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Propagation of Micro-analyticity for Solutions of Pseudo-differential Equations I.

By

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Abstract.

Propagation of micro-analyticity is studied for a solution u of a pseudo-differential equation $P(x,D)u = 0$ of a certain class whose characteristics are not necessarily of constant multiplicity.

Introduction.

Let $P(x,D)u = 0$ be a single pseudo-differential equation of finite order defined in a neighborhood of $x_0^* = (x_0, \sqrt{-1}\xi_0\infty)$, a point in the cosphere bundle $\sqrt{-1}S^*M$ of a real analytic manifold M. This is known to be micro-locally equivalent to the simplest equation $(\partial/\partial x_1)^m v = 0$ provided that $P(x,D)$ is real and with constant multiple characteristics. This fact implies in particular that under the same assumptions micro-analyticity or equivalently the zero of a micro-function solution u of $P(x,D)u = 0$ propagates along bicharacteristic strips of $P(x,D)$. (See Sato, Kawai and Kashiwara [1]; we also refer to Kawai [2], Hörmander [3] and Andersson [4] for linear differential operators with simple characteristics.)

It is the aim of the present article to extend this result on propagation of micro-analyticity to operators whose characteristics are not necessarily of constant multiplicity.

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In this part I we establish lemmas on propagation of micro-analyticity of a solution u of a pseudo-differential equation $P(x,D)u = 0$ for operators whose principal symbol is a product of real and simply characteristic symbols. As an easy corollary of those lemmas we deduce a theorem on analyticity of elementary solutions for linear hyperbolic differential operators with real analytic coefficients. A key to our theorem is the existence of good elementary solutions for micro-hyperbolic operators (established in Kashiwara and Kawai [9]).

A part of the results of this paper has been announced in Miwa [10].

In the case of operators with constant coefficients far reaching results have already been established by several authors (Atiyah, Bott and Gårding [5], Andersson [6], [7] and Bernstein [8]). In the subsequent part II we will extend the present results to the wider class of those pseudo-differential operators whose principal symbols are micro-locally contact-transformable to constant-coefficiented symbols so as to comprise the results of these authors.

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1. Reductive Pseudo-differential Operators and Their Bicharacteristic Strips.

In this section we introduce a class of pseudo-differential operators and define bicharacteristic strips for them.

Definition 1.1. Reductive pseudo-differential operators. Let $P(x,D)$ be a pseudo-differential operator of finite order defined in a neighborhood of $x_0^* = (x_0, \xi_0\infty)$, a point in the cotangential projective bundle P^*X of a complex manifold X. Let

$$(1.1) \qquad \sigma(P)(x,\xi) = p_1^{s_1}(x,\xi)\cdots p_r^{s_r}(x,\xi)$$

be an irreducible decomposition of its principal symbol at x_0^*. We call $P(x,D)$ reductive at x_0^* if each $p_j(x,\xi)$ is simply characteristic, that is, if $d_{(x,\xi)}p_j(x,\xi)$ is not parallel to $\sum_i \xi_i dx_i$ at x_0^*. We call r the number of crossing and $s_1+\cdots+s_r$ the multiplicity.

Examples of reductive symbols are

$$(1.2) \qquad \xi_1^2 - x_1^2(\xi_2^2+\cdots+\xi_n^2), \quad \text{and}$$

$$(1.3) \qquad (\xi_1^2-a(x)\xi_2^2-b(x)\xi_3^2)(\xi_1^2-c(x)\xi_2^2-d(x)\xi_3^2)$$

with $a(x)$, $b(x)$, $c(x)$ and $d(x)$ positive when x is real. Both of (1.2) and (1.3) are reductive on their real characteristic varieties.

Definition 1.2. Bicharacteristic strips.

Let $P(x,D)$ be as in Definition 1.1. We can define a bicharacteristic strip $B_j(x_0^*)$ of $p_j(x,\xi)$ through x_0^* as usual. We call $B_j(x_0^*)$ $(j=1,\cdots,r)$ bicharacteristic strips of $P(x,D)$ through x_0^*.

2. A Survey on Micro-hyperbolic Operators.

In this section we recall the results of Kashiwara and Kawai [9], which we need in section 3.

