数理解析研究所講究録
219巻 1974年 1-18

On Compact 3-manifolds with Infinite Cyclic First Homology Groups

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The knot theory has been discussed by many topologists since the origin of topology. We know not only many interesting examples of knots but also many general properties of them. When we study a tame knot $S^1 \subset S^3$, it is fruitful to take the complement $S^3 - T$ for the open tubular neighborhood $\mathring{T}$ of S^1 in S^3. Then clearly $S^3 - \mathring{T}$ is a compact 3-manifold with $H_1(S^3 - \mathring{T}; Z) = Z$. We will call this manifold the <u>closed knot complement</u> for the knot $S^1 \subset S^3$.

The purpose of this paper is to consider the following problem: <u>To what extent can we generalize the properties of the closed knot complements to those of compact 3-manifolds with infinite cyclic first integral homology groups</u> ?

Section 1 is a classification of compact 3-manifolds with $H_1 = Z$ in the homological sense. In Section 2 the fundamental properties of Alexander polynomials are established. Section 3 shows the existence of Seifert surfaces. The final Section gives examples and questions.

1. Classifications

For simplicity we will deal with only a compact connected 3-manifold M whose boundary is either empty or contains no 2-spheres throughout this paper.

First, we shall show that if $H_1(M;Z)=Z$ then M has the same homology groups as one of the following four types : the orientable handle $S^1 \times S^2$, the non-orientable handle $S^1 \times_\tau S^2$, the solid torus $S^1 \times B^2$ and the solid Klein bottle $S^1 \times_\tau B^2$.

That is,

1.1 <u>Theorem</u>. <u>Let</u> $H_1(M;Z)=Z$. <u>In case</u> $\partial M = \emptyset$, $H_*(M;Z)$ <u>is isomorphic to either</u> $H_*(S^1 \times S^2;Z)$ <u>or</u> $H_*(S^1 \times_\tau S^2;Z)$. <u>In case</u> $\partial M \neq \emptyset$, $H_*(M;Z) \approx H_*(S^1;Z)$. <u>Furthermore</u>, ∂M <u>is connected and is homeomorphic to the torus</u> $S^1 \times S^1$ <u>or the Klein bottle</u> $S^1 \times_\tau S^1$ <u>according as</u> M <u>is orientable or non-orientable</u>.

Proof. If $\partial M = \emptyset$ and M is orientable, then, by the Poincaré duality, we obtain that $H_*(M;Z) \approx H_*(S^1 \times S^2;Z)$. If $\partial M = \emptyset$ and M is non-orientable, we know that $H_3(M;Z)= 0$ and $H^3(M;Z)=Z_2$. Since the Euler characteristic $\chi(M)$ is equal to 0, it follows that $H_2(M;Z)$ is a torsion group. Hence $H_2(M;Z) \approx H^3(M;Z)= Z_2$. This implies that $H_*(M;Z) \approx H_*(S^1 \times_\tau S^2;Z)$. In case $\partial M \neq \emptyset$, we use an infinite cyclic covering $p:\widetilde{M} \longrightarrow M$ associated with natural epimorphism $\gamma: \pi_1(M) \longrightarrow H_1(M;Z)= Z$. By [3, Proposition 4.4] $H_1(\widetilde{M};Z_2)$ is finitely generated over Z_2 because $H_1(M;Z_2)= Z_2$. By the Partial Poincaré Duality[3, Theorem 2.1] we obtain $Z_2= H^0(\widetilde{M};Z_2) \approx H_2(\widetilde{M},\partial\widetilde{M};Z_2)$. Consider the following part of the exact sequence of the pair $(\widetilde{M},\partial\widetilde{M})$

