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The fixed point set of an involution and theorems of the Borsuk-Ulam type

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1. Statement of results. In this note h^* will denote either the unoriented cobordism theory $\mathfrak{N}^*$ or the usual cohomology theory with $\mathbb{Z}_2$-coefficients $H^*(\ ;\mathbb{Z}_2)$. The corresponding equivariant cohomology theory for $\mathbb{Z}_2$-spaces will be denoted by $h^*_{\mathbb{Z}_2}$.

Let M be a manifold and σ an involution on M.[1)] We define an embedding $\Delta : M \longrightarrow M^2 = M \times M$ by $\Delta(x) = (x, \sigma x)$. Then Δ is equivariant with respect to the involution σ on M and the involution T on M^2 which is defined by $T(x_1, x_2) = (x_2, x_1)$. Let $\Delta_! : h^q_{\mathbb{Z}_2}(M) \longrightarrow h^{q+m}_{\mathbb{Z}_2}(M^2)$ denote the Gysin homomorphism for Δ, where $m = \dim M$. We put $\theta(\sigma) = \Delta_!(1) \in h^m_{\mathbb{Z}_2}(M^2)$.

In the present note we shall give an explicit formula for $\theta(\sigma)$ and apply it to get theorems of the Borsuk-Ulam type. Our results generalize those of Nakaoka [3], [4]. From the formula for $\theta(\sigma)$ we shall also derive a sort of integrality theorem concerning the fixed point set of σ; see Theorem 4. Detailed accounts will appear elsewhere.

Let S^∞ be the infinite dimensional sphere with the antipodal

1) In this note we work in the smooth category. All manifolds will be connected, compact and without boundary unless otherwise stated.

involution. The projection $\pi : S^\infty \times M^2 \longrightarrow S^\infty \times_{\mathbb{Z}_2} M^2$ induces the Gysin homomorphism $\pi_! : h^*(M^2) \longrightarrow h^*_{\mathbb{Z}_2}(M^2)$ and the usual homomorphism $\pi^* : h^*_{\mathbb{Z}_2}(M^2) \longrightarrow h^*(M^2)$. Let $d : M \longrightarrow M^2$ be the diagonal map. Since $d(M)$ is the fixed point set of T, $h^*_{\mathbb{Z}_2}(d(M))$ is isomorphic to $h^*_{\mathbb{Z}_2}(pt) \otimes_{h^*(pt)} h^*(M)$ and d induces $d^* : h^*_{\mathbb{Z}_2}(M^2) \longrightarrow h^*_{\mathbb{Z}_2}(pt) \otimes_{h^*(pt)} h^*(M)$.

Lemma 1. <u>The homomorphism</u>

$$\pi^* \oplus d^* : h^*_{\mathbb{Z}_2}(M^2) \longrightarrow h^*(M^2) \oplus (h^*_{\mathbb{Z}_2}(pt) \otimes_{h^*(pt)} h^*(M))$$

<u>is injective</u>.

We denote by S the multiplicative set $\{w_1^k \mid k \geq 1\}$ in $h^*_{\mathbb{Z}_2}(pt) = h^*(P^\infty)$ where w_1 is the universal first Stiefel-Whitney class. If X is a $\mathbb{Z}_2$-space then $h^*_{\mathbb{Z}_2}(X)$ is an $h^*_{\mathbb{Z}_2}(pt)$-module and we can consider the localized ring $S^{-1}h^*_{\mathbb{Z}_2}(X)$ of $h^*_{\mathbb{Z}_2}(X)$ with respect to S. Note that $h^*_{\mathbb{Z}_2}(pt)$ is isomorphic to a formal power series ring $h^*(pt)[[w_1]]$ and $h^*_{\mathbb{Z}_2}(pt) \otimes_{h^*(pt)} h^*(M)$ is canonically embedded in $(S^{-1}h^*_{\mathbb{Z}_2}(pt)) \otimes_{h^*(pt)} h^*(M)$.

