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Abstract.
Propagation of micro-analyticity is studied for a solution \( u \) of a pseudo-differential equation \( P(x,D)u = 0 \) of a certain class whose characteristics are not necessarily of constant multiplicity.

Introduction.
Let \( P(x,D)u = 0 \) be a single pseudo-differential equation of finite order defined in a neighborhood of \( x_0 = (x_0, \sqrt{-1}x_0^\omega) \), a point in the cosphere bundle \( \sqrt{-1}S^\omega M \) of a real analytic manifold \( M \). This is known to be microlocally equivalent to the simplest equation \( (\partial/\partial x_1)^m v = 0 \) provided that \( P(x,D) \) is real and with constant multiple characteristics. This fact implies in particular that under the same assumptions micro-analyticity or equivalently the zero of a micro-function solution \( u \) of \( P(x,D)u = 0 \) propagates along bicharacteristic strips of \( P(x,D) \). (See Sato, Kawai and Kashiwara [1]; we also refer to Kawai [2], Hörmander [3] and Andersson [4] for linear differential operators with simple characteristics.)

It is the aim of the present article to extend this result on propagation of micro-analyticity to operators whose characteristics are not necessarily of constant multiplicity.

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In this part I we establish lemmas on propagation of micro-analyticity of a solution $u$ of a pseudo-differential equation $P(x,D)u = 0$ for operators whose principal symbol is a product of real and simply characteristic symbols. As an easy corollary of those lemmas we deduce a theorem on analyticity of elementary solutions for linear hyperbolic differential operators with real analytic coefficients. A key to our theorem is the existence of good elementary solutions for micro-hyperbolic operators (established in Kashiwara and Kawai [9]).

A part of the results of this paper has been announced in Miwa [10].

In the case of operators with constant coefficients far-reaching results have already been established by several authors (Atiyah, Bott and Gårding [5], Andersson [6], [7] and Bernstein [8]). In the subsequent part II we will extend the present results to the wider class of those pseudo-differential operators whose principal symbols are micro-locally contact-transformable to constant-coefficiented symbols so as to comprise the results of these authors.

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1. Reductive Pseudo-differential Operators
and Their Bicharacteristic Strips.

In this section we introduce a class of pseudo-differential operators and define bicharacteristic strips for them.

Definition 1.1. Reductive pseudo-differential operators.
Let \( P(x,D) \) be a pseudo-differential operator of finite order defined in a neighborhood of \( x_0^* = (x_0, \xi_0^\omega) \), a point in the cotangential projective bundle \( P^*X \) of a complex manifold \( X \). Let

\[
\sigma(P)(x,\xi) = p_{1r}(x,\xi) \cdots p_{rr}(x,\xi)
\]

be an irreducible decomposition of its principal symbol at \( x_0^* \). We call \( P(x,D) \) reductive at \( x_0^* \) if each \( p_j(x,\xi) \) is simply characteristic, that is, \( \det (x,\xi)p_j(x,\xi) \) is not parallel to \( \sum_{k=1}^n d^k \xi_j \). We call \( r \) the number of crossing and \( s_1 + \cdots + s_r \) the multiplicity.

Examples of reductive symbols are

\[
\xi_1^2 - x_1^2(\xi_2^2 + \cdots + \xi_n^2), \quad \text{and}
\]

\[
(\xi_1^2 - a(x)\xi_2^2 - b(x)\xi_3^2)(\xi_1^2 - c(x)\xi_2^2 - d(x)\xi_3^2)
\]

with \( a(x), b(x), c(x) \) and \( d(x) \) positive when \( x \) is real.

Both of (1.2) and (1.3) are reductive on their real characteristic varieties.

Definition 1.2. Bicharacteristic strips.
Let \( P(x,D) \) be as in Definition 1.1. We can define a bicharacteristic strip \( B_j(x_0^*) \) of \( p_j(x,\xi) \) through \( x_0^* \) as usual. We call \( B_j(x_0^*) (j=1, \cdots, r) \) bicharacteristic strips of \( P(x,D) \) through \( x_0^* \).