$H_2(\tilde{M}, \partial\tilde{M}; Z_2) \longrightarrow H_1(\partial\tilde{M}; Z_2) \longrightarrow H_1(\tilde{M}; Z_2)$. Since the both sides are finitely generated, we obtain that $H_1(\partial\tilde{M}; Z_2)$ is finitely generated over Z_2. For each component N of ∂M let $\gamma^*: \pi_1(N) \longrightarrow Z$ be the composite $\pi_1(N) \xrightarrow{\subseteq} \pi_1(M) \xrightarrow{\gamma} Z$. γ^* is a non-trivial homomorphism. Otherwise, by [3, Lemma 3.1], $\partial\tilde{M}$ must contain infinitely many copies of N as components. Because N is not 2-sphere by assumption, $H_1(\partial\tilde{M}; Z_2)$ is not finitely generated over Z_2. This is a contradiction. Therefore γ^* is non-trivial and hence each component $\tilde{N}$ of the preimage $p^{-1}(N)$ is an infinite cyclic covering space over N (See [3, Lemma 3.1]). Using that $H_1(\partial\tilde{M}; Z_2)$ is finitely generated, we obtain that $H_*(\tilde{N}; Z_2)$ is finitely generated. This implies $\chi(N) = 0$ (See J.W.Milnor[9]). Hence $\chi(\partial M) = 0$. By the formula $\chi(\partial M) = 2\chi(M)$, $\chi(M) = 0$. From this we see that $H_2(M; Z)$ is a torsion group. However, $H_2(M; Z)$ is free since $\partial M \neq \emptyset$. Thus, we have $H_*(M; Z) \approx H_*(S^1; Z)$. Furthermore, by the Poincaré duality, $H_1(M, \partial M; Z_2) \approx H^2(M; Z_2) = 0$. This implies $\tilde{H}_0(\partial M; Z_2) = 0$. That is, ∂M is connected. Since $H_3(M, \partial M; Z) \approx H_2(\partial M; Z)$, it follows that the orientability of ∂M coincides with that of M. Therefore, by using $\chi(\partial M) = 0$, we see that ∂M is homeomorphic to $S^1 \times S^1$ or $S^1 \times_\tau S^1$ according as M is orientable or non-orientable. This completes the proof.

Next, we consider the special case $\pi_1(M) = Z$. If Poincaré Conjecture is true, such a manifold M must be homeomorphic to one of the four types: $S^1 \times S^2$, $S^1 \times_\tau S^2$, $S^1 \times B^2$ and $S^1 \times_\tau B^2$.

More precisely, we obtain the following :

1.2 <u>Theorem</u>. <u>If</u> $\pi_1(M) = Z$, <u>then</u> M <u>is homeomorphic to one of the following connected sums</u> $(S^1 \times S^2) \# \tilde{S}^3$, $(S^1 \times_\tau S^2) \# \tilde{S}^3$, $(S^1 \times B^2) \# \tilde{S}^3$ <u>and</u> $(S^1 \times_\tau B^2) \# \tilde{S}^3$, <u>where</u> $\tilde{S}^3$ <u>is a homotopy 3-sphere</u>.

Sketch of Proof. In case $\partial M = \emptyset$, using a result of H.Kneser [6], the sphere theorem in the sense of J.H.C.Whitehead [15] and a technique of J.W.Milnor [8], we obtain that M is homeomorphic to $(S^1 \times S^2) \# \tilde{S}^3$ or $(S^1 \times_\tau S^2) \# \tilde{S}^3$. In case $\partial M \neq \emptyset$, applying the Partial Poincaré Duality [3, Theorem 2.1], we obtain that M is homotopy equivalent to S^1. By the loop theorem [12], M is homeomorphic to $(S^1 \times B^2) \# \tilde{S}^3$ or $(S^1 \times_\tau B^2) \# \tilde{S}^3$. See [5] for details.

2. Alexander Polynomials

Throughout this section we let M be a compact 3-manifold with $H_1(M;Z) = Z$ and $p : \tilde{M} \longrightarrow M$ be an infinite cyclic covering associated with natural epimorphism $\pi_1(M) \longrightarrow H_1(M;Z) = Z$.

By [3, Proposition 4.4], $H_1(\tilde{M};Q)$ is a finitely generated torsion module over the rational group ring $Q[Z]$. Since $Q[Z]$ is a principal ideal domain, $H_1(\tilde{M};Q)$ decomposes into cyclic modules, say,

$$H_1(\tilde{M};Q) \underset{Q[Z]}{\approx} Q[Z]/(f_1(t))_Q \oplus \cdots\cdots \oplus Q[Z]/(f_r(t))_Q$$

, where t denotes a fixed generator of Z and $f_i(t)$ are non-zero polynomials and $(f_i(t))_Q$ are the ideals over $Q[Z]$ generated by $f_i(t)$.

2.1 <u>Definition</u>. A non-zero element $A(t)$ in $Q[Z]$ is called the Alexander polynomial of M, if $A(t)$ is a generator of the product ideal $(f_1(t)f_2(t)\cdots f_r(t))_Q$ and $A(t)$ takes the form of the following $a_0 + a_1t + \cdots + a_mt^m$, where $a_0a_m \neq 0$ and $a_0, a_1, \ldots, a_m$ $(m \geq 0)$ are relatively prime integers. (For example, see J.Levine [7] and J.W.Milnor[9].)

