

Vector Valued Pseudodifferential Operators
and their Applications

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§1. Introduction and the result. In this note we introduce the vector valued pseudodifferential operators where the vector space depends on a parameter (cf. Sjöstrand [3], §4) and take this opportunity to construct global parametrix-like operators for the following operator in R^n :

$$P(x, t, D_x, D_t) = D_t - it^k a(x, t, D_x, D_t) + b(x, t, D_x, D_t),$$

where $(x, t) \in R^n$ with $x \in R^{n-1}$, $t \in R$ and $k \in Z^+$ is odd, and $a(x, t, D_x, D_t)$ and $b(x, t, D_x, D_t)$ are properly supported classical pseudodifferential operators of order 1 and order 0 respectively, and the principal symbol $a_1(x, t, \xi, \tau)$ of $a(x, t, D_x, D_t)$ is positively homogeneous of degree 1 and

$$(A) \quad \operatorname{Re} a_1(x, t, \xi, \tau) \neq 0$$

for $(x, t) \in R^n$, $(\xi, \tau) \neq (0, 0)$, $\xi \in R^{n-1}$, $\tau \in R$ (cf. Sjöstrand [2]).

Introducing the vector valued pseudodifferential operators, we can construct the parametrix-like operators in the above non-elliptic case completely analogous to the elliptic case.

Theorem. (cf. [2], Theorem 1.) Assume that for all $\ell, m \in Z^+ \cup \{0\}$ and multiindices α, β there exists a constant $C =$

$C(\alpha, \ell, \beta, m)$ such that

$$(B) \quad \left| D_x^\alpha D_t^\ell D_\xi^\beta D_\tau^m a_1(x, 0, \xi, \tau) \right| \leq C(1 + |\xi| + |\tau|)^{1 - |\beta| - m}$$

for $x \in \mathbb{R}^{n-1}$, $(\xi, \tau) \neq (0, 0)$.

I. (i) If $\operatorname{Re} a_1(x, t, \xi, \tau) > 0$ for $(x, t) \in \mathbb{R}^n$ and $(\xi, \tau) \neq (0, 0)$, then there exist properly supported operators

$$\mathcal{P}_1 = \begin{pmatrix} P \\ R^+ \end{pmatrix}: \mathcal{D}'(\mathbb{R}^n) \longrightarrow \begin{matrix} \mathcal{D}'(\mathbb{R}^n) \\ \oplus \\ \mathcal{D}'(\mathbb{R}^{n-1}) \end{matrix}$$

$$\mathcal{G}_1 = (G_1, G^+): \begin{matrix} \mathcal{D}'(\mathbb{R}^n) \\ \oplus \\ \mathcal{D}'(\mathbb{R}^{n-1}) \end{matrix} \longrightarrow \mathcal{D}'(\mathbb{R}^n)$$

such that $\mathcal{G}_1 \cdot \mathcal{P}_1 - I$ and $\mathcal{P}_1 \cdot \mathcal{G}_1 - I$ have C^∞ kernels.

(ii) If $\operatorname{Re} a_1(x, t, \xi, \tau) < 0$ for $(x, t) \in \mathbb{R}^n$ and $(\xi, \tau) \neq (0, 0)$, then there exist properly supported operators

$$\mathcal{P}_2 = (P, R^-): \begin{matrix} \mathcal{D}'(\mathbb{R}^n) \\ \oplus \\ \mathcal{D}'(\mathbb{R}^{n-1}) \end{matrix} \longrightarrow \mathcal{D}'(\mathbb{R}^n)$$

$$\mathcal{G}_2 = \begin{pmatrix} G_2 \\ G^- \end{pmatrix}: \mathcal{D}'(\mathbb{R}^n) \longrightarrow \begin{matrix} \mathcal{D}'(\mathbb{R}^n) \\ \oplus \\ \mathcal{D}'(\mathbb{R}^{n-1}) \end{matrix}$$

such that $\mathcal{G}_2 \cdot \mathcal{P}_2 - I$ and $\mathcal{P}_2 \cdot \mathcal{G}_2 - I$ have C^∞ kernels.

