho)\neq 0$ for every (x,ξ) sufficiently close to (x^0, ξ^0) and for $0<\varepsilon\ll 1$. (See K-K §1 for the precise definition. See Gårding [2], [3] also.) Then using the "quantized" contact transformation (Sato-Kawai-Kashiwara [1] Chapter II §3.3) we can easily reduce the problem to the case where P has a matrix form $D_{x_1}-A(x,D')$ so that A is a square matrix of pseudo-differential operators of order $\leqq 1$ which commute with x_1 and that all eigenvalues of its principal symbol $A_1(x, i\xi')$ have non-negative real part. (The above quoted report of Sato, Kawai and Kashiwara will be referred to S-K-K [1] hereafter.)

The first step in our arguments is to construct formally a solution $R(x,D')=\Sigma\, a_\alpha(x)D'^\alpha$ as an infinite sum of pseudo-differential operators so that $PR=0$ and that $R|_{x_1=0}=1$. This part of the proof is not difficult to perform. In fact we need not use the assumption of partial micro-hyperbolicity of P at this stage. Cf. Treves [1]. What we need here is that $\{x_1=0\}$ is non-characteristic with respect to P. (See Proposition 2.2 in K-K §2.) The essential difficulty comes in at the next step. Generally the existence domain of R is so small that G cannot be endowed with the meaning as a kernel microfunction of a fundamental solution of P. In order to overcome this difficulty we rewrite the equation $PR=0$ by using the "defining function" of P and R. (Lemma 4.1 in K-K §4.) Then we try to extend the domain of definition of the defining function G of R by the partial micro-hyperbolicity of P. In extending the domain of definition the following Lemma 1 is crucial. Note that the partial micro-hyperbolicity of $P=D_{x_1}-A(x,D')$ at $(x_1,x'; i(\xi_1,\xi')\infty)=(0,x^{0'}; i(\xi_1,\xi^{0'})\infty)=(0; i(\xi_1,0,\dots,0,1)\infty)$ for every real ξ_1 with respect to the direction x_1 implies that

$$g(x_1,z',\zeta_1,\zeta') = \det(\zeta_1 - A_1(t,z',\zeta'))$$

never vanishes on

$$\{(x_1,z',\zeta_1,\zeta')\in \mathbb{R}\times \mathbb{C}^{n-1}\times \mathbb{C}\times \mathbb{C}^n;\quad 0\leqq x_1\leqq \delta,$$
$$|z'|<\delta,\ |(\zeta_2,\dots,\zeta_{n-1})|=|\zeta''|<\delta|\zeta_n|,\ -\mathrm{Im}(\zeta_1/\zeta_n)$$
$$> M(|y| + \sum_{\nu=2}^{n-1} |\mathrm{Im}(\zeta_\nu/\zeta_n)|.\}$$

Here the assumption of the partial micro-hyperbolicity of P for every real ξ_1 is not restrictive in application because we can easily localize the problem with respect to ξ_1 by the preparation theorem of Weierstrass for pseudo-differential operators (S-K-K [1] Chapter II §2.2.) See the arguments in the proof of Theorem 5.2 in K-K §5 for details.

After the above observation concerning the implication of partial micro-hyperbolicity, we state the following lemma.

Lemma 1. Let $\varphi(x_1, z', \bar{z}')$ be a positive valued real analytic function defined on $U=\{(x_1,z')=(x_1,x'+iy')\ ;\ 0<x_1<\delta_1,$ $|x'|<\delta_2,\ |y'|<\delta_3$ with $\delta_2{}^2+\delta_3{}^2<\delta_1{}^2\}$. Assume that φ satisfies the following:

(1) $$\frac{\partial\varphi}{\partial x_1} > M(|y'| + \varphi + \sum_{\nu=2}^{n-1}|\frac{\partial\varphi}{\partial x_\nu}|)$$

(2) $$\sum_{\nu=2}^{n-1}|\frac{\partial\varphi}{\partial z_\nu}| < \frac{\delta}{2} \quad \text{on } U.$$

Suppose that G can be extended to $V=\{z;\ 0<x_1<\delta_1,$ $|z'|<\delta_2,\ y_n>\varphi(x_1,z',\bar{z}')\}$. Then G can be extended to a holomorphic function defined on an open set V' which contains

$$\{z;\ 0 < x_1 < \delta_1,\ \ |z'| < \delta_2,\ \ y_n \geq \varphi(x_1,z',\bar{z}')\}.$$

Once we have proved this lemma, we can easily prove the following Theorem 2 by a suitable choice of φ, while the proof of the above lemma is reduced to the invertibility of elliptic pseudo-differential operator. (See S-K-K [1] Chapter II §2.1 Theroem 2.1.1. See also the exposition of Kawai of this issue.)