From the above argument, it is easy to see that the Alexander polynomial of M always exists and that it is uniquely determined up to the choice of units.

2.2 Theorem. $A(1) = \pm 1$.

2.3 Theorem. In case M is orientable, then $A(t)=t^mA(t^{-1})$ for some even integer m. In case M is non-orientable and $\partial M = \emptyset$, then $A(t) = (-1)^{m'/2}t^{m'}A(-t^{-1})$ for some even integer m'. In case M is non-orientable and $\partial M \neq \emptyset$, then, under the assumption that $H_1(M, \partial M; Z) = 0$, the same equality holds.

2.4 Remark. For closed knot complements, Theorems 2.2 and 2.3 are well-known, and these Alexander polynomials were called the knot polynomials. For example, see R.H.Crowell and R.H.Fox [2].

2.5 Remark. Now consider the case $\partial M \neq \emptyset$. Then by Theorem 1.1 $H_*(M;Z) \approx H_*(S^1;Z)$ and ∂M is homeomorphic to $S^1 \times S^1$ or $S^1 \times_\tau S^1$. For the proof of Theorem 2.3 $H_1(M, \partial M; Z)$ will have to be vanished. If M is orientable, then, by the Poincaré duality, $H_1(M,\partial M;Z) \approx H^2(M;Z) = 0$. If M is non-orientable, then it is easy to see that $H_1(M,\partial M;Z) \approx Z_d$ for some odd integer d. (Use $H_1(M,\partial M;Z_2) \approx H^2(M;Z_2) = 0$.) In the final section, in fact, we will give an example of a compact 3-manifold M with $H_1(M;Z) = Z$ and $H_1(M,\partial M;Z) = Z_d$ for a given odd integer d. So, in this case, the assumption that $H_1(M,\partial M;Z) = 0$ makes sense.

<u>Proof of Theorem 2.2</u>. We use a technique of Y.Shinohara and D.W.Sumners [11]. Let Λ be the integral group ring of Z : $\Lambda = J[Z]$ (J is the ring of integers.). Consider a presentation matrix $M(t)$ for $H_1(\widetilde{M};Z)$ over Λ i.e. consider an exact sequence $F_1 \longrightarrow F_2 \longrightarrow H_1(\widetilde{M};Z) \longrightarrow 0$ with free Λ-modules F_1, F_2 of ranks r_1, r_2, respectively, and with the homomorphism $F_1 \longrightarrow F_2$ presented by the matrix $M(t)$. Let $E(M(t))$ denotes the ideal over Λ generated by the determinants of $r_2 \times r_2$-submatrices of $M(t)$ ($E(M(t))$ is often called the first elementary ideal of $M(t)$.). And let $E(M(t))_Q$ be the corresponding ideal over $Q[Z]$. It is well-known that $E(M(t))_Q$ is an invariant of the module $H_1(M;Q)$. Hence, by the definition of $A(t)$, we have $E(M(t))_Q = (A(t))_Q$. This implies $E(M(t)) \subset (A(t))$ as ideals over Λ (Use Gauss lemma.). Let $\varepsilon : \Lambda \longrightarrow J$ be the augmentation sending t to 1. By taking a triangulation of M, we obtain a short exact sequence of chain complexes over Λ $0 \longrightarrow C(\widetilde{M};Z) \xrightarrow{t-1} C(\widetilde{M};Z) \xrightarrow{p} C(M;Z) \longrightarrow 0$.

This induces the exact sequence $H_1(\widetilde{M};Z) \xrightarrow{t-1} H_1(\widetilde{M};Z) \longrightarrow 0$, because $H_1(M;Z) = Z$. Therefore $H_1(\widetilde{M};Z)\otimes_\varepsilon Z = 0$. Since $M(1)$ is a presentation matrix for $0 = H_1(\widetilde{M};Z)\otimes_\varepsilon Z$, it follows that $E(M(1)) = J$. Hence $J = E(M(1)) = \varepsilon(E(M(t))) \subset \varepsilon(A(t)) = (A(1))$. Thus, $A(1) = \pm 1$. This completes the outlined proof.

To prove Theorem 2.3, we shall use the following.