In this section we recall the results of Kashiwara and Kawai [9], which we need in section 3.

Let \( M \) be a real analytic manifold of dimension \( n \) and \( X \) be a complex neighborhood. We denote by \( L \) the cosphere bundle \( S^*_M X = \sqrt{-1} S^*_M \) and by \( A \) its complex neighborhood. We take a homogeneous local coordinates \( (x_1, \ldots, x_n, \sqrt{-1}(\xi_1, \ldots, \xi_n) ) \) of \( L \).

**Definition 2.1. Partial micro-hyperbolicity.** Let \( P(x, D) \) be a pseudo-differential operator defined in a neighborhood of \( x_0^* = (x_0, \sqrt{-1}\xi_0) \) and \( V \) be its characteristic variety, that is, \( \{ (x, \xi) \in \Lambda; \sigma(P)(x, \xi) = 0 \} \). Let \( x^* + \sqrt{-1}v \) be a point of \( \sqrt{-1}SL = S_L \Lambda \). We say that \( P(x, D) \) is partially micro-hyperbolic at \( x^* + \sqrt{-1}v \) if \( x^* + \sqrt{-1}v \) is not contained in the closure of \( V - L \) in \( \tilde{L} \Lambda \), the real monoidal transform of \( \Lambda \) with the center \( L \) (see Sato, Kawai and Kashiwara [1]). This is equivalent to saying that

\[
\sigma(P)(y^* + \sqrt{-1}tv) \neq 0
\]

when \( y^* \) is in \( L \) and near \( x^* \) and \( 0 < t < 1 \).

We denote by \( L \times L \) the real analytic manifold \( \sqrt{-1}S^*(M \times M) - M \times \sqrt{-1}S^*_M \times M - \sqrt{-1} S^*_M \times M \) and we take a homogeneous local coordinates \( (x_1, \ldots, x_n, x'_1, \ldots, x'_n, \sqrt{-1}(\xi_1, \ldots, \xi_n, \xi'_1, \ldots, \xi'_n) ) \) of \( L \times L \). We identify \( L \) with the anti-diagonal set of \( L \times L \), that is, \( \{(x, x, \sqrt{-1}(\xi, -\xi)) \in L \times L \} \).

Now let us consider in a coordinate neighborhood of \( L \), where \( \xi_n \neq 0 \). We can use inhomogeneous local coordinates
(x_1,\cdots,x_n,p_1,\cdots,p_{n-1}) of L where p_j = \frac{\xi_j}{\xi_n}(j=1,\cdots,n-1)
and (x_1,\cdots,x_n,x'_1,\cdots,x'_n,p_1,\cdots,p_{n-1},p'_1,\cdots,p'_n) of L \times L
where p_j = \frac{\xi_j}{\xi_n}(j=1,\cdots,n-1) and p'_j = \frac{\xi'_j}{\xi_n}(j=1,\cdots,n). L is
identified with \{(x_1,\cdots,x_n,x'_1,\cdots,x'_n,p_1,\cdots,p_{n-1},-p_1,\cdots,-p_{n-1},1)\} in L \times L.

Let \theta be a subbundle of S^*(L \times L) induced from the
fundamental 1-form on L \times L. Using the local coordinates
\theta can be written as

\[ (2.2) \quad \theta = \{(x,x,p,-p,1,(dx_n - \sum_{j=1}^{n-1} p_j dx_j + \sum_{j=1}^{n-1} p_j dx'_j))^0 \} \].

