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Cartesian product of a homotopy 4-sphere with $E^1$

By
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§0. In this paper we will show that $H^4 \times E^1$ is PL homeomorphic to $S^4 \times E^1$ where $H^4$ is a homotopy 4-sphere which is a PL manifold and $E^1$ is an 1-dim. euclidean space. It is an alternating proof of [9. Th. 6], [10. p. 67]. Throughout this paper we consider PL category of polyhedra and piecewise linear maps (see [8]) if otherwise is stated. $E^n, S^n, D^n$ always mean $n$-dimensional euclidean space, n-dim. PL sphere and PL ball.

§1.

Proposition 1. Let $E^4$ be a PL 4-sphere which is locally flat PL embedded in $S^5$. Then $M$, the closure of one of the complement of $E^4$ in $S^5$, is a PL 5-ball.

Proof. Since $E^4$ is PL locally flat embedded in $S^5$, $M$ is a PL manifold which is (TOP) homeomorphic to $D^5$ [1], [2]. And $\exists M = E^4$ is a (standard) PL 4-sphere. So $(p*\exists M) \cup M$ is a PL manifold which is homeomorphic to $S^5$. Then by the uniqueness of PL structure on $S^5$ [4], $(p*\exists M) \cup M$ is a PL 5-sphere and hence $M \cong S^5 - \text{Int st}(v, S^5)$ is a (standard) PL 5-ball.

Proposition 2 [3, p 89]. Let $K$ be a closed PL subspace in
the interior of a PL manifold M. Then there exists a regular neighborhood of K in M which is unique up to ambient isotopy keeping K fixed.

Lemma 1. Let \( f : S^3 \times E^1 \to E^5 \) be a locally flat PL embedding satisfying the following condition; for any 5-ball \( B^5 \subset E^5 \) containing \( f(S^3 \times \{0\}) \) in its interior there is a positive number \( s = s(B) \) such that \( f(S^3 \times ((-\infty,-s] \cup [s,\infty))) \cap B^5 = \emptyset \). Then \( f(S^3 \times \{0\}) \subset E^5 \) is a PL trivial knot. And there is a locally flat PL embedding \( g : D^4 \to E^5 \) of 4-ball \( D^4 \) such that \( g(\partial D^4) = f(S^3 \times \{0\}) \), \( g(\text{Int } D^4) \cap f(S^3 \times E^1) = \emptyset \).

Proof. Let \( B^5 \subset E^5 \) be a 5-ball with \( \text{Int } B^5 \supset f(S^3 \times \{0\}) \). Then by the assumption there is a \( s = s(B^5) > 0 \) such that
\[
f(S^3 \times ((-\infty,-s] \cup [s,\infty))) \cap B^5 = \emptyset.
\]
So \( f(\{x\} \times [0,s]) \cap \partial B^5 = f(\{x\} \times \{s_1\}) \cup \cdots \cup f(\{x\} \times \{s_m\}) \)
where \( x \in S^3 \), \( 0 < s_1 < \cdots < s_m < s \) and \( m = 2p + 1 \). Now if \( B^5_1 \)
is a 5-ball in \( E^5 \) with \( \text{Int } B^5_1 \supset B^5 \cup f(S^3 \times [0,s]) \), by the assumption there is a \( t = t(B^5_1) > 0 \) such that
\[
f(S^3 \times ((-\infty,-t] \cup [t,\infty))) \cap B^5_1 = \emptyset.
\]
Since \( f(\{x\} \times \{s_{2r-1}, s_{2r}\}) \cap B^5 = \emptyset \), \( 1 \leq r \leq p \), we take a simple are \( \gamma_r \) on \( \partial B^5 \) joining \( f(\{x\} \times \{s_{2r-1}\}) \) with \( f(\{x\} \times \{s_{2r}\}) \) where \( \gamma_1 \cap \gamma_j = \emptyset \) (\( i \neq j \)). Then the simple closed curve \( f(\{x\} \times [s_{2r-1}, s_{2r}]) \) \( \cup \gamma_r \) is homotopic to constant in \( B^5_1 - \text{Int } B^5 \cong S^4 \times I \). Then using general position technique there
are non-singular 2-balls \( \delta_r \) \((1 \leq r \leq p)\) such that

1. \( \text{Int} \; \delta_r \subset B_{l}^5 - \text{Int} \; B^5 \)
2. \( \partial \delta_r = f(\{x\} \times [s_{2r-1}, s_{2r}]) \cup \gamma_r \)
3. \( \delta_r \cap B^5 = \gamma_r \)
4. \( \delta_i \cap \delta_j = \emptyset \) \((i \neq j)\).