2.6 Theorem. $\widetilde{M}$ is always orientable.

Proof. By Theorem 1.1 and [3], $H_*(\widetilde{M};Q)$ and $H_*(\partial\widetilde{M};Q)$ are finitely generated over Q. Thus, by [4], $\widetilde{M}$ is orientable if and only if $H_2(\widetilde{M},\partial\widetilde{M};Z) \neq 0$. If M is orientable, so is $\widetilde{M}$. Hence it suffices to prove for the case that M is non-orientable. First, we consider the case $\partial M = \emptyset$. Then from the following short exact sequence of chain modules over $J[Z]$

$$0 \longrightarrow C(\widetilde{M};Z) \xrightarrow{t-1} C(\widetilde{M};Z) \xrightarrow{p} C(M;Z) \longrightarrow 0,$$ we obtain the exact sequence $$0 \longrightarrow H_2(\widetilde{M};Z) \xrightarrow{t-1} H_2(\widetilde{M};Z) \xrightarrow{p} \underset{\underset{Z_2}{\wr\wr}}{H_2(M;Z)} \longrightarrow 0$$

, because $H_*(M;Z) \approx H_*(S^1 \times_\tau S^2;Z)$ (Use the fact that $H_*(\widetilde{M};Z)$ is a Noetherian module and that any surjective endomorphism of a Noetherian module is actually an automorphism.). Hence, in particular, $H_2(\widetilde{M};Z) \neq 0$. This implies that $\widetilde{M}$ is orientable. Second, we consider the case $\partial M \neq \emptyset$. Then $H_1(M,\partial M;Z) \approx Z_d$ for some odd integer $d \geq 1$. Let R be the ring consisting of all $r \in Q$ such that $r = i/d^j$, where $i, j \in J$. From the universal coefficient theorem, we have $H_1(M,\partial M;R) \approx Z_d \otimes R = 0$, $H_2(M,\partial M;R) \approx Z_2 \otimes R \approx Z_2$ and $H_3(M,\partial M;R) = 0$. Since $R[Z]$ is a

Noetherian ring, $H_1(\widetilde{M},\partial\widetilde{M};R)$ is a Noetherian module over $R[Z]$. Hence from the short exact sequence

$$0 \longrightarrow C(\widetilde{M},\partial\widetilde{M};R) \xrightarrow{t-1} C(\widetilde{M},\partial\widetilde{M};R) \xrightarrow{p} C(M,\partial M;R) \longrightarrow 0$$

, we obtain the exact sequence

$$0 \longrightarrow H_2(\widetilde{M},\partial\widetilde{M};R) \xrightarrow{t-1} H_2(\widetilde{M},\partial\widetilde{M};R) \xrightarrow{p} \underset{\underset{Z_2}{\wr\wr}}{H_2(M,\partial M;R)} \longrightarrow 0.$$

In particular, $H_2(\widetilde{M},\partial\widetilde{M};R) \neq 0$. But $H_2(\widetilde{M},\partial\widetilde{M};R) \approx H_2(\widetilde{M},\partial\widetilde{M};Z)\otimes R$. Therefore $H_2(\widetilde{M},\partial\widetilde{M};Z) \neq 0$. This implies that $\widetilde{M}$ is orientable.

2.7 <u>Remark</u>. By Theorem 2.6 and [3, Theorem 2.1], there is a duality $H_2(\widetilde{M},\partial\widetilde{M};Z) \approx H^0(\widetilde{M};Z) = Z$. Then t induces the automorphism of $H_2(\widetilde{M},\partial\widetilde{M};Z) \approx Z$ of degree 1 or -1 according as the original manifold M is orientable or non-orientable. In fact, if M is orientable, we have an exact sequence

$$\underset{\underset{Z}{\wr\wr}}{H_3(M,\partial M;Z)} \xrightarrow{\approx} \underset{\underset{Z}{\wr\wr}}{H_2(\widetilde{M},\partial\widetilde{M};Z)} \xrightarrow{t-1} \underset{\underset{Z}{\wr\wr}}{H_2(\widetilde{M},\partial\widetilde{M};Z)} \xrightarrow{\approx} \underset{\underset{Z}{\wr\wr}}{H_2(M,\partial M;Z)}$$

Hence $t-1\colon H_2(\widetilde{M},\partial\widetilde{M};Z) \longrightarrow H_2(\widetilde{M},\partial\widetilde{M};Z)$ is a trivial homomorphism. This implies that t induces the identity homomorphism. If M is non-orientable, from the proof of Theorem 2.6, it is easy to see that there is an exact sequence

$$0 \longrightarrow \underset{\underset{Z}{\wr\wr}}{H_2(\widetilde{M},\partial\widetilde{M};Z)} \xrightarrow{t-1} \underset{\underset{Z}{\wr\wr}}{H_2(\widetilde{M},\partial\widetilde{M};Z)} \longrightarrow Z_2 \longrightarrow 0.$$

This implies that t induces the automorphism of degree -1.