We consider S^*(L) as a subbundle of S^*(L \times L) of codimension
1 by the map

\[ (x_1,\cdots,x_n,p_1,\cdots,p_{n-1},(\sum_{j=1}^{n} a_j dx_j + \sum_{j=1}^{n-1} b_j dp_j)^0) \]
\[ \mapsto (x,x,p,-p,1,(\sum_{j=1}^{n} a_j dx_j - \sum_{j=1}^{n} a_j dx'_j + \sum_{j=1}^{n-1} b_j dp_j + \sum_{j=1}^{n-1} b_j dp'_j + (\sum_{j=1}^{n} b_j p_j dp_j)^0). \]

Definition 2.2. Canonical map \( H \). There is a canonical
map \( H \) from \( S^*(L \times L) \) to \( \sqrt{-1}SL \) induced from the
fundamental 1-form on L. \( H \) maps \( (x,x,p,-p,1,\)
\[ (\sum_{j=1}^{n} a_j (\partial_3 x_j - \sum_{j=1}^{n} a_j dx'_j + \sum_{j=1}^{n-1} b_j dp_j + \sum_{j=1}^{n-1} b_j dp'_j)^0, \]
\[ (\sum_{j=1}^{n} b_j (b/3x_j) - \sum_{j=1}^{n-1} (a_i p_j + a_j)(b/3p_j))))^0. \]

Lemma 2.1. Let \( \theta_1, \theta_2 \in S^*(L) \). Then we have

\[ \langle H(\theta_1), \theta_2 \rangle = -\langle H(\theta_2), \theta_1 \rangle. \]

If we denote by \([,]\) the Lagrangean bracket (see Sato, Kawai
and Kashiwara [1]),

$$\langle H(da_l), da_r \rangle = [a_l^\gamma, a_r^\gamma]$$

is valid.

Now we explain the notion of normal set and conormal set. Let $M$ be a real analytic manifold, $N$ be its submanifold and $G$ be a closed subset of $M$. The normal set of $G$ along $N$ is the intersection of $S_{\bar{N}}^M$ and the closure of $G-N$ in $S_N^M$ and we denote it by $S_{\bar{N}}^G$. The polar of $S_{\bar{N}}^G$, that is,

$$(2.3) \quad \{(x, \xi^0) \in S_{\bar{N}}^M; \langle \xi, \nu \rangle \leq 0 \text{ for any } x+\nu \in S_N^G\}$$

is called the conormal set of $G$ along $N$ and we denote it by $S_{\bar{N}}^{\bar{G}}$.

Definition 2.3. $\mathcal{A}^\Delta$ and $\mathcal{H}_{\Delta}$. Let $x^\#$ be a point of $L$ and $\Gamma$ be a subset in $S_x^L(\hat{L} \times L)$ over $x^\#$. We denote $\mathcal{A}_{\Gamma}$ the set of all germs $K$ of $C_{\alpha \times M}^{(0, n)}$ at $x^\#$ such that the fiber over $x^\#$ of the conormal set of the support of $K$ along $L \subset \hat{L} \times L$ contains a neighborhood of the antipodal set of $\Gamma$. If $\Delta$ is a subset of $\sqrt{-1S}_L^x$, then $\mathcal{A}_{\Delta}$ is denoted by $\mathcal{H}_{\Delta}$.

Proposition 2.1. Ring structure of $\mathcal{A}_{\Gamma}$. $\mathcal{A}_{\Gamma}$ is a ring by the operation $(K_1(x, x)dx, K_2(x, x)dx) \mapsto \{\int K_1(x, x)K_2(x, x)dx\}dx'$ if $\Gamma$ is not empty.

Definition 2.4. $\mathcal{M}_{\Gamma}$. Let $x^\#$ be a point of $L$ and $\Gamma$ be a subset of $S_x^L$. The set of all germs $u(x)$ of $C_M$ at $x^\#$ satisfying the following condition is denoted

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by $\mathcal{M}_\Gamma$: the normal set of the support of $u(x)$ along $x^\#$
does not intersect the polar of $\Gamma$.

Proposition 2.2. $\mathcal{M}_\Gamma$ is an $\mathcal{A}_\Gamma$ module by the operation
$(K(x,x^\#)dx^\#, u(x)) \mapsto (Ku)(x) = \int K(x,x^\#)u(x^\#)dx^\#$.