To state our main theorem we need some notations. Let $P : h^q(M) \longrightarrow h^{2q}_{\mathbb{Z}_2}(M^2)$ be the Steenrod-tom Dieck operation; see [4], [6]. For $u \in h^q(M)$ we define $P_0(u)$ to be $d^*P(u)/w_1^{2q}$. Then P_0 is extended to a ring homomorphism $P_0 : h^*(M) \longrightarrow (S^{-1}h^*_{\mathbb{Z}_2}(pt)) \otimes_{h^*(pt)} h^*(M)$. For a real vector bundle ξ over a CW-complex X its h^*-theory Wu classes $v_\alpha(\xi) \in h^*(X)$ are defined in a similar way as in [5]. The Wu classes of the tangent bundle of a manifold X will be denoted by $v_\alpha(X)$. Finally we define $a_j(x) \in h^*(pt)[[x]]$ by

$$F(x, y) = \sum_{0\leq j} a_j(x)y^j$$

where F is the formal group law of the theory h^*. For a multi-index $\alpha = (\alpha_1, \alpha_2, \cdots)$ we put $a^\alpha(x) = \prod_{1\leq j} a_j^{\alpha_j}(x)$, $\ell(\alpha) = \sum_j \alpha_j$ and $|\alpha| = \sum_j j\alpha_j$, cf. [6].

Theorem 2. Let M be a manifold and σ an involution on M. Let F be the fixed point set of σ. F is a disjoint union of submanifolds $F_1, \cdots, F_\ell$.

i) $\pi^*\theta(\sigma) \in h^*(M^2)$ is given by

$$\pi^*\theta(\sigma) = \Delta_!(1)$$

where the $\Delta_!$ on the right-hand side is the usual Gysin homomorphism $h^*(M) \longrightarrow h^*(M^2)$. If $\{u_i\}$ is a homogeneous $h^*(pt)$ basis of $h^*(M)$ and $\Delta_!(1) = \sum a_{ij}u_i \times u_j$ with $a_{ij} \in h^*(pt)$ then the a_{ij}'s satisfy the relation

$$\sum_j a_{ij}c_{jk} = \delta_{ik} \quad \text{(the Kronecker } \delta \text{)}$$

where $c_{jk} = p_!(u_j \cup \sigma^* u_k)$ with $p : M \longrightarrow pt$.

ii) $d^*\theta(\sigma) \in h^*_{\mathbb{Z}_2}(pt) \otimes_{h^*(pt)} h^*(M) \subset (S^{-1}h^*_{\mathbb{Z}_2}(pt)) \otimes_{h^*(pt)} h^*(M)$ is given by

$$d^*\theta(\sigma) = w_1^m \frac{\sum_{i=1}^{\ell} \sum_\alpha w_1^{2(-\ell(\alpha)+|\alpha|)} a^{2\alpha}(w_1)P_0(j_!(v_\alpha(F_i)^2)}{\sum_\alpha w_1^{-\ell(\alpha)+|\alpha|} a^\alpha(w_1)P_0(v_\alpha(M))}$$

where $j_!$ is the Gysin homomorphism of the inclusion $j : F \subset M$

and $m = \dim M$.

Remark 3. *In Theorem 2, when the theory* h^* *is the usual cohomology theory* $H^*(\ ;\mathbb{Z}_2)$, *the formula for* $d^*\theta(\sigma)$ *reduces to*

$$d^*\theta(\sigma) = w_1^m P_0\left(\left\{\sum_{i=1}^{\ell}\sum_{s=0}^{[\frac{f_i}{2}]} j_!(v_s(F_i)^2)\right\}/\left\{\sum_{s=0}^{[\frac{m}{2}]} v_s(M)\right\}\right)$$

where $f_i = \dim F_i$.

Theorem 4. *Let* M, σ *and* F_i *be as in Theorem* 2. *Suppose that* $h^* = H^*(\ ;\mathbb{Z}_2)$. *If we write*

$$\sum_{i=1}^{\ell}\sum_{s=0}^{[\frac{f_i}{2}]} j_!(v_s(F_i)^2)/\sum_{s=0}^{[\frac{m}{2}]} v_s(M) = \sum_{i=0}^{m} u_i$$

where $u_i \in H^i(M;\mathbb{Z}_2)$ *then we must have*

$$u_i = 0 \quad \text{for} \quad i > \frac{m}{2}.$$

Corollary 5. *Under the situation of Theorem 4 the element* $\theta(\sigma) \in H^m_{\mathbb{Z}_2}(M^2;\mathbb{Z}_2)$ *is given by*

$$\theta(\sigma) = \sum_{i=0}^{[\frac{m}{2}]} w_1^{m-2i}P(u_i) + \theta_1$$

where θ_1 *is characterized by the conditions*

a) $\rho \in \pi_!$*-image*

and

b) $\pi^*\rho = \Delta_!(1) + u_{\frac{m}{2}} \times u_{\frac{m}{2}}$.

