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Cartesian product of a homotopy 4-sphere with E^1

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§0. In this paper we will show that $H^4 \times E^1$ is PL homeomorphic to $S^4 \times E^1$ where H^4 is a homotopy 4-sphere which is a PL manifold and E^1 is an 1-dim. euclidean space. It is an alternating proof of [9. Th. 6], [10. p. 67]. Throughout this paper we consider PL category of polyhedra and piecewise linear maps (see [8]) if otherwise is stated. E^n, S^n, D^n always mean n-dimensional euclidean space, n-dim. PL sphere and PL ball.

§1.

Proposition 1. Let Σ^4 be a PL 4-sphere which is locally flat PL embedded in S^5. Then M, the closure of one of the complement of Σ^4 in S^5, is a PL 5-ball.

Proof. Since Σ^4 is PL locally flat embedded in S^5, M is a PL manifold which is (TOP) homeomorphic to D^5 [1], [2]. And $\partial M = \Sigma^4$ is a (standard) PL 4-sphere. So $(p*\partial M) \cup M$ is a PL manifold which is homeomorphic to S^5. Then by the uniqueness of PL structure on S^5 [4], $(p*\partial M) \cup M$ is a PL 5-sphere and hence $M \underset{PL}{\cong} S^5 - \mathrm{Int}\, st(v, S^5)$ is a (standard) PL 5-ball.

Proposition 2 [3, p 89]. Let K be a closed PL subspace in the interior of a PL manifold M. Then there exists a regular neighborhood of K in M which is unique up to ambient isotopy keeping K fixed.

Lemma 1. Let $f : S^3 \times E^1 \to E^5$ be a locally flat PL embedding satisfying the following condition; for any 5-ball $B^5 \subset E^5$ containing $f(S^3 \times \{0\})$ in its interior there is a positive number $s = s(B)$ such that $f(S^3 \times ((-\infty,-s] \cup [s,\infty))) \cap B^5 = \phi$. Then $(f(S^3 \times \{0\}) \subset E^5)$ is a PL trivial knot. And there is a locally flat PL embedding $g : D^4 \to E^5$ of 4-ball D^4 such that $g(\partial D^4) = f(S^3 \times \{0\})$, $g(\mathrm{Int}\ D^4) \cap f(S^3 \times E^1) = \phi$.

Proof. Let $B^5 \subset E^5$ be a 5-ball with $\mathrm{Int}\ B^5 \supset f(S^3 \times \{0\})$. Then by the assumption there is a $s = s(B^5) > 0$ such that

$$f(S^3 \times ((-\infty,-s] \cup [s,\infty))) \cap B^5 = \phi.$$

So $f(\{x\} \times [0,s]) \cap \partial B^5 = f(\{x\} \times \{s_1\}) \cup \cdots \cup f(\{x\} \times \{s_m\})$ where $x \in S^3$, $0 < s_1 < \cdots < s_m < s$ and $m = 2p + 1$. Now if B_1^5 is a 5-ball in E^5 with $\mathrm{Int}\ B_1^5 \supset B^5 \cup f(S^3 \times [0,s])$, by the assumption there is a $t = t(B_1) > 0$ such that

$$f(S^3 \times ((-\infty,-t] \cup [t,\infty))) \cap B_1^5 = \phi.$$

Since $f(\{x\} \times (s_{2r-1}, s_{2r})) \cap B^5 = \phi$, $1 \leqq r \leqq p$, we take a simple are γ_r on ∂B^5 joining $f(\{x\} \times \{s_{2r-1}\})$ with $f(\{x\} \times \{s_{2r}\})$ where $\gamma_i \cap \gamma_j = \phi$ $(i \neq j)$. Then the simple closed curve $f(\{x\} \times [s_{2r-1}, s_{2r}]) \cup \gamma_r$ is homotopic to constant in $B_1^5 - \mathrm{Int}\ B^5 \cong S^4 \times I$. Then using general position technique there are non-singular 2-balls δ_r $(1 \leq r \leq p)$ such that

① $\mathrm{Int}\ \delta_r \subset B_1^5 - \mathrm{Int}\ B^5$

② $\partial\delta_r = f(\{x\} \times [s_{2r-1}, s_{2r}]) \cup \gamma_r$

③ $\delta_r \cap B^5 = \gamma_r$

④ $\delta_i \cap \delta_j = \phi$ $(i \neq j)$.