<u>Proof of Theorem 2.3</u>. Let $Z \in H_2(\widetilde{M},\partial\widetilde{M};Z)$ be a generator. By the Duality Theorem of J.W.Milnor [9] or [3, Theorem 2.1]

or [4], there is a duality $\cap Z : H^1(\tilde{M};Q) \approx H_1(\tilde{M},\partial\tilde{M};Q)$, where $\cap$ denotes the cap product operation. If M is orientable, from the remark 2.7, we obtain the formula $t[(tu)\cap Z] = u\cap(tZ) = u \cap Z$. Hence the following diagram is commutative:

$$\begin{tikzcd} H^1(\tilde{M};Q) \arrow[r, "\approx", "\cap Z"'] \arrow[d, "\approx"', "t"] & H_1(\tilde{M},\partial\tilde{M};Q) \arrow[d, "\approx"', "t^{-1}"] & H_1(\tilde{M};Q) \arrow[l, "\approx"', "\text{inclusion}"] \arrow[d, "\approx"', "t^{-1}"] \\ H^1(\tilde{M};Q) \arrow[r, "\approx", "\cap Z"'] & H_1(\tilde{M},\partial\tilde{M};Q) & H_1(\tilde{M};Q) \arrow[l, "\approx"', "\text{inclusion}"] \end{tikzcd}$$

(We note that in case $\partial M \neq \emptyset$, the fact that $H_1(M,\partial M;Z) = 0$ is used. In fact, by using this, the inclusion homomorphism $H_1(\partial M;Z) \longrightarrow H_1(M;Z)$ is onto. By [3, Lemma 3.1], $\partial\tilde{M}$ is connected. Hence the inclusion homomorphism $H_1(\tilde{M};Q) \longrightarrow H_1(\tilde{M},\partial\tilde{M};Q)$ is an isomorphism.)

This diagram implies that if $H^1(\tilde{M};Q)$ is isomorphic to $Q[Z]/(f_1(t))_Q \oplus Q[Z]/(f_2(t))_Q \oplus \cdots\cdots \oplus Q[Z]/(f_r(t))_Q$, then $H_1(\tilde{M};Q)$ is isomorphic to $Q[Z]/(f_1(t^{-1}))_Q \oplus \cdots\cdots \oplus Q[Z]/(f_r(t^{-1}))_Q$ (as $Q[Z]$-modules). On the other hand, since $H^1(\tilde{M};Q) = \mathrm{Hom}[H_1(\tilde{M},Q),Q]$, $H_1(\tilde{M};Q)$ and $H^1(\tilde{M};Q)$ are isomorphic as $Q[Z]$-modules. Thus,

$$(f_1(t)\ldots\ldots f_r(t))_Q = (f_1(t^{-1})\ldots\ldots f_r(t^{-1}))_Q.$$

Using Gauss lemma, $A(t) = \pm\, t^m A(t^{-1})$ for some integer m. By Theorem 2.2, $A(1) \neq 0$. So we have $A(t) = t^m A(t^{-1})$. A standard argument implies that m is an even integer (See R.H.Crowell and R.H.Fox [2].). If M is non-orientable, from the remark 2.7, we obtain the formula $t[(tu)\cap Z] = u\cap(tZ) = -u\cap Z$. Hence the following diagram is commutative:

$$\begin{tikzcd}
H^1(\tilde{M};Q) \arrow[r, "\approx", "\cap Z"'] \arrow[d, "t", "\approx"'] & H_1(\tilde{M},\partial\tilde{M};Q) \arrow[d, "-t^{-1}", "\approx"'] & H_1(\tilde{M};Q) \arrow[l, "\approx"', "\text{inclusion}"] \arrow[d, "-t^{-1}", "\approx"'] \\
H^1(\tilde{M};Q) \arrow[r, "\approx", "\cap Z"'] & H_1(\tilde{M},\partial\tilde{M};Q) & H_1(\tilde{M};Q) \arrow[l, "\approx"', "\text{inclusion}"]
\end{tikzcd}$$

(We note that in case $\partial M \neq \emptyset$, the assumption that $H_1(M,\partial M;Z)=0$ is used. In fact, by using this, the same argument as in the orientable case asserts that the inclusion homomorphism $H_1(\tilde{M};Q) \longrightarrow H_1(\tilde{M},\partial\tilde{M};Q)$ is an isomorphism.)

