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Structure of cohomology groups whose coefficients are microfunction solution sheaves of systems of pseudo-differential equations with multiple characteristics. I.

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In this note we investigate the structure of cohomology groups whose coefficients are the microfunction solution sheaf of a system $\mathcal{M}$ of linear pseudo-differential equations, when the characteristic variety V of $\mathcal{M}$ has singularities of rather limited type, i.e., V has the form $V_1 \cup V_2$, where at least either one of them is regular. The precise conditions on V and $\mathcal{M}$ will be given in the below. Our method consists in two steps: firstly we investigate the structure of the system $\mathcal{M}$ itself in the complex domain and secondly we calculate the cohomology groups by making use of a special expression of sheaf $\mathcal{C}$ of microfunctions. In the first step we rely on the results obtained in Chapter II of Sato-Kawai-Kashiwara [9] (hereafter referred to as S-K-K [9]) and in the second step we resort to the theory of boundary value problems for elliptic system of linear differential equations, which is expounded in Kashiwara-Kawai [7]. These arguments will be also used in Kashiwara [6].

Note that the investigation of cohomology groups whose coefficients are the microfunction solution sheaf of the system of pseudo-differential equations are fully investigated in S-K-K [9] under the assumption that its characteristic variety is regular, that is, V is non-singular as a variety and $\omega|_V \neq 0$ for the canonical 1-form ω of the contact manifold in which V lies. See also Oshima [8] for the case of maximally degenerate system. We also note that great efforts have recently

been made to clarify the situation we encounter in the last half part of this note by Grušin [5], Treves [12], Boutet de Monvel-Treves [1], [2], Boutet de Monvel [13], Sjöstrand [10], Folland [3], Folland-Stein [4], Taira [11], and others in the case of determined systems, though they only discuss the (micro-local) C^{∞}-regularity with the exception of Grušin [5], where the (global) C^{ω}-regularity is also descussed.

In order to present all theorems in the simplest form as possible as we can, we have restricted the investigation to the simplest and the most typical cases in this note. The detailed arguments and the full scope of the applicability of our method will appear elsewhere.

We now begin with the following theorem, which tells that a new difficulty appears only in the situations treated in Theorems 3, 4 and 5 in the below as long as the geometrical structure of the crossing is "generic". In fact we need not seek for any new means other than those employed in S-K-K [9], when we deal with the equations treated in Theorem 2 in the below.

<u>Theorem 1.</u> Let $\mathfrak{M}$ be an admissible system of pseudo-differential equations. Assume that its characteristic variety V is contained in $V_1 \cup V_2$, where V_j is a regular submanifold (i.e., non-singular and $\omega|_{V_j} \neq 0$ for the canonical 1-form ω). Assume further that $\mathrm{codim}_{\mathfrak{b}_j}(\mathfrak{b}_j \cap V_k)$ $(j \neq k,\ j,k = 1,2)$ is greater than or equal to 2 for any bicharacteristic $\mathfrak{b}_j$ of V_j $(j=1,2)$. Then $\mathfrak{M}$ is canonically decomposed into the direct sum $\mathfrak{M}_1 \oplus \mathfrak{M}_2$, where $\mathfrak{M}_j$ is an admissible system of pseudo-differential equations whose support is contained in V_j ($j=1,2$, respectively).

The proof of this theorem relies on the fundamental structure theorem for systems of pseudo-differential equations in complex domain (Theorem 5.3.1 in Chap. II of S-K-K [9]) and an extension theorem of

Hartogs' type for systems of pseudo-differential equations.

Remark. When we only assume that V_1 is regular and that $\operatorname{codim}_{\mathcal{b}_1}(\mathcal{b}_1 \cap V_2)$ is greater than or equal to 2 for any bicharacteristic $\mathcal{b}_1$ of V_1, the conclusion of Theorem 1 still holds under the assumption that $\operatorname{proj\,dim}\mathcal{M} \leq \operatorname{codim} V_1$.