Let M be a real analytic manifold of dimention n and X be a complex neighborhood of it. We denote by L the cosphere bundle $S_M^*X = \sqrt{-1}S^*M$ and by Λ its complex neighborhood. We take a homogeneous local coordinates $(x_1,\cdots,x_n, \sqrt{-1}(\xi_1,\cdots,\xi_n)\infty)$ of L.

Definition 2.1. Partial micro-hyperbolicity. Let $P(x,D)$ be a pseudo-differential operator defined in a neighborhood of $x_0^* = (x_0, \sqrt{-1}\xi_0\infty)$ and V be its characteristic variety, that is, $\{(x,\xi\infty) \in \Lambda; \sigma(P)(x,\xi) = 0\}$. Let $x^* + \sqrt{-1}v0$ be a point of $\sqrt{-1}SL = S_L\Lambda$. We say that $P(x,D)$ is partially micro-hyperbolic at $x^* + \sqrt{-1}v0$ if $x^* + \sqrt{-1}v0$ is not contained in the closure of $V - L$ in ${}^L\widetilde{\Lambda}$, the real monoidal transform of Λ with the center L (see Sato, Kawai and Kashiwara [1]). This is equivalent to saying that

$$\sigma(P)(y^*+\sqrt{-1}tv) \neq 0 \tag{2.1}$$

when y^* is in L and near x^* and $0 < t < 1$.

We denote by $L \hat{\times} L$ the real analytic manifold $\sqrt{-1}S^*(M \times M) - M \times \sqrt{-1}S^*M - \sqrt{-1}\, S^*M \times M$ and we take a homogeneous local coordinates $(x_1,\cdots,x_n,x_1',\cdots,x_n', \sqrt{-1}(\xi_1,\cdots,\xi_n,\xi_1',\cdots,\xi_n')\infty)$ of $L \hat{\times} L$. We identify L with the anti-diagonal set of $L \hat{\times} L$, that is, with $\{(x,x,\sqrt{-1}(\xi,-\xi)\infty) \in L \hat{\times} L\}$.

Now let us consider in a coordinate neighborhood of L, where $\xi_n \neq 0$. We can use inhomogeneous local coordinates

$(x_1,\cdots,x_n,p_1,\cdots,p_{n-1})$ of L where $p_j = -\frac{\xi_j}{\xi_n}(j=1,\cdots,n-1)$ and $(x_1,\cdots,x_n,x_1',\cdots,x_n',p_1,\cdots,p_{n-1},p_1',\cdots,p_n')$ of $L \hat{\times} L$ where $p_j = -\frac{\xi_j}{\xi_n}(j=1,\cdots,n-1)$ and $p_j' = -\frac{\xi_j'}{\xi_n}(j=1,\cdots,n)$. L is identified with $\{(x_1,\cdots,x_n,x_1,\cdots,x_n,p_1,\cdots,p_{n-1},-p_1,\cdots,-p_{n-1},1)\}$ in $L \hat{\times} L$.

Let Θ be a subbundle of $S^*_L(L \hat{\times} L)$ induced from the fundamental 1-form on $L \hat{\times} L$. Using the local coordinates Θ can be written as

$$(2.2) \qquad \Theta = \{(x,x,p,-p,1,(dx_n - \sum_{j=1}^{n-1} p_j dx_j - dx_n' + \sum_{j=1}^{n-1} p_j dx_j')\infty)\}.$$

We consider $\hat{S}^*L$ as a subbundle of $S^*_L(L \hat{\times} L)$ of codimension 1 by the map

$$(x_1,\cdots,x_n,p_1,\cdots,p_{n-1},(\sum_{j=1}^{n} a_j dx_j + \sum_{j=1}^{n-1} b_j dp_j)\infty)$$

$$\mapsto (x,x,p,-p,1,(\sum_{j=1}^{n} a_j dx_j - \sum_{j=1}^{n} a_j dx_j' + \sum_{j=1}^{n-1} b_j dp_j + \sum_{j=1}^{n-1} b_j dp_j' + (\sum_{j=1}^{n-1} b_j p_j)dp_n')\infty).$$

Definition 2.2. Canonical map H. There is a canonical map H from $S^*_L(L \hat{\times} L) - \Theta$ to $\sqrt{-1}SL$ induced from the fundamental 1-form on L. H maps $(x,x,p,-p,1,(\sum_{j=1}^{n} a_j dx_j - \sum_{j=1}^{n} a_j dx_j' + \sum_{j=1}^{n-1} b_j dp_j + \sum_{j=1}^{n} b_j dp_j')\infty)$ to $(x,p,\sqrt{-1}(\sum_{j=1}^{n} b_j(\partial/\partial x_j) - \sum_{j=1}^{n-1}(a_n p_j + a_j)(\partial/\partial p_j))0)$.