II. For all $s \in \mathbb{R}$,

$$G_1, G_2: H_s^{\text{loc}}(\mathbb{R}^n) \longrightarrow H_{s + \frac{1}{1+k}}^{\text{loc}}(\mathbb{R}^n),$$

$$G^+: H_s^{\text{loc}}(\mathbb{R}^{n-1}) \longrightarrow H_{s + \frac{1}{1+k}}^{\text{loc}}(\mathbb{R}^n),$$

$$G^-: H_S^{\text{loc}}(\mathbb{R}^n) \longrightarrow H_S^{\text{loc}}(\mathbb{R}^{n-1})$$

are continuous.

$$\begin{aligned} \text{III. } WF'(G_1), WF'(G_2) &\subset \{((x, t, \xi, \tau), (x, t, \xi, \tau)) \in (T^*(\mathbb{R}^n) \setminus 0) \\ &\times (T^*(\mathbb{R}^n) \setminus 0)\}; \\ WF'(R^-), WF'(G^+) &\subset \{((x, 0, \xi, 0), (x, \xi)) \in (T^*(\mathbb{R}^n) \setminus 0) \times (T^*(\mathbb{R}^{n-1}) \setminus 0)\}; \\ WF'(R^+), WF'(G^-) &\subset \{((x, \xi), (x, 0, \xi, 0)) \in (T^*(\mathbb{R}^{n-1}) \setminus 0) \times (T^*(\mathbb{R}^n) \setminus 0)\}. \end{aligned}$$

§2. Vector valued pseudodifferential operators. (cf. Treves [4], Theorem 4.1.) Let H_1 and H_2 be complex Hilbert spaces and let $\mathcal{L}(H_1, H_2)$ be the Banach space of bounded linear operators $H_1 \longrightarrow H_2$. We define $S^m(\mathbb{R}^n \times \mathbb{R}^n; H_1, H_2)$ as the space of C^∞ functions $p(x, \xi)$ on $\mathbb{R}^n \times \mathbb{R}^n$ with values in $\mathcal{L}(H_1, H_2)$ such that for all $K \subset\subset \mathbb{R}^n$ and multiindices α, β there exists a constant $C = C(\alpha, \beta, K)$ such that

$$\| D_x^\alpha D_\xi^\beta p(x, \xi) \|_{\mathcal{L}(H_1, H_2)} \leq C(1 + |\xi|)^{m - |\beta|}$$

for all $(x, \xi) \in K \times \mathbb{R}^n$. With such symbols we define $L^m(\mathbb{R}^n; H_1, H_2)$ to be the space of pseudodifferential operators $P(x, D_x): C_0^\infty(\mathbb{R}^n; H_1) \longrightarrow C^\infty(\mathbb{R}^n; H_2)$.

We shall consider the case that H_1 or H_2 is equal to the space $D_\xi^k(\mathbb{R})$ with $k \in \mathbb{Z}^+$, $\xi \in \mathbb{R}^{n-1}$, which is a subspace of $H^1(\mathbb{R})$, given by the norm:

$$\| u \|_{D_\xi^k}^2 = (1 + |\xi|)^{\frac{2}{1+k}} \int_{-\infty}^{\infty} |u(t)|^2 dt +$$

(continued)

$$(1 + |\xi|)^2 \int_{-\infty}^{\infty} t^{2k} |u(t)|^2 dt + \int_{-\infty}^{\infty} |D_t u(t)|^2 dt.$$

(cf. [3], §4.)

In this case the norm $\| \cdot \|_{\mathcal{L}(H_1, H_2)}$ depends on $\xi \in \mathbb{R}^{n-1}$, but all the calculus for scalar operators (cf. Hörmander [1]) extends to the vector valued case, in particular we have the usual composition formula and the results about H_S -continuity because we have the inequality:

$$\|u\|_{D^k} \leq \|u\|_{D_{\xi}^k} \leq (1 + |\xi|) \|u\|_{D^k}$$

and hence

$$L^m(\mathbb{R}^{n-1}; H_1, D_{\xi}^k(\mathbb{R})) \subset L^m(\mathbb{R}^{n-1}; H_1, D^k(\mathbb{R})),$$

$$L^m(\mathbb{R}^{n-1}; D_{\xi}^k(\mathbb{R}), H_2) \subset L^{m+1}(\mathbb{R}^{n-1}; D^k(\mathbb{R}), H_2),$$

where $D^k(\mathbb{R})$ is the space $D_{\xi}^k(\mathbb{R})$ with $\xi = 0$.

We define $T^m(\mathbb{R}^n)$ to be the space of "pseudodifferential operators" $a(x, t, D_x): C_0^{\infty}(\mathbb{R}^n) \longrightarrow C^{\infty}(\mathbb{R}^n)$ where $a(x, t, \xi) \in S^m(\mathbb{R}^n \times \mathbb{R}^{n-1})$ (see [2], Appendix).

Under the assumptions (A), (B), we can reduce the proof of Theorem to the proof of

Proposition. (cf. [2], Proposition 3.6.) Let $L(x, t, D_x, D_t) = D_t - it^k r(x, t, D_x) + s(x, t, D_x)$, where $r(x, t, D_x) \in T^1(\mathbb{R}^n)$ (resp. $s(x, t, D_x) \in T^0(\mathbb{R}^n)$) is properly supported and its symbol $r(x, t, \xi)$ is positively homogeneous of degree 1 and $r(x, t, \xi)$ (resp.