Theorem 1 combined with Theorem 2.3.10 in Chap. III of S-K-K [9] easily proves the following

Theorem 2. Assume that $\mathcal{M}$ defined on an open set Ω in $\sqrt{-1}S^*M$ satisfies conditions in Theorem 1 in a complex neighborhood of Ω. Assume further that the generalized Levi form L_j of $V_j (j=1,2)$ has at least $(q+1)$ negative eigenvalues or at least $(d_{x^*}-q+1)$ positive eigenvalues at any point x^* in $V_j \cap V_j^c$ $(j=1,2$, respectively). Here d_{x^*} denotes the projective dimension of $\mathcal{M}$ at x^*. Then

(1) $\operatorname{Ext}^q_{\mathcal{P},Z}(\Omega;\mathcal{M}, \mathcal{C}_M) = 0$

holds for any locally closed subset Z of Ω.

Taking account of Theorem 1 we investigate in detail the case where $V_1 \cap V_2$ has codimension 1 both in V_1 and in V_2. In this case an interesting complexity appears as the works of Treves and others referred to at the beginning of this notes suggest.

Before stating the precise conditions on $\mathcal{M}$ we prepare the following notion concerning the structure of a submanifold V of a contact manifold with a canonical 1-form ω.

If $(\omega \wedge (d\omega)^r)(x^*) \neq 0$ and $(\omega \wedge (d\omega)^{r+1})(x^*) = 0$, then x^* is said to be a point of rank $2r$. If each point x^* in V is of rank $2r$, then V is said to be of rank $2r$.

If we use this terminology, then saying that $\operatorname{rank} V$ is equal to $2\dim V - \dim X - 1$ is clearly equivalent to saying that V is regular.

If X is a purely imaginary contact manifold, then the Poisson bracket induces a skew symmetric form $\{\xi,\eta\}$ on the cotangent vector

space $T^*_{x^*}X$ up to positive constant multiple. We define the hermition form on $\mathbb{C} \otimes_{\mathbb{R}} T^*_{x^*}X = T^*_{x^*}X^{\mathbb{C}}$ by $\{\xi,\bar{\eta}\}$. Note that the generalized Levi form of a submanifold V of the complexification $X^{\mathbb{C}}$ of X is nothing but the restriction of this hermitian form onto the conormal vector space $(T^*_V X^{\mathbb{C}})_{x^*}$.

Now we state the precise conditions on $\mathcal{M}$.

The system $\mathcal{M}$ of pseudo-differential equations we will study satisfies the following conditions (2) ~ (5)

(2) $\mathcal{M}$ has the form $\mathcal{P}^f/\mathcal{J}$, where symbol ideal J of $\mathcal{J}$ is reduced.

(3) $\mathrm{Supp}\,\mathcal{M} = V_1 \cup V_2$, where V_1 and V_2 are regular and V_1 and V_2 intersect regularly along $V_{12} = V_1 \cap V_2$, i.e.,

$T_{x^*}V_1 \cap T_{x^*}V_2 = T_{x^*}V_{12}$ for any x^* in $V_1 \cap V_2$.

(4) $\dim V_1 = \dim V_2 = \dim V_{12}+1$ and $\mathrm{rank}\, V_1 = \mathrm{rank}\, V_2 = \mathrm{rank}\, V_{12}$.

By virtue of conditions (2), (3) and (4), $\mathcal{J}$ contains a pseudo-differential operator P which has the form P_1P_2+Q where P_j and Q are pseudo-differential operators of order m_j and m_1+m_2-1 respectively satisfying the following:

$$\sigma(P_j)|_{V_j} = 0 \quad \text{and} \quad \{\sigma(P_1),\sigma(P_2)\}|_{V_1 \cap V_2}$$

never vanishes. Here $\sigma(P_j)$ denotes the principal symbol of P_j. Then the last condition is the following:

(5) $\left(d\left(\dfrac{\sigma(Q)}{\{\sigma(P_2),\sigma(P_1)\}}\right) \wedge \omega\right)|_{V_1 \cap V_2}$ never vanishes.