Theorem 2. There exist $\delta_0>0$ and M_1 such that $G(x_1,z')$ is holomorphic

$$\{(x_1,z')\in \mathbb{R}\times\mathbb{C}^n;\quad 0<x_1<\delta_0,\quad |z'|<\delta_0,$$
$$\operatorname{Im} z_n > M_1 x_1\left(\sum_{\nu=2}^{n-1}|\operatorname{Im} z_\nu|\right)\}$$

As for the details of the proof of Lemma 1 and Theorem 2 we refer to K-K §4.

Now Theorem 2 allows us to define the boundary value $w(x)$ of a hyperfunction $G^+(x_1,z')=Y(x_1)G(x_1,z')$ with holomorphic parameters z' defined on

$$\{(x_1,z');\quad |x_1|,\ |z'|<\delta,\quad \operatorname{Im} z_n > M|x_1|\left(\sum_{\nu=2}^{n-1}|\operatorname{Im} z_\nu|\right)\}.$$

(See S-K-K Chapter I §3.2 about the notion of taking the boundary value of hyperfunctions with holomorphic parameters.)

It is readily verified that the singular spectrum $u(x)$ of $w(x)$ satisfies $Pu=\delta(x)$ and that $\operatorname{Supp} u\subset\{(x;\, i\xi\infty);\ x_1\geqq 0,$ $|\xi_\nu|\leqq Mx_1|\xi_n|\ (\nu=2,\dots,n-1),\ |x_n|\leqq vx_1\}$. Using this fact one can easily show the existence of fundamental solution of the partially micro-hyperbolic pseudo-differential operator P. (See Theorem 5.2 in K-K §5.)

Once one gets a fundamental solution, it is easy to show the existence or (propagation of) regularity of solutions. The results are listed up in §6 of K-K and we omit the details here. However, we would like to touch the following theorem without proof. This theorem will show why we have treated the *partial* micro-hyperbolic operators, not the micro-hyperbolic operators. In fact, $D_{x_1}+ix_1^{2k}D_{x_2}$ (Mizohata [3]), the easiest and most typical example that can be covered by Theorem 3, is not micro-hyperbolic, though it is partially micro-hyperbolic. (See Sato-Kawai-Kashiwara [2] also.)

Theorem 3. Assume that the real characteristic variety V of $P(x,D_x)$ is defined by $a(x,\eta)+\sqrt{-1}b(x,\eta)=0$ where $(\sqrt{-1})^{-m}a(x,\sqrt{-1}\eta)$ and $(\sqrt{-1})^{-m}b(x,\sqrt{-1}\eta)$ are real for $(x,\sqrt{-1}\eta)$ in $\sqrt{-1}S^*M$ near $x_0^*=(x_0,\sqrt{-1}\eta_0)$ and that $\mathrm{grad}_{(x,\eta)}a(x,\eta)$ and ω are linearly independent there. (Here m denote the degree of a and b with respect to η.) Assume further that $(\sqrt{-1})^{-m}b(x,\sqrt{-1}\eta)$ is positive (or negative) on each real bicharacteristic strip of $(\sqrt{-1})^{-m}a(x,\sqrt{-1}\eta)$ and not identically zero there. Then $P(x,D_x)$ has an inverse in the ring of micro-local operators.

We refer to S-K-K [1] Chapter I §2.5 and the exposition of Kawai of this issue about the notion of micro-local operators. Note that the above theorem implies not only the micro-local solvability of the equation $Pu=f$ but also the "micro-local" analytic-hypoellipticity of P. We also note that a more general result is given in K-K. (Theorem 6.6 in §6.)

At the end of this exposition the speakers would like to lay stress on the following point as a summary:

The employment of hyperfunctions and microfunctions has made the theory of linear hyperbolic differential equations very lucid and thrown the light to the nature of a class of hypoelliptic operators from the view-point of "hyperbolicity." The essential idea in showing these is "taking the boundary value of pseudo-differential operators defined in the complex domain." In fact $P(x,D_x)$ is invertible when $p_m(x,\eta)\neq 0$ and the partial micro-hyperbolicity of $P(x,D_x)$ means the invertibility of P on a conical set which is tangent to the real axis. Therefore what we have done may be summarized as a justification of the procedure of "taking the boundary value of pseudo-differential operators."

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