By the same argument as in the orientable case, we obtain that $A(t) = \varepsilon t^{m'}A(-t^{-1})$ for some integer m', where $\varepsilon = 1$ or -1. Now we shall show that m' is an even number and $\varepsilon = (-1)^{m'/2}$. Let $A(t) = a_0 + a_1 t + \ldots\ldots + a_{m'}t^{m'}$, $a_i \in J$ and $a_0 a_{m'} \neq 0$. By the equality $A(t) = \varepsilon t^{m'}A(-t^{-1})$, we have $a_0 = (-1)^{m'}\varepsilon a_{m'}$ and $a_{m'} = \varepsilon a_0$. If m' is odd, $a_0 = a_{m'} = 0$, which contradicts to $a_0 a_{m'} \neq 0$. Hence m' is even. Then, by using $A(1) = \pm 1$ and the equality $A(t) = \varepsilon t^{m'}A(-t^{-1})$, we obtain $(1+\varepsilon)a_0 + (1+(-1)^1\varepsilon)a_1 + \ldots + (1+(-1)^{m'/2-1}\varepsilon)a_{m'/2-1} + a_{m'/2} = \pm 1$ and $a_{m'/2} = \varepsilon(-1)^{m'/2}a_{m'/2}$. The first equality implies $a_{m'/2} \equiv 1$ mod 2, and hence, in particular, $a_{m'/2} \neq 0$. By the second equality, $\varepsilon(-1)^{m'/2} = 1$. Therefore, $\varepsilon = (-1)^{m'/2}$. This completes the proof.(The essential part of the above proof is due to R.C. Blanchfield [1]. Also, see J.W.Milnor [9].)

3. Seifert Surfaces

3.1 Definition. A proper, connected, collared surface F in M is called a Seifert surface of M, if $M - F$ is connected.

The following may be considered as the EXISTENCE THEOREM OF A SEIFERT SURFACE for a 3-manifold with $H_1 = Z$.

3.2 Theorem. Let $\Upsilon : \pi_1(M) \longrightarrow Z$ be an epimorphism. If $H_1(M;Z) = Z$, then there exists a P.L.map $f: M \longrightarrow S^1$ such that

(1) $f_{\#} = \Upsilon : \pi_1(M) \longrightarrow \pi_1(S^1) = Z$

(2) For some point $p \in S^1$, $F = f^{-1}(p)$ is a proper, connected, orientable, collared surface

(3) $M - F$ is connected and orientable.

Proof. Since there is a one-to-one correspondence between the homotopy class $[M, S^1]$ and the induced homomorphism class $\mathrm{Hom}[\pi_1(M), \pi_1(S^1)]$, we can choose a map $f: M \longrightarrow S^1$ which induces Υ. We may consider that M and S^1 are simplicial complexes and that f is a simplicial map. Let $p \in S^1$ be a point which is not a vertex of this triangulation. Then $f^{-1}(p) = T$ is a proper, collared surface which need not be connected. Let $T_1, T_2, \ldots\ldots, T_n$ be the components of T. Pick points $a_i \in T_i$. Then J.Stallings [13] showed that there is a map $\rho : M \longrightarrow \Gamma$ from M to a connected graph Γ in M containing points a_i such that the composite $\Gamma \subset M \xrightarrow{\rho} \Gamma$ is homotopic to the identity; such that $\rho^{-1}(a_i) = T_i$ for all i ;and such that the following triangle is homotopy- commutative:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "\rho"'] & & S^1 \\ & \Gamma \arrow[ur, "f|\Gamma"'] & \end{tikzcd}$$

Passing to the homology, we get a commutative triangle

$$\begin{tikzcd} H_1(M;Z) \arrow[rr, "f_*"] \arrow[dr, "\rho_*"'] & & H_1(S^1;Z) \\ & H_1(\Gamma;Z) \arrow[ur, "(f|\Gamma)_*"'] & \end{tikzcd}$$

f_* is an isomorphism because f_* is onto and $H_1(M;Z) = Z$. We note that ρ_* is onto. Then $(f|\Gamma)_* : H_1(\Gamma;Z) \longrightarrow H_1(S^1;Z)$ is an isomorphism. Since Γ is a connected graph, $f|\Gamma$ is homotopic to a map $h : \Gamma \longrightarrow S^1$ so that $h^{-1}(p)$ consists of just one of the points a_i, say a_{i_o}. Then $h\rho$ is a map from M to S^1 which is homotopic to f and such that $(h\rho)^{-1}(p)=T_{i_o}$. Next, we let M^* be a manifold obtained by splitting along T_{i_o}. Now, we use the fact that the infinite cyclic covering space $\widetilde{M}$ over M associated with $\gamma: \pi_1(M) \longrightarrow \pi_1(S^1)=Z$ can be constructed from M^* (See L.P.Neuwirth [10].). By Theorem 2.6, $\widetilde{M}$ is orientable. Hence M^* is orientable. Thus, it follows that $M - T_{i_o}$ and T_{i_o} are orientable. This completes the proof.