<u>Remark.</u> The analytic function $\kappa = \sigma(Q)/\{\sigma(P_2),\sigma(P_1)\}$ restricted onto $V_{12} = V_1 \cap V_2$ is well-defined by $\mathcal{J}$ up to the permutation between V_1 and V_2. The permutation will bring κ into $1-\kappa$. See also Boutet de Monvel-Treves [1].

Under these assumptions we may find a canonical form in complex domain of such a system of pseudo-differential equations by a "quantized" contact transformation, that is, we have the following

theorem.

Theorem 3. A system $\mathcal{M}$ of pseudo-differential equations satisfying conditions (2)$\sim$(5) may be micro-locally brought to the following system $\mathcal{N}$ in complex domain by a suitable contact transformation.

$$\mathcal{N}:\begin{cases}(z_1\frac{\partial u}{\partial z_1} - z_{n-1})u = 0\\ \frac{\partial u}{\partial z_j}u = 0,\ j = 2,\dots,d\ (d < n-1).\end{cases}$$

Here we consider $\mathcal{N}$ near $(z;\zeta_1,\dots,\zeta_{n-1},\zeta_n) = (0;\ 0,\dots,0,1)$.

The proof of this theorem is performed in the following two steps: firstly, conditions (2), (3) and (4) allow us to bring $\mathcal{M}$ micro-locally into the following system $\mathcal{M}'$ by the aid of a suitable "quantized" contact transformation and by the application of the division theorem of Weierstrass' type for pseudo-differential operators.

$$\mathcal{M}':\begin{cases}(z_1\frac{\partial u}{\partial z_1} + Q(z',\ D'))u = 0\\ \frac{\partial u}{\partial z_j} = 0,\ j = 2,\dots,d\ .\end{cases}$$

Here z' and D' denote $(z_{d+1},\dots,z_n)$ and $(\frac{\partial}{\partial z_{d+1}},\dots,\frac{\partial}{\partial z_n})$ and $Q(z',\ D')$ is a pseudo-differential operator of order 0.

Nextly condition (5b) allows us to find a "quantized" contact transformation which commutes with z_1 and $\frac{\partial}{\partial z_1}$ and which brings $\mathcal{M}'$ micro-locally into the following system

$$\mathcal{N}:\begin{cases}(z_1\frac{\partial}{\partial z_1} - z_{n-1})u = 0\\ \frac{\partial u}{\partial z_j} = 0,\ j = 2,\dots,d.\end{cases}$$

Now we apply Theorem 3 to calculate the cohomology groups having as coefficients the microfunction solution sheaf of $\mathcal{M}$ which satisfies the following conditions (6)$\sim$(8) beside conditions (2)$\sim$(5).

(6) The generalized Levi forms of V_1 and V_2 are non-degenerate, and the hermitian form $\{\xi,\overline{\eta}\}$ is non-degenate on $(T^*_{V_1}X^{\mathbb{C}})_{x^*} \cap (T^*_{V_2}X^{\mathbb{C}})_{x^*}$, where $X^{\mathbb{C}}$ is the complexification of $\sqrt{-1}S^*M$.

(7) $V_1 \cap V_1^c = V_2 \cap V_2^c$. We denote by W the real locus of this manifold, which turns out to be a purely imaginary contact manifold.

(8) $\{\sigma(Q)|_W, \sigma(Q)^c|_W\} \neq 0$

(9) $\left.\dfrac{\sigma(Q)}{\{\sigma(P_2),\sigma(P_1)\}}\right|_{V_1 \cap V_2}$ never takes integral values.

Remark. Condition (9) is imposed here in order to make the argument simpler. In our subsequent note we discuss the case where condition (9) fails. Condition (8) is a technical one and we hope we will be able to drop it.

Under these assumptions we separate the problem into the following two cases. We denote by L_j the generalized Levi form of V_j ($j = 1, 2$, respectively). Its signature (p,q) means by definition the pair of the numbers of its positive eigenvalues and its negative eigenvalues.

Case A. The signature of L_1 is different from that of L_2.