Lemma 2.1. Let $\theta_1, \theta_2 \in S^*_{x_0^*}L$. Then we have

$$\langle H(\theta_1), \theta_2\rangle = -\langle H(\theta_2), \theta_1\rangle.$$

If we denote by [,] the Lagrangean bracket (see Sato, Kawai and Kashiwara [1]),

$$\langle H(da_1), da_2\rangle = [a_1, a_2]$$

is valid.

Now we explain the notion of normal set and conormal set. Let M be a real analytic manifold, N be its submanifold and G be a closed subset of M. The normal set of G along N is the intersection of $S_N M$ and the closure of $G-N$ in $\widetilde{N_M}$ and we denote it by $S_N G$. The polar of $S_N G$, that is,

(2.3) $\{(x,\xi\infty) \in S_N^* M; \langle\xi,v\rangle \leq 0$ for any $x+vo \in S_N G\}$

is called the conormal set of G along N and we denote it by $S_N^* G$.

Definition 2.3. $\mathcal{A}_\Gamma$ and $\mathcal{H}_\Delta$. Let x^* be a point of L and Γ be a subset in $S_L^*(L \hat{\times} L)$ over x^*. We denote $\mathcal{A}_\Gamma$ the set of all germs K of $\mathcal{C}_{M\times M}^{(0,n)}$ at x^* such that the fiber over x^* of the conormal set of the support of K along $L \subset L \hat{\times} L$ contains a neighborhood of the antipodal set of Γ. If Δ is a subset of $\sqrt{-1}S_{x^*}L$, then $\mathcal{A}_{H^{-1}(\Delta)}$ is denoted by $\mathcal{H}_\Delta$.

Proposition 2.1. Ring structure of $\mathcal{A}_\Gamma$. $\mathcal{A}_\Gamma$ is a ring by the operation $(K_1(x',x)dx, K_2(x',x)dx) \mapsto \{\int K_1(x,x'')K_2(x'',x')dx''\}dx'$ if Γ is not empty.

Definition 2.4. $\mathcal{M}_\Gamma$. Let x^* be a point of L and Γ be a subset of $S_{x^*}^*L$. The set of all germs $u(x)$ of $\mathcal{C}_M$ at x^* satisfying the following condition is denoted by $\mathcal{M}_{\Gamma}$: the normal set of the support of $u(x)$ along x^* does not intersect the polar of Γ.

Proposition 2.2. $\mathcal{M}_{\Gamma}$ is an $\mathcal{A}_{\Gamma}$ module by the operation $(K(x,x')dx', u(x)) \mapsto (Ku)(x) = \int K(x,x')u(x')dx'$.

Now we can state the main theorem in Kashiwara and Kawai [9].

Theorem 2.1. Existence of a good elementary solution. Let $P(x,D)$ be a single pseudo-differential operator of finite order which is partially micro-hyperbolic at $x^*+\sqrt{-1}v_0$. Then there exists a unique elementary solution E of P in $\mathcal{H}_{x^*+\sqrt{-1}v_0}$, that is, an element E in $\mathcal{H}_{x^*+\sqrt{-1}v_0}$ satisfying

(2.4) $$PE = EP = 1.$$

3. Lemmas on Propagation of Micro-analyticity.

In this section we prove lemmas on propagation of micro-analyticity for solutions of pseudo-differential equations.

The following lemma is another expression of Lemma 3.1.5 and Lemma 3.1.6 of Sato, Kawai and Kashiwara [1].

Lemma 3.1. Normal set of a non-singular hypersurface. Let $\varphi(x)$ be a real analytic function defined in a neighborhood of x_0, a point of a real analytic minifold M. Assume that $d\varphi(x_0) \neq 0$. We can rewrite as

$$\varphi(x) = g(x)(a(x) + \sqrt{-1}b(x))$$

where

$$g(x_0) \neq 0,$$
$$da(x_0) \neq 0$$

and $b(x) \not\equiv 0$ on $\{x; a(x) = 0\}$ or $b(x) \equiv 0$.