Corollary 6. *Under the situation of Theorem 4 assume moreover that* $\dim F_i < \dim M/2$ *for all* i. *Then*

$$\sum_{i=1}^{\ell}\sum_{s=0}^{[\frac{f_i}{2}]} j_!(v_s(F_i)^2) = 0$$

and $\theta(\sigma) \in H^*_{\mathbb{Z}_2}(M^2;\ Z_2)$ is characterized by the conditions

a) $\theta(\sigma) \in \pi_!$-image

and

b) $\pi^*\theta(\sigma) = \Delta_!(1)$.

Corollary 7. Let M be an m-manifold which is a $\mathbb{Z}_2$-homology sphere and σ an involution on M. Then, in the usual homology theory $H^*(\ ;\ \mathbb{Z}_2)$, the element $\theta(\sigma) \in H^m_{\mathbb{Z}_2}(M^2;\ \mathbb{Z}_2)$ is given by

$$\theta(\sigma) = \begin{cases} \pi_!(1\times\mu) & \text{if } \sigma \text{ is not trivial,} \\ w_1^m + \pi_!(1\times\mu) & \text{if } \sigma \text{ is trivial,} \end{cases}$$

where $\mu \in H^m(M;\ \mathbb{Z}_2)$ is the cofundamental class.

Now let N be another manifold with an involution τ and $f : N \longrightarrow M$ a continuous map. We put

$$A(f) = \{y \mid y \in N,\ \ f\tau(y) = \sigma f(y)\}$$

and define an equivariant map $\hat{f} : N \longrightarrow M^2$ by $\hat{f}(y) = (f(y),\ f\tau(y))$. The following is fundamental for our theorems of the Borsuk-Ulam type.

Theorem 8. If $A(f) = \phi$ then the class $\hat{f}^*\theta(\sigma) \in h^m_{\mathbb{Z}_2}(N)$ vanishes.

Corollary 9. Let $\bar{f}$ denote the restriction of f on the fixed point set $F(\tau)$ of τ. Suppose that we have

$$\bar{f}^*(\sum_{i=1}^{\ell}\sum_{s=0}^{[\frac{f_i}{2}]} j_!(v_s(F_i)^2)) \neq 0$$

<u>in</u> $H^*_{\mathbb{Z}_2}(pt) \otimes H^*(F(\tau); \mathbb{Z}_2)$ <u>then the set</u> $A(f)$ <u>is not empty</u>.

When the involution τ on N is free the module $h^*_{\mathbb{Z}_2}(N)$ is canonically identified with $h^*(N/\mathbb{Z}_2)$.

Corollary 10. <u>Let</u> M <u>and</u> N <u>be manifolds of the same dimension</u> m. <u>Let</u> σ <u>be an involution on</u> M <u>such that</u> $\dim F_i < \frac{m}{2}$ <u>for all components</u> F_i <u>of the fixed point set of</u> σ. <u>Let</u> τ <u>be a free involution on</u> N <u>and</u> $f : N \longrightarrow M$ <u>a continuous map</u>. <u>Then, in the usual cohomology, the evaluation of the class</u> $\hat{f}^*\theta(\sigma) \in H^m(N/\mathbb{Z}_2)$ <u>on the fundamental class</u> $[N/\mathbb{Z}_2]$ <u>is given by</u>

$$\langle [N/\mathbb{Z}_2], \hat{f}^*\theta(\sigma)\rangle = \hat{\chi}(f)$$

<u>where</u> $\hat{\chi}(f)$ <u>is the equivariant Lefschetz number of</u> f <u>as defined in</u> [3]. <u>Consequently if</u> $\hat{\chi}(f) \neq 0$ <u>then</u> $A(f) \neq \phi$.

Corollary 11. <u>Let</u> M <u>be an</u> m-<u>manifold which is a</u> $\mathbb{Z}_2$-<u>homology sphere with an involution</u> σ. <u>Let</u> N <u>be an</u> m-<u>manifold with a free involution</u> τ <u>and</u> $f : N \longrightarrow M$ <u>a map</u>. <u>Then we have</u>

$$\langle [N/\mathbb{Z}_2], \hat{f}^*\theta(\sigma)\rangle = \begin{cases} 1 + \deg f & \underline{\text{if}}\ \sigma\ \underline{\text{is trivial}}, \\ \deg f & \underline{\text{if}}\ \sigma\ \underline{\text{is not trivial.}} \end{cases}$$

<u>Consequently if</u> σ <u>is not trivial and</u> $\deg f \neq 0$ <u>then</u> $A(f) \neq \phi$.