Now we can state the main theorem in Kashiwara and Kawai [9].

Theorem 2.1. Existence of a good elementary solution.

Let $P(x,D)$ be a single pseudo-differential operator of
finite order which is partially micro-hyperbolic at $x^\# + \sqrt{-1}v_0$.

Then there exists a unique elementary solution $E$ of $P$ in
$\mathcal{K}_{x^\# + \sqrt{-1}v_0}$, that is, an element $E$ in $\mathcal{K}_{x^\# + \sqrt{-1}v_0}$ satisfying

(2.4) $PE = EP = 1.$

In this section we prove lemmas on propagation of micro-analyticity for solutions of pseudo-differential equations.

The following lemma is another expression of Lemma 3.1.5 and Lemma 3.1.6 of Sato, Kawai and Kashiwara [1].

Lemma 3.1. Normal set of a non-singular hypersurface. Let \( \mathcal{P}(x) \) be a real analytic function defined in a neighborhood of \( x_0 \), a point of a real analytic manifold \( M \). Assume that \( d\mathcal{P}(x_0) \neq 0 \). We can rewrite as

\[
\mathcal{P}(x) = g(x)(a(x) + \sqrt{-1}b(x))
\]

where

\[
g(x_0) \neq 0, \\
da(x_0) \neq 0
\]

and \( b(x) \neq 0 \) on \( \{x; a(x) = 0\} \) or \( b(x) \equiv 0 \).

Then the normal set \( (S_M^V)_x_0 \) of \( V = \{x \in X; \mathcal{P}(x) = 0\} \) is as follows.

1) If \( b(x) \equiv 0 \), then \( (S_M^V)_x_0 = \{v \in (S_M^X)_x_0; \langle v, da(x_0) \rangle = 0\} \).

ii) If \( b(x) \geq 0 \) on \( \{x \in M; a(x) = 0\} \), then

\[
(S_M^V)_x_0 = \{v \in (S_M^X)_x_0; \langle v, da(x_0) \rangle \leq 0\}.
\]

iii) If \( b(x) \leq 0 \) on \( \{x \in M; a(x) = 0\} \), then

\[
(S_M^V)_x_0 = \{v \in (S_M^X)_x_0; \langle v, da(x_0) \rangle \geq 0\}.
\]

iv) Otherwise \( (S_M^V)_x_0 = (S_M^X)_x_0 \).
Let $P(x,D)$ be a pseudo-differential operator defined in a neighborhood of $x^\#_0 = (0,\ldots,0;\sqrt{-1}(0,\ldots,1)^\infty) \in L$. We assume that $P(x,D)$ is reductive at $x^\#_0$ and (1.1) is an irreducible decomposition of its principal symbol. Moreover we assume that each $p^i(x,\xi)$ is real and partially micro-hyperbolic at $x^\#_0 + \sqrt{-1}v$ for such an element $v$ that $\langle v,\omega \rangle = 0$ where $\omega$ is the fundamental $l$-form on $L$.

From Lemma 3.1, we have $\langle v, dp^i(x,\xi) \rangle \neq 0$. By a suitable contact transformation of $L$, we can transform $v$ into $\partial/\partial p^i$. An associated quantized contact transformation transform $P(x,D)$ so that we can take

$$p^i(x,p) = p^i - q^i(x,p^1) \quad (i = 1,\ldots,r).$$

Therefore r-bicharacteristic strips through $x^\#_0$ can be parametrized by $x_1$. We denote by $B^+_i$ the positive (respectively negative) part, that is to say, $x_1 < 0$, of i-th bicharacteristic strip through $x^\#_0$.

Our first lemma on propagation of micro-analyticity is as follows.

Lemma 3.2. Assume that $r = 2$ and that

$$\forall \nu \quad \{p^1(x,\xi), p^2(x,\xi)\} \neq 0$$

at $x^\#_0$. If $u$ is a microfunction solution of the equation $P(x,D)u = 0$ in a neighborhood of $x^\#_0$, and micro-analytic on each $B^\pm_i$ ($i = 1,2$), then $u$ is micro-analytic at $x^\#_0$.