Then the normal set $S_M V$ of $V = \{x \in X; \varphi(x) = 0\}$ is as follows.

i) If $b(x) \equiv 0$, then $(S_M V)_{x_0} = \{v \in (S_M X)_{x_0}; \langle v, da(x_0)\rangle = 0\}$.

ii) If $b(x) \geq 0$ on $\{x \in M; a(x) = 0\}$, then
$(S_M V)_{x_0} = \{v \in (S_M X)_{x_0}; \langle v, da(x_0)\rangle \leq 0\}$.

iii) If $b(x) \leq 0$ on $\{x \in M; a(x) = 0\}$, then
$(S_M V)_{x_0} = \{v \in (S_M X)_{x_0}; \langle v, da(x_0)\rangle \geq 0\}$.

iv) Otherwise $(S_M V)_{x_0} = (S_M X)_{x_0}$.

Let $P(x,D)$ be a pseudo-differential operator defined in a neighborhood of $x_0^* = (0,\cdots,0;\sqrt{-1}(0,\cdots,1)\infty) \in L$. We assume that $P(x,D)$ is reductive at x_0^* and (1.1) is an irreducible decomposition of its principal symbol. Moreover we assume that each $p_i(x,\xi)$ is real and partially micro-hyperbolic at $x_0^* + \sqrt{-1}v0$ for such an element v that $\langle v,\omega\rangle = 0$ where ω is the fundamental 1-form on L.

From Lemma 3.1, we have $\langle v,dp_i(x,\xi)\rangle \neq 0$. By a suitable contact transformation of L, we can transform v into $\partial/\partial p_1$. An associated quantized contact transformation transform $P(x,D)$ so that we can take

$$p_i(x,p) = p_1 - q_i(x,p') \quad (i = 1,\cdots,r).$$

Therefore r-bicharacteristic strips through x_0^* can be parametrized by x_1. We denote by $B_i^{\pm}$ the positive (respectively negative) part, that is to say, (the part where) $x_1 \gtrless 0$, of i-th bicharacteristic strip through x_0^*.

Our first lemma on propagation of micro-analyticity is as follows.

Lemma 3.2. Assume that $r = 2$ and that

(3.1) $$\{p_1(x,\xi), p_2(x,\xi)\} \neq 0$$

at x_0^*. If u is a microfunction solution of the equation $P(x,D)u = 0$ in a neighborhood of x_0^*, and micro-analytic on each B_i^- $(i = 1,2)$, then u is micro-analytic at x_0^*.

For arbitrary r we have only a partial result. We do not know whether the following condition (3.3) is necessary or not.

Lemma 3.3. Assume that

(3.2) $\{p_i(x,\xi),\ p_j(x,\xi)\} \neq 0$ for $i \neq j$

at x_0^*. Moreover we assume that the skew-symmetric matrix

$$W = (\{p_i(x,\xi),\ p_j(x,\xi)\}) \quad i,j = 1,\cdots,r$$

at x_0^* satisfies the following condition.

(3.3) For any principal minor

$$W_I = (\{p_i(x,\xi),\ p_j(x,\xi)\}) \quad i,j \in I \subset \{1,\cdots,r\}$$

such that $\#(I) \geq 2$ and for any vector $\lambda = (\lambda_1,\cdots,\lambda_{\#(I)})$ such that $\lambda_1 \geq 0,\ \cdots,\lambda_{\#(I)} \geq 0$

$$W_I\lambda = 0$$

if and only if $\lambda = (0,\cdots,0)$.

Then the conclusion of Lemma 3.2 is also valid for $r \geq 3$.