$s(x, t, \xi)$ is equal to $r(x, 0, \xi)$ (resp. $s(x, 0, \xi)$) when $|t| \geq C$ for some constant $C > 0$ and $\operatorname{Re} r(x, t, \xi) \neq 0$ for $(x, t) \in \mathbb{R}^n$ and $\xi \neq 0$.

(i) If $\operatorname{Re} r(x, t, \xi) > 0$ for $(x, t) \in \mathbb{R}^n$ and $\xi \neq 0$, then there exist properly supported operators

$$\mathcal{L}_1(x, D_x) = \begin{pmatrix} L(x, t, D_x, D_t) \\ R^+(x, D_x) \end{pmatrix} \in L^0(\mathbb{R}^{n-1}; D_\xi^k(\mathbb{R}), L^2(\mathbb{R}) \oplus \mathbb{C}),$$

$$\mathcal{E}_1(x, D_x) = (E_1(x, D_x), E^+(x, D_x)) \in L^0(\mathbb{R}^{n-1}; L^2(\mathbb{R}) \oplus \mathbb{C}, D_\xi^k(\mathbb{R}))$$

such that

$$\mathcal{L}_1(x, D_x) \cdot \mathcal{E}_1(x, D_x) \equiv I \bmod L^{-\infty}(\mathbb{R}^{n-1}; L^2(\mathbb{R}) \oplus \mathbb{C}, L^2(\mathbb{R}) \oplus \mathbb{C}),$$

$$\mathcal{E}_1(x, D_x) \cdot \mathcal{L}_1(x, D_x) \equiv I \bmod L^{-\infty}(\mathbb{R}^{n-1}; D_\xi^k(\mathbb{R}), D_\xi^k(\mathbb{R})).$$

(ii) If $\operatorname{Re} r(x, t, \xi) < 0$ for $(x, t) \in \mathbb{R}^n$ and $\xi \neq 0$, then there exist properly supported operators

$$\mathcal{L}_2(x, D_x) = (L(x, t, D_x, D_t), R^-(x, D_x)) \in L^0(\mathbb{R}^{n-1}; D_\xi^k(\mathbb{R}) \oplus \mathbb{C}, L^2(\mathbb{R})),$$

$$\mathcal{E}_2(x, D_x) = \begin{pmatrix} E_2(x, D_x) \\ E^-(x, D_x) \end{pmatrix} \in L^0(\mathbb{R}^{n-1}; L^2(\mathbb{R}), D_\xi^k(\mathbb{R}) \oplus \mathbb{C})$$

such that

$$\mathcal{L}_2(x, D_x) \cdot \mathcal{E}_2(x, D_x) \equiv I \bmod L^{-\infty}(\mathbb{R}^{n-1}; L^2(\mathbb{R}), L^2(\mathbb{R})),$$

$$\mathcal{E}_2(x, D_x) \cdot \mathcal{L}_2(x, D_x) \equiv I \bmod L^{-\infty}(\mathbb{R}^{n-1}; D_\xi^k(\mathbb{R}) \oplus \mathbb{C}, D_\xi^k(\mathbb{R}) \oplus \mathbb{C}).$$

§3. Sketch of the proof of Proposition.

Lemma 1. Let $L(x, \xi) = L_0(x, \xi) + L_1(x, \xi)$, where $L_0(x, \xi) = D_t - it^k r(x, t, \xi)$, $L_1(x, \xi) = s(x, t, \xi)$. Then we have

$$L_0(x, \xi) \in S^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; D_\xi^k(\mathbb{R}), L^2(\mathbb{R})),$$

$$L_1(x, \xi) \in S^{-\frac{1}{1+k}}(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; D_\xi^k(\mathbb{R}), L^2(\mathbb{R})).$$

In particular, we can regard $L(x, t, D_x, D_t)$ as an element of $L^0(\mathbb{R}^{n-1}; D_\xi^k(\mathbb{R}), L^2(\mathbb{R}))$.

The next lemma is the essential step in our proof of Proposition.

Lemma 2. Let $B(x, t, s, \xi) = - \int_s^t \theta^k r(x, \theta, \xi) d\theta$.