Using \( \delta_r \) \((1 \leq r \leq p)\) we can engulf \( f(\{x\} \times [s_{2r-1}, s_{2r}]) \) into \( B^5 \) by an ambient isotopy i.e. there is a level preserving PL homeomorphism \( F : E^5 \times I \rightarrow E^5 \times I \) such that \( F|_{(E^5-B_{l}^5)} \times I = \text{id.} \), \( F_0 = \text{id.} \) and \( F_1 f(\{x\} \times [s_{2r-1}, s_{2r}]) \subset \text{Int} \; B^5 \) \((1 \leq r \leq p)\). Then

\[
F_1 f(\{x\} \times [0,s]) \cap \partial B^5 = F_1 f(\{x\} \times [0,t]) \cap \partial B^5
= F_1 f(\{x\} \times \{s_{m}\}).
\]

Let

\[
F_1 f(S^3 \times \{0\}) \cup F_1 f(N(x) \times [0,t]) \cup F_1 f(S^3 \times \{t\})
- F_1 f(\text{Int} \; N(x) \times [0, t]) = E^3.
\]

Then \((E^3 \subset E^5)\) is a knot which is the sum of the knots \((f(S^3 \times \{0\}) \subset E^5)\) and \((f(S^3 \times \{t\}) \subset E^5)\) using \( \partial B^5 \) and it is trivial because \( E^3 \) bounds a 4-ball \( f((S^3-\text{Int} \; N(x)) \times [0,t]) \) in \( E^5 \). So \((f(S^3 \times \{0\}) \subset E^5)\), \((f(S^3 \times \{t\}) \subset E^5)\) are both topologically trivial by [5] and then piecewise linearly trivial by [7].

Now we define an embedding \( g : D^4 \rightarrow E^5 \) satisfying \( g(\partial D^4) = f(S^3 \times \{0\}) \) and \( g(\text{Int} \; D^4) \cap f(S^3 \times E^1) = \emptyset \). Since \((f(S^3 \times \{0\}) \subset E^5)\) is trivial, \( f(S^3 \times \{0\}) \) bounds a locally flat 4-ball \( B^4_0 \).
in $E^5$ and $(f(S^3 \times \{t\}) \subset E^5)$ is trivial for any $t$ by using the infinite cylinder $f(S^3 \times E^1)$. Furthermore $(f(S^3 \times \{0\}) \cup f(S^3 \times \{t\}) \subset E^5)$ is a split link by the assumption for $f$. So $(f(S^3 \times \{0\}) \cup f(S^3 \times \{t\}) \subset E^5)$ is a trivial link for any $t \in E^1$ and there is $\varepsilon > 0$ such that $f(S^3 \times [-\varepsilon, \varepsilon]) \cap \text{Int } B^n_0 = \phi$. And hence for any $t > 0$ there is a 4-ball $B^n_t$ in $E^5$ such that $\partial B^n_t = f(S^3 \times \{0\})$ and $\text{Int } B^n_t \cap f(S^3 \times [-t, t]) = \phi$. So there is a 4-ball $B^n_4$ in $E^5$ satisfying $\partial B^n_4 = f(S^3 \times \{0\})$, $\text{Int } B^n_4 \cap f(S^3 \times E^1) = \phi$. We may define $g : D^4 \times E^5$ by $g(D^4) = B^n_4$.

Let $H^4$ be a homotopy 4-sphere which is a PL manifold and $V^4 = H^4 - \text{Int } \sigma^4$ where $\sigma^4$ is a 4-simplex.

Lemma 2. If $f : S^3 \times E^1 \to V^4 \times E^1$ is a PL homeomorphism, there is a PL homeomorphism $g : D^4 \times E^1 \to V^4 \times E^1$ which is an extension of $f$.

Proof. Let $\tilde{\sigma}_1 : \partial D^4 \times I \to D^4$, $\tilde{\sigma}_2 : \partial V^4 \times I \to V^4$ ($I = [0, 1]$) be boundary collars i.e. $\tilde{\sigma}_1, \tilde{\sigma}_2$ are embeddings such that $\tilde{\sigma}_1(x, 0) = x$ ($x \in \partial D^4$) and $\tilde{\sigma}_2(y, 0) = y$ ($y \in \partial V^4$). And let $c_1 : \partial D^4 \times I \times E^1 \to D^4 \times E^1$, $c_2 : \partial V^4 \times I \times E^1 \to V^4 \times E^1$ be $c_1(x, s, t) = (\tilde{\sigma}_1(x, s), t)$, $c_2(y, s, t) = (\tilde{\sigma}_2(y, s), t)$. Let $f_1 : c_1(\partial D^4 \times I \times E^1) \cup c_2(\partial V^4 \times I \times E^1)$ be $f_1 \circ c_1(p, s, t) = c_2(p', s, t')$ where $f_1 \circ c_1(p, 0, t) = f_0 \circ c_1(p, 0, t) = c_2(p, 0, t')$. Since $\text{Int } V^4 \times E^1 \cong E^5$ by [6], let $\mathcal{J} : \text{Int } V^4 \times E^1 \to E^5$ be a PL homeomorphism. Then $\mathcal{J} \circ c_1 | \partial D^4 \times \{1\} \times E^1 : \partial D^4 \times \{1\} \times E^1$
$E^5$ satisfies the condition for $f$ of Lemma 1, i.e. for any 5-ball $B^5 \subset E^5$ containing $f_1c_1(\partial D^4 \times \{1\} \times \{0\})$ in its interior there is a positive number $s = s(B)$ such that $f_1c_1(\partial D^4 \times ((-\infty, -s] \cup [s,\infty))) \cap B^5 = \phi$. Because $f^{-1}(B^5) \subset \text{Int} \ V^4 \times (-s', s')$ for some $s' > 0$ and $f^{-1}(B^5) \cap c_2(\partial V^4 \times \{1\} \times ((-\infty, -s') \cup [s', \infty))) = \phi$. Hence there is a number $s > 0$ such that