For arbitrary $r$ we have only a partial result. We do not know whether the following condition (3.3) is necessary
or not.

Lemma 3.3. Assume that

\[(3.2) \quad \{ p_{\Delta}(x, \xi), p_{\Delta|j}(x, \xi) \} \neq 0 \quad \text{for} \quad i \neq j \]

at $x_{\Delta}$. Moreover we assume that the skew-symmetric matrix

\[ W = \{ p_{\Delta}(x, \xi), p_{\Delta|j}(x, \xi) \} \quad i, j = 1, \ldots, r \]

at $x_{\Delta}$ satisfies the following condition.

\[(3.3) \quad \text{For any principal minor} \]

\[ W_I = \{ p_{\Delta}(x, \xi), p_{\Delta|j}(x, \xi) \} \quad i, j \in I \subset \{1, \ldots, r\} \]

such that $\#(I) \geq 2$ and for any vector $\lambda = (\lambda_1, \ldots, \lambda_{\#(I)})$

such that $\lambda_1 \geq 0, \ldots, \lambda_{\#(I)} \geq 0$

\[ W_I \lambda = 0 \]

if and only if $\lambda = (0, \ldots, 0)$.

Then the conclusion of Lemma 3.2 is also valid for $r \geq 3$.

Proof. We prove Lemma 3.3 under the condition (3.3) by the induction on $r$. Let

\[ (dp_1-dq_1)^\prime = \{ v \in S_{x_0}^* L; < v, dp_1-dq_1 > < 0 \}. \]

We also denote by $(dp_1-dq_1)^\prime$ (respectively $(dp_1-dq_1)^\prime_e$) the set

\[ \{ v \in S_{x_0}^* L; < v, dp_1-dq_1 > > 0 \} \quad (\text{respectively} \quad \{ v \in S_{x_0}^* L; < v, dp_1-dq_1 >= 0 \}). \]
\[ P(x,D) \text{ is partially micro-hyperbolic at } x_0^\# + \sqrt{-1} v \]

for any \( v \) in \( \Delta = \bigcap_{i=1}^r (dp_{i1} - dq_{i1}) \). Let

\[ \Gamma = H^{-1}(\Delta) \cap \mathbb{S}^\#_{x_0^\#} L = \bigcap_{i=1}^r H^{-1}(dp_{i1} - dq_{i1}) \cap \mathbb{S}^\#_{x_0^\#} L \]

and \( \mathbb{P}^0 \) be its polar in \( \mathbb{S}^\#_{x_0^\#} L \).

From Lemma 2.1

\[ \langle -H(dp_{i1} - dq_{i1}), H^{-1}(dp_{i1} - dq_{i1}) \rangle < 0. \]

Hence \( \bigcap_{i=1}^r H^{-1}(dp_{i1} - dq_{i1}) \cap \mathbb{S}^\#_{x_0^\#} L \) is the support of \( u \). If we can show that

\[ \bigcap_{i=1}^r \mathbb{S}^\#_{x_0^\#} L \cap H^0 = \phi, \]

then a good elementary solution \( E \) for \( P(x,D) \) operates on \( u \). Hence we have

\[ u = (EP)u = E(Pu) = 0. \]

To prove (3.4), first we recall that micro-analyticity propagates along a bicharacteristic strip if \( P(x,D) \) is with constant multiple characteristics.

Let us take a point \( x_1^\# \) on \( B_{11}^- \) near \( x_0^\# \). From condition (3.2), we may assume that \( p_j - q_j(x,p) \neq 0 \) at \( x_1^\# \) for \( j \neq i \), that is, \( P(x,D) \) is of constant multiplicity at \( x_1^\# \). If \( y_1^\# \) is a point on \( V_1 = \{ p_j - q_j(x,p) = 0 \} \) near
\(x^\#_j\), \(P(x,D)\) is of constant multiplicity at \(y^\#_j\) also. From our assumptions, \(u\) is micro-analytic at \(x^\#_j\) and hence at \(y^\#_j\). If we consider the bicharacteristic strip \(B_j^j(y^\#_j)\) through \(y^\#_j\), micro-analyticity of \(u\) propagates along \(B_j^j(y^\#_j)\) until when \(B_j^j(y^\#_j)\) meets other characteristic varieties \(V_j^\#(j \neq 1)\).