Case B. The signature of L_1 is equal to that of L_2. We denote it by $(d-q,q)$.

Theorem 4. Assume that $\mathcal{M}$ satisfies conditions (2)~(9). Then, in Case A, we have

(10) $\mathcal{E}xt^j_{\mathcal{P}}(\mathcal{M},\mathcal{C}) = 0$

for any j.

Theorem 5. Assume that $\mathcal{M}$ satisfies conditions (2)~(9). Then, in Case B,

(11) $\mathcal{E}xt^j_{\mathcal{P}}(\mathcal{M},\mathcal{C}) = 0$

holds for $j \neq q$ and the remaining q-th cohomology group is isomorphic to $\mathcal{C}^2_W$. Here $\mathcal{C}_W$ denotes the sheaf of microfunctions on $W = V_1 \cap \sqrt{-1}S^*M = V_2 \cap \sqrt{-1}S^*M$.

In order to prove these theorems we realize sheaf $\mathcal{C}$ of microfunctions on the conormal bundle S_H^*Y of a hypersurface H of real codimension 1 in a complex manifold of Y and investigate the structure of holomorphic solutions of the system $\mathcal{N}$ on Y.

Remark. We may discuss the micro-local solvability and regularity for a wider class of pseudo-differential equations other than the equations discussed here. The equation $[\sum_{k=1}^{n}(Z_k\overline{Z}_k + \overline{Z}_kZ_k) - i\alpha\frac{\partial}{\partial t}]u(x,t) = 0$ with $Z_k = \frac{1}{2}(\frac{\partial}{\partial x_k} + i\frac{\partial}{\partial x_{n+k}} - 2i(x_k + ix_{n+k})\frac{\partial}{\partial t})$ $(k = 1, \cdots, n)$ and $\alpha \in \mathbb{C}$ falls into such a class. This is the equation of the type discussed in Folland [3]. See also Folland-Stein [4]. We note that not only the C^∞-regularity but also C^ω-regularity does hold for the solutions of this equation if $\alpha \neq \pm n, \pm(n+2), \pm(n+4), \cdots$.

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Structure of cohomology groups whose coefficients are microfunction solution sheaves of systems of pseudo-differential equations with multiple characteristics. II.

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This is a continuation of our preceding note Kashiwara-Kawai-Oshima [1], hereafter referred to as K-K-O [1]. The purpose of this note is to investigate the structure of cohomology groups whose coefficients are microfunction solution sheaf of a system $\mathcal{M}$ of pseudo-differential equations which satisfies conditions (2) $\sim$ (8) in K-K-O [1], but does not necessarily satisfy condition (9) in general. The details of this note will appear elsewhere.

In this note we use the same notations as in K-K-O [1]. For example, W denotes the real locus of $V_1 \cap V_1^c = V_2 \cap V_2^c$. Since W acquires canonically the structure of a purely imaginary contact manifold by condition (6) in K-K-O [1], sheaf $\mathcal{C}_W$ of microfunctions and sheaf $\mathcal{P}_W$ of pseudo-differential operators can be defined on W.

When $\kappa = \left.\dfrac{\sigma(Q)}{\{\sigma(P_2),\sigma(P_1)\}}\right|_{V_1 \cap V_2}$ takes an integral value, the structure of κ plays an important role in calculating the cohomology groups. So we give the following preparatory consideration concerning lower order terms.

Let R be a pseudo-differential operator on W whose principal symbol is κ. Such a pseudo-differential operator R is uniquely determined up to inner automorphism of $\mathcal{P}_W$ by condition (5) in K-K-O [1]. (See Theorem 2.1.2 in Chap. II of Sato-Kawai-Kashiwara [2].) Taking account of this fact, we denote by $\mathcal{L}_\ell$ the pseudo-differential equation $(R-\ell)u = 0$ on W for $\ell \in \mathbb{Z}$.