Using δ_r $(1 \leq r \leq p)$ we can engulf $f(\{x\} \times [s_{2r-1}, s_{2r}])$ into B^5 by an ambient isotopy i.e. there is a level preserving PL homeomorphism $F : E^5 \times I \to E^5 \times I$ such that $F|(E^5-B_1^5) \times I = \mathrm{id.}$, $F_0 = \mathrm{id.}$ and $F_1 f(\{x\} \times [s_{2r-1}, s_{2r}]) \subset \mathrm{Int}\ B^5$ $(1 \leq r \leq p)$.
Then

$$F_1 f(\{x\} \times [0,s]) \cap \partial B^5 = F_1 f(\{x\} \times [0,t]) \cap \partial B^5$$
$$= F_1 f(\{x\} \times \{s_m\}).$$

Let

$$F_1 f(S^3 \times \{0\}) \cup F_1 f(N(x) \times [0,t]) \cup F_1 f(S^3 \times \{t\})$$
$$- F_1 f(\mathrm{Int}\ N(x) \times [0, t]) = \Sigma^3.$$

Then $(\Sigma^3 \subset E^5)$ is a knot which is the sum of the knots $(f(S^3 \times (0)) \subset E^5)$ anf $(f(S^3 \times (t)) \subset E^5)$ using ∂B^5 and it is trivial because Σ^3 bounds a 4-ball $f((S^3 - \mathrm{Int}\ N(x)) \times [0,t])$ in E^5. So $(f(S^3 \times \{0\}) \subset E^5)$, $(f(S^3 \times \{t\}) \subset E^5)$ are both topologically trivial by [5] and then piecewise linearly trivial by [7].

Now we define an embedding $g : D^4 \to E^5$ satisfying $g(\partial D^4) = f(S^3 \times \{0\})$ and $g(\mathrm{Int}\ D^4) \cap f(S^3 \times E^1) = \phi$. Since $(f(S^3 \times \{0\}) \subset E^5)$ is trivial, $f(S^3 \times \{0\})$ bounds a locally flat 4-ball B_0^4

in E^5 and $(f(S^3\times\{t\})\subset E^5)$ is trivial for any t by using the infinite cylinder $f(S^3 \times E^1)$. Furthermore $(f(S^3\times\{0\})\cup f(S^3\times\{t\}) \subset E^5)$ is a split link by the assumption for f. So $(f(S^3\times\{0\}) \cup f(S^3\times\{t\})\subset E^5)$ is a trivial link for any $t \in E^1$ and there is $\varepsilon > 0$ such that $f(S^3 \times [-\varepsilon,\varepsilon]) \cap \mathrm{Int}\, B_0^4 = \phi$. And hence for any $t > 0$ there is a 4-ball B_t^4 in E^5 such that $\partial B_t^4 = f(S^3 \times \{0\})$ and $\mathrm{Int}\, B_t^4 \cap f(S^3 \times [-t, t]) = \phi$. So there is a 4-ball B^4 in E^5 satisfying $\partial B^4 = f(S^3 \times \{0\})$, $\mathrm{Int}\, B^4 \cap f(S^3 \times E^1) = \phi$. We may define $g : D^4 \to E^5$ by $g(D^4) = B^4$.

Let H^4 be a homotopy 4-sphere which is a PL manifold and $V^4 = H^4 - \mathrm{Int}\ \sigma^4$ where σ^4 is a 4-simplex.

Lemma 2. If $f : S^3 \times E^1 \to \partial V^4 \times E^1$ is a PL homeomorphism, there is a PL homeomorphism $g : D^4 \times E^1 \to V^4 \times E^1$ which is an extension of f.

Proof. Let $\tilde{c}_1 : \partial D^4 \times I \to D^4$, $\tilde{c}_2 : \partial V^4 \times I \to V^4$ ($I = [0, 1]$) be boundary collars i.e. $\tilde{c}_1$, $\tilde{c}_2$ are embeddings such that $\tilde{c}_1(x, 0) = x$ $(x\in \partial D^4)$ and $\tilde{c}_2(y, 0) = y$ $(y \in \partial V^4)$. And let $c_1 : \partial D^4 \times I \times E^1 \to D^4 \times E^1$, $c_2 : \partial V^4 \times I \times E^1 \to V^4 \times E^1$ be $c_1(x, s, t) = (\tilde{c}_1(x,s), t)$, $c_2(y, s, t) = (\tilde{c}_2(y,s), t)$. Let $f_1 : c_1(\partial D^4 \times I \times E^1) \to c_2(\partial V^4 \times I \times E^1)$ be $f_1c_1(p, s, t) = c_2(p', s, t')$ where $f_1c_1(p, 0, t) = fc_1(p, 0, t) = c_2(p', 0, t')$. Since $\mathrm{Int}\, V^4 \times E^1 \cong E^5$ by [6], let $\mathcal{f} : \mathrm{Int}\, V^4 \times E^1 \to E^5$ be a PL homeomorphism. Then $\mathcal{f} f_1c_1|\partial D^4 \times \{1\} \times E^1 : \partial D^4 \times \{1\} \times E^1$