Micro-analyticity of \(u\) propagates along bicharacteristic strips on \(V_j^\#\) as well as on \(V_k^\#\) hence from our induction hypothesis, \(u\) is still micro-analytic if the crossing point is of multiplicity \(\leq r-1\). (Note that from \(\Box\) assumption (3.2), a bicharacteristic strip on \(V_k^\#\) meets other \(V_k^\#\) only once near \(x^\#_0\). Hence if we put \(T = \bigcup_{j=1}^r V_j^\#\), \(u\) is micro-analytic outside of \(T \cup B_j^+(T)\) near \(x^\#_0\), where \(B_j^+(T) = \bigcup_{j=1,\ldots,r} B_j^+(x^\#_j)\).

Hence

\[
S_{x^\#_0} \cap V_1 \subset S_{x^\#_0} \cap V_1 \cap \cdots \cap V_1 \cap V_j^\# \neq \phi.
\]

It follows that if \(S_{x^\#_0} \cap V_1 \cup \cdots \cup V_j^\# \neq \phi\),

\[
v_{x^\#_0} \cap \cdots \cap v_{x^\#_0} \cup S_{x^\#_0} \cap T \neq \phi.
\]

This implies that for some \(\lambda = (\lambda_1, \ldots, \lambda_r) \neq (0, \ldots, 0)\) such that \(\lambda_1 \geq 0, \ldots, \lambda_r \geq 0\)

\[
\langle - \sum_{j=1}^r \lambda_j H(dp_{\lambda_j} dq_{\lambda_j}) \rangle = 0 \quad (k = 1, \ldots, r).
\]

From Lemma 2.1 this contradicts \(\Box\) assumption (3.3).

It is easy to see that if \(n = 2 \Box\) condition (3.2) implies \(\Box\) condition (3.3).
Lemma 3.4. Assume that \( n = 2 \) and (3.2) holds, then the conclusion of Lemma 3.2 is valid.

Next we treat the case when \( r = 2 \) and the Poisson bracket vanishes at \( x_0^* \) but not identically. The case when the Poisson bracket vanishes identically will be treated in the subsequent part II.

Lemma 3.5. Assume that \( r = 2 \) and that

\[
\{p_1(x, \xi), p_2(x, \xi)\} = 0
\]

at \( x_0^* \). Further assume the following.

(3.5) There exist positive integers \( m_1 \) and \( m_2 \) such that

\[
\{p_1, \{p_1, \ldots, \{p_1, p_2\}\}\} \neq 0 \quad \text{\( m_1 \)-times}
\]

\[
\{p_2, \{p_2, \ldots, \{p_2, p_1\}\}\} \neq 0 \quad \text{\( m_2 \)-times}
\]

at \( x_0^* \) and

(3.6) \( \{p_1, \{p_1, p_2\}\}\{p_2, \{p_1, p_2\}\} \geq 0 \)

near \( x_0^* \). Then the conclusion of Lemma 3.2 is valid.

Proof. Let \((m_1(x^*), m_2(x^*))\) be the smallest choice of the integers satisfying (3.5) at \( x_0^* \). We prove this lemma by the induction on \( m_1(x^*) \) and \( m_2(x^*) \).

As in the proof of Lemma 3.3 it is sufficient to show that

(3.7) \( v_1 \vee v_2 \cap S_{x_0^*}^T = \emptyset \)

where \( v_i = -H(dp_{i1}) \) for \( i = 1, 2 \).
and \( T = \{ x^\# \in L/m_1(x^\#) = m_2(x^\#) : i = 1,2 \} \).