2. Indication of proofs. Lemma 1 is a consequence of the following structure theorem for $h^*_{\mathbb{Z}_2}(M^2)$ and a localization theorem due to tom Dieck [2] applied to the diagonal map d.

Theorem 12. <u>In</u> $h^*_{\mathbb{Z}_2}(M^2)$ <u>the union</u> $\bigcup_{k\geq 1}(\cup w_1^k\text{-}\underline{\text{kernel}})$ <u>coincides with</u> $\pi_!$-<u>image which is isomorphic to</u> $h^*(M^2)/h^*(M^2)^T$ <u>through</u> $\pi_!$.

The homomorphism π^* restricted on $\pi_!$-image is injective. The quotient $h^*_{\mathbb{Z}_2}(M^2)/(\pi_!\text{-image})$ is a free $h^*_{\mathbb{Z}_2}(pt)$-module and is generated by P-image. Its rank is equal to the rank of the $h^*(pt)$-module $h^*(M)$.

Theorem 12 is proved using the Gysin exact sequence of the double covering $\pi : S^\infty \times M^2 \longrightarrow S^\infty \times_{\mathbb{Z}_2} M^2$ and the following properties of $\pi_!$, π^* and P :

$$\pi^* \pi_!(u \times v) = u \times v + v \times u ,$$

$$\pi^* P(u) = u \times u .$$

Part i) of Theorem 2 follows from the commutativity of the diagram

$$\begin{array}{ccc} h^*(M) & \xrightarrow{\Delta_!} & h^*(M^2) \\ \pi^* \uparrow & & \uparrow \pi^* \\ h^*_{\mathbb{Z}_2}(M) & \xrightarrow{\Delta_!} & h^*_{\mathbb{Z}_2}(M^2) \end{array}$$

which holds since π is a covering projection.

In order to prove Part ii) we consider the submanifolds $\Delta(M)$ and $d(M)$ of M^2. They are invariant under the action T . Their intersection is canonically identified with F. Let $j' : F \subset \Delta(M)$ and $j : F \subset d(M)$ be the inclusions. Let $\nu_{j'}$ and ν_d be the normal bundles of j' and d respectively. We see that $\Delta(M)$ and $d(M)$ cut each other cleanly along F, that is, $\nu_{j'}$ is a subbundle of $j^* \nu_d$. Thus we have the excess bundle $E = j^* \nu_d / \nu_{j'}$, and it follows from the clean intersection formula (cf. [6]) that

$$d^*\Delta_!(1) = j_!(e(E))$$

where $e(E) \in h^*_{\mathbb{Z}_2}(pt) \underset{h^*(pt)}{\otimes} h^*(F)$ is the h^*-theory Euler class of the bundle E with $\mathbb{Z}_2$-action. In our situation we have

Lemma 13. <u>The bundle</u> E <u>is isomorphic to the normal bundle</u> $\nu_{d'}$ <u>of the diagonal map</u> $d' \colon F \longrightarrow F^2$ <u>where the</u> $\mathbb{Z}_2$<u>-action on</u> $\nu_{d'}$ <u>is induced from</u> T.

From Lemma 13 and the clean intersection formula applied to the commutative diagram

$$\begin{array}{ccc} F & \xrightarrow{d'} & F^2 \\ \big\downarrow{\scriptstyle j} & & \big\downarrow{\scriptstyle j^2} \\ M & \xrightarrow{d} & M^2 \end{array}$$

we infer that

$$(*) \qquad d^*\Delta_!(1) = d^*\left(\frac{(j^2)_!(d'_!(1)^2)}{d_!(1)}\right),$$

in $(S^{-1}h^*_{\mathbb{Z}_2}(pt) \underset{h^*(pt)}{\otimes} h^*(M)$. But we have a formula due to Nakaoka [5] which expresses $d_!(1)$ in terms of $v_\alpha(M)$, P_0 and $a^\alpha(w_1)$ and a similar one for $d'_!(1)$. Using these in (*) we obtain the formula in Part ii) of Theorem 2.

Finally Theorem 8 follows from the fact that $\hat{f}^*\theta(\sigma)$ is the Poincaré dual (in the equivariant cohomology) of $\hat{f}^{-1}(\Delta(M)) = A(f)$ in N.

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