4. Examples and Questions

4.1. Now we shall show that, given any odd $m \geq 1$, there is a 3-manifold M with $\partial M = S^1 \times_\tau S^1$ and $H_1(M;Z) = Z$ and $H_1(M,\partial M;Z) = Z_m$. For $m = 1$, the solid Klein bottle is such an example. So, we may consider $m > 1$. Consider a 2-sphere D with m holes and let $C_1, C_2, \ldots\ldots, C_m$ be components of ∂D. Take the product $D \times I$, where I is the unit interval, and let $D \times I$ be oriented. We choose orientations of $D \times 0$ and $D \times 1$ induced from that of $D \times I$. Let M be the manifold obtained by attaching $D \times 0$ to $D \times 1$ by an <u>orientation-preserving</u> homeomorphism sending $C_1 \times 0$ to $C_2 \times 1$, $C_2 \times 0$ to $C_3 \times 1, \ldots\ldots$, $C_{m-1} \times 0$ to $C_m \times 1$ and $C_m \times 0$ to $C_1 \times 1$. Since m is odd, it follows that $\partial M = S^1 \times_\tau S^1$. The first homology group $H_1(M;Z)$ is an abelian group generated by an infinite order element t and cycles $C_1, \ldots\ldots, C_m$ obtained from components of ∂D with relations $C_1 = -C_2$, $C_2 = -C_3, \ldots\ldots, C_{m-1} = -C_m$, $C_m = -C_1$ and $C_1 + C_2 + \ldots\ldots + C_m = 0$. Because m is odd, $m-1$ is even. So,

$$
\begin{aligned}
-C_m &= C_1 + C_2 + \ldots\ldots + C_m \\
&= (C_1 + C_2) + , \ldots\ldots + (C_{m-2} + C_{m-1}) \\
&= 0.
\end{aligned}
$$

Thus, $C_1 = C_2 = \ldots\ldots\ldots = C_m = 0$. That is, $H_1(M;Z)$ is an infinite cyclic group generated by t. Since the infinite cyclic covering space $\widetilde{M}$ associated with $\pi_1(M) \longrightarrow H_1(M;Z) = Z$ can be constructed from $D \times I$, it follows that ∂M consists of m components. Hence, by [3, Lemma 3.1],

$$H_1(M;Z)/\mathrm{Im}[\,H_1(\partial M;Z) \longrightarrow H_1(M;Z)\,] \approx Z_m .$$

Using the exact sequence of the pair $(M,\partial M)$, we obtain that $H_1(M,\partial M;Z) \approx Z_m$. Also, it is easily seen that the Alexander polynomial of this example is given by $t^m+1/\ t+1$.

4.2. By the classical knot theory, it is well-known that, given an integral polynomial $A(t)$ with $A(1)=\pm 1$ and $A(t) = t^m A(t^{-1})$, there exists a closed knot complement whose Alexander polynomial is $A(t)$. Accordingly, the following question occurs naturally: <u>Given</u> $A(t)$ <u>with</u> $A(1)=\pm 1$ <u>and</u> $A(t) = t^m A(t^{-1})$, <u>then does there exist a closed 3-manifold with</u> $H_1 = Z$ <u>whose Alexander polynomial coincides with</u> $A(t)$?

Of course, by Theorem 2.3, such a manifold must be orientable , unless $A(t)$ is a constant.

The question for non-orientable 3-manifolds is as follows: <u>Given a polynomial</u> $A(t)$ (<u>with integral coefficients</u>)<u>satisfying</u> $A(1)=\pm 1$ <u>and</u> $A(t) = (-1)^{m/2} t^m A(-t^{-1})$, <u>then do there exist closed and bounded 3-manifolds with</u> $H_1 = Z$ <u>whose Alexander polynomials are</u> $A(t)$?

Of course, by Theorem 2.3, such manifolds are always non-orientable, unless $A(t)$ is a constant.

4.3. We consider a compact, connected, orientable surface F with genus $g \geq 1$ whose boundary is either empty or homeomorphic to the circle. To characterize a homeomorphism between surfaces, H.Terasaka [14] introduced a skew-orthogonal matrix.