Let us begin with the case when \( m_1(x^\#) = m_2(x^\#) = 2 \).

Then

\[
T = \{ p_{1\#}, p_{2\#} = \{ p_{1\#}, p_{2\#} \} = 0 \}.
\]

If (3.7) is not valid, there exists a pair \( \{ \lambda_{1\#}, \lambda_{2\#} \} \neq (0,0) \)
where \( \lambda_{i\#} \geq 0 \) \((i = 1,2)\) such that

\[
-\{ \lambda_{1\#} p_{1\#} + \lambda_{2\#} p_{2\#}, p_{i\#} \} = 0 \quad i = 1,2
\]

and

\[
-\{ \lambda_{1\#} p_{1\#} + \lambda_{2\#} p_{2\#}, \{ p_{1\#}, p_{2\#} \} \} = 0
\]

at \( x^\#_0 \). Since we have assumed (3.6), this is a contradiction.

Next we proceed \( \leftarrow \) to the case when \( m_1(x^\#) = 2 \) and
\( m_2(x^\#) = 3 \). Then

\[
T = \{ p_{1\#} = p_{2\#} = \{ p_{1\#}, p_{2\#} \} = \{ p_{2\#}, \{ p_{1\#}, p_{2\#} \} \} = 0 \}.
\]

Hence if (3.7) is not valid, there exists a pair \( \{ \lambda_{1\#}, \lambda_{2\#} \} \neq (0,0) \), where \( \lambda_{i\#} \geq 0 \) \((i = 1,2)\) such that

\[
-\{ \lambda_{1\#} p_{1\#} + \lambda_{2\#} p_{2\#}, p_{i\#} \} = 0 \quad i = 1,2,
\]

(3.8) \[
-\{ \lambda_{1\#} p_{1\#} + \lambda_{2\#} p_{2\#}, \{ p_{1\#}, p_{2\#} \} \} = 0
\]

and

(3.9) \[
-\{ \lambda_{1\#} p_{1\#} + \lambda_{2\#} p_{2\#}, \{ p_{2\#}, \{ p_{1\#}, p_{2\#} \} \} \} = 0.
\]

Since \( \{ p_{1\#}, \{ p_{1\#}, p_{2\#} \} \} \neq 0 \), (3.8) implies that \( \lambda_{1\#} = 0 \). Then, since \( \{ p_{2\#}, \{ p_{2\#}, \{ p_{1\#}, p_{2\#} \} \} \} \neq 0 \), (3.9) implies that \( \lambda_{2\#} = 0 \).

This is a contradiction. In the same way we can proceed with the step of the induction and prove the lemma.

In this section we define characteristic cones for operators treated in section 3 and show that the elementary solutions for those operators are analytic outside the characteristic cones.

Let $P(x,D)$ be a pseudo-differential operator satisfying the conditions stated in section 3. We assume that $P_i(x,p)$ $(i=1,\ldots,r)$ satisfy the assumptions of one of the lemmas 3.2 \sim 3.5.

Now we define "bicharacteristic closure" operation for $P(x,D)$.

**Definition 4.1. Bicharacteristic closure.** Let $x_0^p$ be a point in a neighborhood where $P(x,D)$ is defined. Let us pursue a bicharacteristic strip $B_j^+(x_0^p)$ through $x_0^p$. It may fall across a point $x_1^p$ of $r(x_1^p) \geq 2$. Then we proceed with pursuing $B_j^+(x_1^p)$, one of the $r(x_1^p)$-bicharacteristic strips through $x_1^p$. It may fall across again a point $x_2^p$ of $r(x_2^p) \geq 2$. Then we pursue some $B_j^+(x_2^p)$ and so on. The union of these bicharacteristic strips is denoted by $B^+(x_0^p)$ and the union of $B^+(x_0^p)$ and $x_0^p$ is called the positive bicharacteristic closure of $x_0^p$ for $P(x,D)$. Likewise we define $B^-(x_0^p)$ and the negative bicharacteristic closure. We denote by $B(x_0^p)$ the union of $B^+(x_0^p)$ and $x_0^p$ and call it the bicharacteristic closure of $x_0^p$ for $P(x,D)$.