Proof. We prove Lemma 3.3 under トル ~~the~~ condition (3.3) by the induction on r. Let

$$(dp_1-dq_i)^{-} = \{v \in S_{x_0^*}L;\ \langle v, dp_1-dq_i\rangle < 0\}.$$

We also denote by $(dp_1-dq_i)^{+}$ (respectively $(dp_1-dq_i)^{\perp}$) the set

$$\{v \in S_{x_0^*}L;\ \langle v, dp_1-dq_i\rangle > 0\} \text{ (respectively}$$

$$\{v \in S_{x_0^*}L;\ \langle v, dp_1-dq_i\rangle = 0\}\ .$$

$P(x,D)$ is partially micro-hyperbolic at $x_0^* + \sqrt{-1}v_0$ for any v in $\Delta = \bigcap_{i=1}^{r} (dp_1 - dq_i)^{\vee}$. Let

$$\Gamma = H^{-1}(\Delta) \cap S^*_{x_0^*}L = (\bigcap_{i=1}^{r} H^{-1}(dp_1 - dq_i)^{\vee}) \cap S^*_{x_0^*}L$$

and Γ^0 be its polar in $S_{x_0^*}L$.

From Lemma 2.1

$$\langle -H(dp_1 - dq_i),\ H^{-1}(dp_1 - dq_i)^{\vee} \rangle = \langle dp_1 - dq_i,\ (dp_1 - dq_i)^{\vee} \rangle < 0.$$

Hence $\{H^{-1}(dp_1 - dq_i)^{\vee} \cap S^*_{x_0^*}L\}^0 = -H(dp_1 - dq_i)$.

Put $v_i = -H(dp_1 - dq_i)$. Then

$$\Gamma^0 = v_1 \vee \cdots \vee v_r = \{ \sum_{i=1}^{r} \lambda_i v_i ; \lambda_i \geq 0 \quad i=1,\cdots,r \text{ and } (\lambda_1,\cdots,\lambda_r) \neq (0,\cdots,0)\}.$$

We denote by G the support of u. If we can show that

$$(3.4) \qquad S_{x_0^*}G \cap \Gamma^0 = \phi,$$

then a good elementary solution E for $P(x,D)$ operates on u. Hence we have

$$u = (EP)u = E(Pu) = 0.$$

To prove (3.4), first we recall that micro-analyticity propagates along a bicharacteristic strip if $P(x,D)$ is with constant multiple characteristics.

Let us take a point x_1^* on B_i^- near x_0^*. From condition (3.2), we may assume that $p_1 - q_j(x,p) \neq 0$ at x_1^* for $j \neq i$, that is, $P(x,D)$ is of constant multiplicity at x_1^*. If y_1^* is a point on $V_i = \{p_1 - q_i(x,p) = 0\}$ near

x_1^*, $P(x,D)$ is of constant multiplicity at y_1^* also. From our assumptions, u is micro-analytic at x_1^* and hence at y_1^*. If we consider the bicharacteristic strip $B_i(y_1^*)$ through y_1^*, micro-analyticity of u propagates along $B_i(y_1^*)$ until when $B_i(y_1^*)$ meets other characteristic varieties V_j $(j \neq i)$.

Micro-analyticity of u propagates along bicharacteristic strips on V_j as well as on V_i, hence from our induction hypothesis, u is still micro-analytic if the crossing point is of multiplicity $\leq r-1$. (Note that from assumption (3.2), a bicharacteristic strip on V_k meets other $V_{k'}$ only once near x_0^*. Hence if we put $T = \bigcap_{j=1}^{r} V_j$, u is micro-analytic outside of $T \cup B^+(T)$ near x_0^*, where $B^+(T) = \bigcup_{\substack{x^* \in T \\ j=1,\cdots,r}} B_j^+(x^*)$.

Hence

$$S_{x_0^*}G \cap V_i \subset S_{x_0^*}T \vee (-v_i).$$

It follows that if $S_{x_0^*}G \cap \Gamma^0 \cap V_i \neq \phi$,

$$v_1 \vee \cdots \vee v_r \cap S_{x_0^*}T \neq \phi.$$

This implies that for some $\lambda = (\lambda_1,\cdots,\lambda_r) \neq (0,\cdots,0)$ such that $\lambda_1 \geq 0,\cdots,\lambda_r \geq 0$

$$\langle -\sum_{j=1}^{r} \lambda_j H(dp_1-dq_j),\ dp_1-dq_k \rangle = 0 \quad (k = 1,\cdots,r).$$

From Lemma 2.1 this contradicts assumption (3.3).

It is easy to see that if $n = 2$ condition (3.2) implies condition (3.3).

Lemma 3.4. Assume that $n = 2$ and (3.2) holds, then the conclusion of Lemma 3.2 is valid.