<u>Definition</u> (Terasaka). A <u>skew-orthogonal matrix</u> is an

integral $(2g)\times(2g)$-matrix A satisfying $A.\tilde{A} = \varepsilon E$, where if

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ \cdots & & & & \\ \cdots & & & & \end{pmatrix}$$ then $\tilde{A}$ denotes the matrix

$$\begin{pmatrix} a_{22} & -a_{12} & \cdots \\ -a_{21} & a_{11} & \cdots \\ a_{24} & -a_{14} & \cdots \\ -a_{23} & a_{13} & \cdots \\ \cdots & & \end{pmatrix}$$, and $\varepsilon = 1$ or -1, and E is the unit matrix.

Note that any integral 2×2-matrix whose determinant is ± 1 is a skew-orthogonal matrix.

Let F be oriented, and choose a standard basis $\langle a_1, b_1, \ldots\ldots, a_g, b_g\rangle$ for $H_1(F;Z)$ with intersection numbers $a_i.b_i = 1$, $a_i.b_j=0$ $(i \neq j)$ and $a_i.a_j = b_i.b_j = 0$ (all i, j). Then H.Terasaka [14] showed that, given a skew-orthogonal matrix A, then there is an auto-homeomorphism $h: F \longrightarrow F$ such that the automorphism $h_*: H_1(F;Z) \longrightarrow H_1(F;Z)$ represents A with the basis $\langle a_1, b_1, \ldots\ldots, a_g, b_g\rangle$,and conversely. In this case, we can see that h is orientation-preserving or orientation-reversing according as $\varepsilon = 1$ or -1.

Now take the orientation of the real line R^1 so that the map $R^1 \longrightarrow R^1$, $y \longrightarrow y+1$ is orientation-priserving, and let $\tilde{M} = F\times R^1$ be oriented by the orientation induced from those of F and R^1. The transformation $t : \tilde{M} \longrightarrow \tilde{M}$ defined by $t(x,y)=(hx,y+1)$ induces the automorphism $t_*: H_1(\tilde{M};Z) \longrightarrow H_1(\tilde{M};Z)$ representing the matrix A with the basis $\langle a_1, b_1, \ldots, a_g, b_g\rangle$. Clearly, t is orientation-preserving or orientation-reversing according as $\varepsilon = 1$ or -1. Since the infinite cyclic group Z generated by t acts $\tilde{M}$ properly discontinuously, the orbits space $M = \tilde{M}/Z$ is a compact manifold so that the natural projection $\tilde{M} \longrightarrow M$ is an infinite cyclic covering whose covering transformation group is Z. By Remark 2.7, we know that M is orientable or non-orientable according as $\varepsilon = 1$ or -1. Since $H_1(M;Z) \approx Z \oplus H_1(\tilde{M};Z)/(E-A)H_1(\tilde{M};Z)$, it follows that $H_1(M;Z) \approx Z$ is equivalent to $\det(E-A) = \pm 1$. Note that, from construction, M is a fibered manifold over S^1 with fiber F. Since the Alexander polynomial of M is $\det(tE-A)$, we showed the following:

<u>Theorem</u>. <u>Given a skew-orthogonal matrix</u> A <u>with</u> $\det(E-A)=\pm 1$, <u>then there exist fibered</u>, <u>both closed and bounded</u>, <u>compact 3-manifolds with</u> $H_1 = Z$ <u>whose Alexander polynomials are equal to</u> $\det(tE-A)$. <u>Such manifolds can be chosen to be orientable or non-orientable according as</u> $\varepsilon = 1$ <u>or</u> -1.

For example, let $A_0 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$ and consider a $(2g)\times(2g)$-matrix

$$A = \begin{pmatrix} A_0 & & & 0 \\ & A_0 & & \\ & & A_0 & \\ 0 & & & \ddots \end{pmatrix}. \text{ Then } A.\tilde{A} = E \text{ and } \det(E-A) = 1.$$

Hence there are orientable, fibered, both closed and bounded manifolds with $H_1 = Z$ whose Alexander polynomials are $(t^2-t+1)^g$.

Similarly, if we let $B_0 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} B_0 & & 0 \\ 0 & B_0 & \\ & & \ddots \end{pmatrix}$

we see that there are non-orientable, fibered, closed and bounded manifolds with $H_1 = Z$ whose Alexander polynomials are $\det(tE - B) = (t^2 - t - 1)^g$, because $B.\tilde{B} = -E$ and $\det(E-B) = (-1)^g$.

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