$$f^{-1}(B^5) \cap f_1c_1(\partial D^4 \times \{1\} \times ((-\infty, -s] \cup [s,\infty))) = \phi$$

and so

$$B^5 \cap f_1c_1(\partial D^4 \times \{1\} \times ((-\infty, -s] \cup [s,\infty))) = \phi.$$  

So by Lemma 1 ($f_1c_1(\partial D^4 \times \{1\} \times \{0\}) \subset E^5$) is a trivial knot and it bounds a locally flat 4-ball $\bar{B}^4_0$ with $\text{Int} \ B^4_0 \cap f_1c_1(\partial D^4 \times \{1\} \times E^1) = \phi$. So $f_1c_1(\partial D^4 \times \{1\} \times \{0\})$ bounds a locally flat PL 4-ball $B^4_0 = f^{-1}(\partial B^4_0)$ such that

$$\text{Int} \ B^4_0 \cap f_1c_1(\partial D^4 \times \{1\} \times E^1) = \text{Int} \ B^4_0 \cap c_2(\partial V^4 \times \{1\} \times E^1) = \phi.$$  

Similarly we may assume there are 4-balls $B^4_t \ (t \in \mathbb{Z}: \text{integer})$ such that $\partial B^4_t = f_1c_1(\partial D^4 \times \{1\} \times \{t\})$ and $\text{Int} \ B^4_t \cap c_2(\partial V^4 \times \{1\} \times E^1) = \phi$. And we may assume $B^4_t \cap B^4_{t+1} = \phi \ (t \in \mathbb{Z})$. So we can extend $f_1$ to

$$f_2 : c_1(\partial D^4 \times I \times E^1) \cup (D^4 \times \mathbb{Z}) \rightarrow c_2(\partial V^4 \times I \times E^1) \cup B^4_t \quad t \in \mathbb{Z}$$

by a cone extension. Since

$$f_1c_1(\partial D^4 \times \{1\} \times [t, t+1]) \cup B^4_t \cup B^4_{t+1} \ (t \in \mathbb{Z})$$

is a PL 4-sphere which is locally flat embedded in $\text{Int} \ V^4 \times E^1 \cong E^5$, it bounds
a PL 5-ball $B^5_t$ by Proposition 1. So we can extend $f_2$ to a required PL homeomorphism

$$g : D^4 \times E^1 \cup \bigcup_{t \in Z} E^5_t (= V^4 \times E^1)$$

by a cone extension.

Theorem. $H^4 \times E^1$ is PL homeomorphic to $S^4 \times E^1$ where $H^4$ is a homotopy 4-sphere which is a PL manifold.

Proof. Since any regular neighborhood $N$ of $p \times E^1$ in $H^4 \times E^1$ is PL homeomorphic to $D^4 \times E^1$, we identify $N = D^4 \times E^1 \subset H^4 \times E^1$ using Proposition 2. So $H^4 \times E^1 - \text{Int} N$ is PL homeomorphic to $V^4 \times E^1$. Let $\mathcal{Y}_1 : D^4 \times E^1 \to N$ be a PL homeomorphism. We may consider $S^4 \times E^1 = (D^4 \times E^1) \cup \partial (D^4 \times E^1)$ where the one of $D^4 \times E^1$ is a regular neighborhood of $q \times E^1$ for some $q \in S^4$. Then by Lemma 2 we can extend $\partial \mathcal{Y}_1 = (\mathcal{Y}_1 | \partial D^4 \times E^1)$ to another $D^4 \times E^1$ and so we can get a PL homeomorphism $\mathcal{Y} : S^4 \times E^1 \to H^4 \times E^1$. 

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References