Next we treat the case when $r = 2$ and the Poisson bracket vanishes at x_0^* but not identically. The case when the Poisson bracket vanishes identically will be treated in the subsequent part II.

Lemma 3.5. Assume that $r = 2$ and that

$$\{p_1(x,\xi),\ p_2(x,\xi)\} = 0$$

at x_0^*. Further assume the following.

(3.5) There exist positive integers m_1 and m_2 such that

$$\underbrace{\{p_1,\{p_1,\cdots,\{p_1,p_2\}\}\}}_{m_1\text{-times}} \neq 0$$

$$\underbrace{\{p_2,\{p_2,\cdots,\{p_2,p_1\}\}\}}_{m_2\text{-times}} \neq 0$$

at x_0^* and

(3.6) $$\{p_1,\{p_1,p_2\}\}\{p_2,\{p_1,p_2\}\} \geqq 0$$

near x_0^*. Then the conclusion of Lemma 3.2 is valid.

Proof. Let $(m_1(x^*),\ m_2(x^*))$ be the smallest choice of the integers satisfying (3.5) at x_0^*. We prove this lemma by the induction on $m_1(x^*)$ and $m_2(x^*)$.

As in the proof of Lemma 3.3 it is sufficient to show that

(3.7) $$v_1 \vee v_2 \cap S_{x_0^*}T = \phi$$

where $v_i = -H(dp_i)$ $i=1,2$

and $T = \{x^* \in L / m_i(x^*) = m_i(x_0^*) \quad i = 1,2\}$.

Let us begin with the case when $m_1(x_0^*) = m_2(x_0^*) = 2$. Then

$$T = \{p_1 = p_2 = \{p_1,p_2\} = 0\}.$$

If (3.7) is not valid, there exists a pair $(\lambda_1,\lambda_2) \neq (0,0)$ where $\lambda_i \geq 0$ $(i = 1,2)$ such that

$$-\{\lambda_1 p_1 + \lambda_2 p_2,\ p_i\} = 0 \quad i = 1,2$$

and

$$-\{\lambda_1 p_1 + \lambda_2 p_2,\ \{p_1,p_2\}\} = 0$$

at x_0^*. Since we have assumed (3.6), this is a contradiction.

Next we proceed to the case when $m_1(x_0^*) = 2$ and $m_2(x_0^*) = 3$. Then

$$T = \{p_1 = p_2 = \{p_1,p_2\} = \{p_2,\{p_1,p_2\}\} = 0\}.$$

Hence if (3.7) is not valid, there exists a pair $(\lambda_1,\lambda_2) \neq (0,0)$, where $\lambda_i \geq 0$ $(i = 1,2)$ such that

$$-\{\lambda_1 p_1 + \lambda_2 p_2,\ p_i\} = 0 \quad i=1,2,$$

$$(3.8) \qquad -\{\lambda_1 p_1 + \lambda_2 p_2,\ \{p_1,p_2\}\} = 0$$

and

$$(3.9) \qquad -\{\lambda_1 p_1 + \lambda_2 p_2, \{p_2,\{p_1,p_2\}\}\} = 0.$$

Since $\{p_1,\{p_1,p_2\}\} \neq 0$, (3.8) implies that $\lambda_1 = 0$. Then, since $\{p_2,\{p_2,\{p_1,p_2\}\}\} \neq 0$, (3.9) implies that $\lambda_2 = 0$. This is a contradiction. In the same way we can proceed with the step of the induction and prove the lemma.

4. Analyticity of Elementary Solutions.

In this section we define characteristic conoids for operators treated in section 3 and show that the elementary solutions for those operators are analytic outside the characteristic conoids.

Let $P(x,D)$ be a pseudo-differential operator satisfying the conditions stated in section 3. We assume that $p_i(x,p)$ $(i=1,\cdots,r)$ satisfy the assumptions of one of the lemmas 3.2 ∿ 3.5.

Now we define "bicharacteristic closure" operation for $P(x,D)$.