In order to calculate the cohomology groups when κ takes an integral value, we should study in the following four cases classified according to the signatures of the generalized Levi forms of V_1, V_2 and $T^*_{V_1}X^{\mathbb{C}} \cap T^*_{V_2}X^{\mathbb{C}}$. We denote by L_j the generalized Levi form of V_j (j=1, 2, respectively) and by L the hermitian form $\{\xi, \bar{\eta}\}$ on $(T^*_{V_1}X^{\mathbb{C}})_{X^*} \cap (T^*_{V_2}X^{\mathbb{C}})_{X^*}$.

Case A_1. The signature of L_1 is $(d-q-1, q+1)$, that of L_2 is $(d-q, q)$ and that of L is $(d-q-1, q)$.

Case A_2. The signature of L_2 is $(d-q-1, q+1)$, that of L_1 is $(d-q, q)$ and that of L is $(d-q-1, q)$.

Case B_1. Both the signature of L_1 and L_2 are $(d-q, q)$ and the signature of L is $(d-q, q-1)$. $(1 \leqq q \leqq d)$

Case B_2. Both the signature of L_1 and L_2 are $(d-q, q)$ and the signature of L is $(d-q-1, q)$. $(0 \leqq q \leqq d-1)$

<u>Theorem 1.</u> In Case A_1, we have

$$\mathcal{E}xt^j_{\mathcal{P}}(\mathcal{M}, \mathcal{C}) \cong \bigoplus_{\ell=0,-1,-2,\cdots} \mathcal{E}xt^{j-q}_{\mathcal{P}_W}(\mathcal{L}_\ell, \mathcal{C}_W)$$

for every j,

and, in Case A_2, we have

$$\mathcal{E}xt^j_{\mathcal{P}}(\mathcal{M}, \mathcal{C}) \cong \bigoplus_{\ell=1,2,3,\cdots} \mathcal{E}xt^{j-q}_{\mathcal{P}_W}(\mathcal{L}_\ell, \mathcal{C}_W)$$

for every j.

<u>Theorem 2.</u> In Case B_1, we have

$$\mathcal{E}xt^{j}_{\mathcal{P}}(\mathcal{M}, \mathcal{C}) = 0 \quad \text{for} \quad j \neq q-1,\ q$$

and the following exact sequence holds:

$$0 \longrightarrow \mathcal{E}xt^{q-1}_{\mathcal{P}}(\mathcal{M}, \mathcal{C}) \longrightarrow \bigoplus_{\ell=0,-1,-2,\ldots} \mathcal{H}om_{\mathcal{P}_W}(\mathcal{L}_\ell, \mathcal{C}_W) \longrightarrow \mathcal{C}^2_W$$

$$\longrightarrow \mathcal{E}xt^{q}_{\mathcal{P}}(\mathcal{M}, \mathcal{C}) \longrightarrow \bigoplus_{\ell=0,-1,-2,\ldots} \mathcal{E}xt^{1}_{\mathcal{P}_W}(\mathcal{L}_\ell, \mathcal{C}_W) \longrightarrow 0.$$

<u>Remark.</u> We conjecture that $\mathcal{E}xt^{q-1}_{\mathcal{P}}(\mathcal{M}, \mathcal{C}) = 0$ holds in this case.

<u>Theorem 3.</u> In Case B_2, we have

$$\mathcal{E}xt^{j}_{\mathcal{P}}(\mathcal{M}, \mathcal{C}) = 0 \quad \text{for} \quad j \neq q,$$

and the following exact sequence holds:

$$0 \longrightarrow \bigoplus_{\ell=0,-1,-2,\ldots} \mathcal{H}om_{\mathcal{P}_W}(\mathcal{L}_\ell, \mathcal{C}_W) \longrightarrow \mathcal{C}^2_W \longrightarrow \mathcal{E}xt^{q}_{\mathcal{P}}(\mathcal{M}, \mathcal{C})$$

$$\longrightarrow \bigoplus_{\ell=0,-1,-2,\ldots} \mathcal{E}xt^{1}_{\mathcal{P}_W}(\mathcal{L}_\ell, \mathcal{C}_W) \longrightarrow 0.$$

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