$\to E^5$ satisfies the condition for f of Lemma 1, i.e. for any 5-ball $B^5 \subset E^5$ containing $\mathcal{f} f_1 c_1(\partial D^4 \times \{1\} \times \{0\})$ in its interior there is a positive number $s = s(B)$ such that $\mathcal{f} f_1 c_1(\partial D^4 \times ((-\infty, -s] \cup [s,\infty))) \cap B^5 = \phi$. Because $\mathcal{f}^{-1}(B^5) \subset \mathrm{Int}\, V^4 \times (-s', s')$ for some $s' > 0$ and $\mathcal{f}^{-1}(B^5) \cap c_2(\partial V^4 \times \{1\} \times ((-\infty,-s'] \cup [s', \infty))) = \phi$. Hence there is a number $s > 0$ such that

$$\mathcal{f}^{-1}(B^5) \cap f_1 c_1(\partial D^4 \times \{1\} \times ((-\infty,-s] \cup [s,\infty))) = \phi$$

and so

$$B^5 \cap \mathcal{f} f_1 c_1(\partial D^4 \times \{1\} \times ((-\infty,-s] \cup [s,\infty))) = \phi.$$

So by Lemma 1 $(\mathcal{f} f_1 c_1(\partial D^4 \times \{1\} \times \{0\}) \subset E^5)$ is a trivial knot and it bounds a locally flat 4-ball $\widetilde{B}_0^4$ with $\mathrm{Int}\, \widetilde{B}_0^4 \cap \mathcal{f} f_1 c_1(\partial D^4 \times \{1\} \times E^1) = \phi$. So $f_1 c_1(\partial D^4 \times \{1\} \times \{0\})$ bounds a locally flat PL 4-ball $B_0^4 = \mathcal{f}^{-1}(\widetilde{B}_0^4)$ such that

$$\mathrm{Int}\, B_0^4 \cap f_1 c_1(\partial D^4 \times \{1\} \times E^1) = \mathrm{Int}\, B_0^4 \cap c_2(\partial V^4 \times \{1\} \times E^1) = \phi.$$

Similary we may assume there are 4-balls B_t^4 ($t \in Z$: integer) such that $\partial B_t^4 = f_1 c_1(\partial D^4 \times \{1\} \times \{t\})$ and $\mathrm{Int}\, B_t^4 \cap c_2(\partial V^4 \times \{1\} \times E^1) = \phi$. And we may assume $B_t^4 \cap B_{t+1}^4 = \phi$ $(t \in Z)$. So we can extend f_1 to

$$f_2 : c_1(\partial D^4 \times I \times E^1) \cup (D^4 \times Z) \longrightarrow c_2(\partial V^4 \times I \times E^1) \cup \bigcup_{t \in Z} B_t^4$$

by a cone extension. Since

$f_1 c_1(\partial D^4 \times \{1\} \times [t,t+1]) \cup B_t^4 \cup B_{t+1}^4$ $(t \in Z)$ is a PL 4-sphere which is locally flat embedded in $\mathrm{Int}\, V^4 \times E^1 \cong E^5$, it bounds a PL 5-ball B_t^5 by Proposition 1. So we can extend f_2 to a required PL homeomorphism

$$g : D^4 \times E^1 \to c_2(\partial V^4 \times I \times E^1) \cup \bigcup_{t\in Z} B_t^5 \ (= V^4 \times E^1)$$

by a cone extension.

Theorem. $H^4 \times E^1$ is PL homeomorphic to $S^4 \times E^1$ where H^4 is a homotopy 4-sphere which is a PL manifold.

Proof. Since any regular neighborhood N of $p \times E^1$ in $H^4 \times E^1$ is PL homeomorphic to $D^4 \times E^1$, we identify $N = D^4 \times E^1 \subset H^4 \times E^1$ using Proposition 2. So $H^4 \times E^1 - \operatorname{Int} N$ is PL homeomorphic to $V^4 \times E^1$. Let $\mathscr{S}_1 : D^4 \times E^1 \to N$ be a PL homeomorphism. We may consider $S^4 \times E^1 = (D^4 \times E^1) \bigcup_{\partial} (D^4 \times E^1)$ where the one of $D^4 \times E^1$ is a regular neighborhood of $q \times E^1$ for some $q \in S^4$. Then by Lemma 2 we can extend $\partial\mathscr{S}_1 = (\mathscr{S}_1 | \partial D^4 \times E^1)$ to another $D^4 \times E^1$ and so we can get a PL homeomorphism $\mathscr{S} : S^4 \times E^1 \to H^4 \times E^1$.

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