Now let $K \in C_{\mathbb{M} \times \mathbb{M}}^{(0,n)}$ be the kernel function of a good elementary solution $E$ of Kashiwara and Kawai [9]. Then the following theorem is a trivial consequence of lemmas in section 3.
Theorem 4.1. Micro-analyticity of elementary solutions. 
\[ \text{supp} \, K \subset \{(x,y,\sqrt{-1}(\xi,\eta)) \in \sqrt{-1}\mathbb{S}^\#(M \times M); \sigma(P)(x,\xi) = \sigma(P)(y,-\eta) \] 
= 0 \text{ and } (x,\xi) \text{ belongs to the positive bicharacteristic closure of } (y,-\eta) \cup \{(x,y,\sqrt{-1}(\xi,\eta)) \in \sqrt{-1}\mathbb{S}^\#(M \times M); x=y, \xi=-\eta \}. 

Now let \( P(x,D) \) be a \( m \)-th order linear differential operator hyperbolic with respect to the direction \((1,\cdots,0)\). We assume that \( P(x,D) \) is reductive on the real characteristic variety \( \sqrt{\mathbb{R}} \), and satisfies the assumptions of one of the lemmas 3.2 \sim 3.5. We define the characteristic conoid for \( P(x,D) \).

Definition 4.2. Characteristic conoid. Let \( y \) be a point in a neighborhood where \( P(x,D) \) is defined. Let 
\[ \hat{W}^+(y) = \pi(y \in \bigcup_{\mathcal{A}} \mathcal{L}(y) \cap \mathcal{Y}_{\mathcal{B}}(y^\#)) \cup y, \text{ where } \pi: \sqrt{-1}\mathbb{S}^\#M \to M. \] 
This is called the positive characteristic conoid through \( y \) for \( P(x,D) \). Likewise we define \( \hat{W}^-(y) \) and \( W(y) \). Moreover we denote by \( \hat{W}^\pm(y) \) (respectively \( \hat{W}(y) \)) the compliment of the connected component of the compliment of \( W^+(y) \) (respectively \( W(y) \)) which contains the germ of a set 
\[ \{x, |x-y| < 1, x_\perp = y_\perp, x \neq y \}. \]

Now the following theorem is a consequence of Theorem 4.1 and the fundamental exact sequence 
\[ 0 \to A \to B \to \pi^\#(C) \to 0. \]

Theorem 4.2. Analyticity of elementary solutions.
Let $E(x,y)$ be a good elementary solution for $P(x,D)$, that is to say $E(x,y)$ is a hyperfunction satisfying

$$P(x,D_x)E(x,y) = \delta(x-y)$$

and

$$\text{supp } E(x,y) \subseteq \{(x,y) : x_1 \geq y_1\},$$

and let $E_\alpha(x,y)$ ($j = 1, \ldots, m$) be the elementary solution of the Cauchy problem for $P(x,D)$, that is, $E_\alpha(x,y)$ is a hyperfunction satisfying

$$P(x,D_x)E_\alpha(x,y) = 0$$

and

$$\left(\frac{\partial}{\partial x_1} \int_1^y E_j(x,y) \right)_{x_1 = y_1} = \delta(x-y).$$

Then we have

$$\text{sing supp } E(x,y) \subseteq \{(x,y) : x \in W^+(y)\},$$

$$\text{supp } E(x,y) \subseteq \{(x,y) : x \in \hat{W}^+(y)\},$$

$$\text{sing supp } E_\alpha(x,y) \subseteq \{(x,y) : x \in \hat{W}(y)\}$$

and

$$\text{supp } E_\alpha(x,y) \subseteq \{(x,y) : x \in \hat{W}(y)\}.$$
References