Definition 4.1. Bicharacteristic closure. Let x^*_0 be a point in a neighborhood where $P(x,D)$ is defined. Let us pursue a bicharacteristic strip $B^+_{j_0}(x^*_0)$ through x^*_0. It may fall across a point x^*_1 of $r(x^*_1) \geq 2$. Then we proceed with pursuing $B^+_{j_1}(x^*_1)$, one of the $r(x^*_1)$-bicharacteristic strips through x^*_1. It may fall across again a point x^*_2 of $r(x^*_2) \geq 2$. Then we pursue some $B^+_{j_2}(x^*_2)$ and so on. The union of these bicharacteristic strips is denoted by $B^+(x^*_0)$ and the union of $B^+(x^*_0)$ and x^*_0 is called the positive bicharacteristic closure of x^*_0 for $P(x,D)$. Likewise we define $B^-(x^*_0)$ and the negative bicharacteristic closure. We denote by $B(x^*_0)$ the union of $B^\pm(x^*_0)$ and x^*_0 and call it the bicharacteristic closure of x^*_0 for $P(x,D)$.

Now let $K \in \mathcal{C}^{(0,n)}_{M\times M}$ be the kernel function of a good elementary solution E of Kashiwara and Kawai [9]. Then the following theorem is a trivial consequence of lemmas in section 3.

Theorem 4.1. Micro-analyticity of elementary solutions.

supp $K \subset \{(x,y,\sqrt{-1}(\xi,\eta)) \in \sqrt{-1}S^*(M\times M); \sigma(P)(x,\xi) = \sigma(P)(y,-\eta) = 0$ and (x,ξ) belongs to the positive bicharacteristic closure of $(y,-\eta)\} \cup \{(x,y,\sqrt{-1}(\xi,\eta)) \in \sqrt{-1}S^*(M\times M); x=y,\ \xi=-\eta\}$.

Now let $P(x,D)$ be a m-th order linear differential operator hyperbolic with respect to the direction $(1,\cdots,0)$. We assume that $P(x,D)$ is reductive on the real characteristic variety $V_{\mathbb{R}}$, and satisfies the assumptions of one of the lemmas 3.2 $\sim$ 3.5. We define the characteristic conoid for $P(x,D)$.

Definition 4.2. Characteristic conoid. Let y be a point in a neighborhood where $P(x,D)$ is defined. Let

$$W^{+}(y) = \pi\Big(\bigcup_{y^*\in\pi^{-1}(y)\cap V_{\mathbb{R}}} B^{+}(y^*)\Big)\cup y, \text{ where } \pi: \sqrt{-1}S^*M \to M.$$

This is called the positive characteristic conoid through y for $P(x,D)$. Likewise we define $W^{-}(y)$ and $W(y)$. Moreover we denote by $\hat{W}^{+}(y)$ (respectively $\hat{W}(y)$) the compliment of the connected component of the compliment of $W^{+}(y)$ (respectively $W(y)$) which contains the germ of a set

$$\{x; |x-y| \ll 1,\ x_1 = y_1,\ x \neq y\}.$$

Now the following theorem is a consequence of Theorem 4.1 and the fundamental exact sequence

$$0 \to \mathcal{A} \to \mathcal{B} \to \pi_*(\mathcal{C}) \to 0.$$

Theorem 4.2. Analyticity of elementary solutions.

Let $E(x,y)$ be a good elementary solution for $P(x,D)$, that is to say $E(x,y)$ is a hyperfunction satisfying

$$P(x,D_x)E(x,y) = P^*(y,D_y)E(x,y) = \delta(x-y)$$

and

$$\mathrm{supp}\ E(x,y) \subset \{(x,y);\, x_1 \geqq y_1\},$$

and let $E_j(x,y)$ $(j = 1,\cdots,m)$ be the elementary solution of the Cauchy problem for $P(x,D)$, that is, $E_j(x,y)$ is a hyperfunction satisfying

$$P(x,D_x)E_j(x,y) = 0$$

and

$$(\partial/\partial x_1)^{i-1}E_j(x,y)\big|_{x_1=y_1} = \delta_{ij}\delta(x-y).$$

Then we have

$$\text{sing supp}\ E(x,y) \subset \{(x,y);\, x \in W^+(y)\},$$

$$\mathrm{supp}\ E(x,y) \subset \{(x,y);\, x \in \hat{W}^+(y)\},$$

$$\text{sing supp}\ E_j(x,y) \subset \{(x,y);\, x \in W(y)\}$$

and

$$\mathrm{supp}\ E_j(x,y) \subset \{(x,y);\, x \in \hat{W}(y)\}.$$

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