(i) When $\operatorname{Re} r(x, t, \xi) > 0$ for $(x, t) \in \mathbb{R}^n$, $\xi \neq 0$, we define the kernel $K_1(x, t, s, \xi)$ by

$$K_1(x, t, s, \xi) = \begin{cases} i \exp [B(x, t, s, \xi)] & 0 \leq s \leq t, \\ -i \exp [B(x, t, s, \xi)] & t \leq s \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(ii) When $\operatorname{Re} r(x, t, \xi) < 0$ for $(x, t) \in \mathbb{R}^n$, $\xi \neq 0$, we define the kernel $K_2(x, t, s, \xi)$ by

$$K_2(x, t, s, \xi) = \begin{cases} -i \exp [B(x, t, s, \xi)] & 0 \leq t \leq s, \\ i \exp [B(x, t, s, \xi)] & s \leq t \leq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then for $K_j(x, t, s, \xi)$ ($j=1, 2$) we have the following estimates:

$$(1) \sup_t \int_{-\infty}^{\infty} |K_j(x, t, s, \xi)| ds = O(|\xi|^{-\frac{1}{1+k}}), \quad \xi \rightarrow \infty,$$

$$\sup_s \int_{-\infty}^{\infty} |K_j(x, t, s, \xi)| dt = O(|\xi|^{-\frac{1}{1+k}}), \quad \xi \rightarrow \infty,$$

uniformly when x belongs to any compact subset of \mathbb{R}^{n-1} .

$$(2) \sup_t \int_{-\infty}^{\infty} |t|^k |K_j(x, t, s, \xi)| ds = O(|\xi|^{-1}), \quad \xi \rightarrow \infty,$$

$$\sup_s \int_{-\infty}^{\infty} |t|^k |K_j(x, t, s, \xi)| dt = O(|\xi|^{-1}), \quad \xi \rightarrow \infty,$$

uniformly when x belongs to any compact subset of \mathbb{R}^{n-1} .

Lemma 2 follows from the following two facts (cf. Treves [4],

Lemma C.1):

(a) There exists a constant $C_1 > 0$ such that

$$|t - s|^{k+1} \leq C_1 |t^{k+1} - s^{k+1}|$$

for all $t, s \geq 0$ when $k \in \mathbb{Z}^+$ is odd.

(b) There exists a constant $C_2 > 0$ such that

$$|t|^k |t - s| \leq C_2 |t^{k+1} - s^{k+1}|$$

for all $t, s \geq 0$ when $k \in \mathbb{Z}^+$ is odd.

Combining Corollary in [4], p. 94 and Lemma 2, we can prove

Lemma 3. Let $\operatorname{Re} r(x, t, \xi) > 0$. We define for $|\xi| \geq 1$

$$R^+(x, \xi): D_{\xi}^k(\mathbb{R}) \longrightarrow \mathbb{C},$$

$$E_0^+(x, \xi): \mathbb{C} \longrightarrow D_{\xi}^k(\mathbb{R}),$$

$$E_{10}(x, \xi): L^2(\mathbb{R}) \longrightarrow D_{\xi}^k(\mathbb{R}),$$

by

$$R^+(x, \xi)u = |\xi|^{\frac{1}{1+k}} \int_{-\infty}^{\infty} u(t) \overline{\varphi(x, \xi, t)} dt,$$

$$E_0^+(x, \xi)z = |\xi|^{-\frac{1}{1+k}} \varphi(x, \xi, t)z,$$

$$E_{10}(x, \xi)f = \int_{-\infty}^{\infty} K_1(x, t, s, \xi) f(s) ds - E_0^+ R^+ K_1 f,$$

respectively, where

$$\varphi(x, \xi, t) = \exp \left[- \int_0^t \theta^k \operatorname{Re} r(x, \theta, \xi) d\theta \right] / \left(\int_{-\infty}^{\infty} \exp \left[-2 \int_0^t \theta^k \operatorname{Re} r(x, \theta, \xi) d\theta \right] dt \right)^{\frac{1}{2}}.$$

Then, after having been suitably modified for small ξ ,

$$\mathcal{L}_{10}(x, \xi) = \begin{pmatrix} L_0(x, \xi) \\ R^+(x, \xi) \end{pmatrix} \in S^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; D_{\xi}^k(\mathbb{R}), L^2(\mathbb{R}) \oplus \mathbb{C}),$$

$$\mathcal{E}_{10}(x, \xi) = (E_{10}(x, \xi), E_0^+(x, \xi)) \in S^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; L^2(\mathbb{R}) \oplus \mathbb{C}, D_{\xi}^k(\mathbb{R}))$$

and $\mathcal{E}_{10}(x, \xi)$ is the inverse of $\mathcal{L}_{10}(x, \xi)$ for $|\xi| \geq 1$.

Lemma 4. Let $\operatorname{Re} r(x, t, \xi) < 0$. We define for $|\xi| \geq 1$

$$R^-(x, \xi): \mathbb{C} \longrightarrow L^2(\mathbb{R}),$$

$$E_0^-(x, \xi): L^2(\mathbb{R}) \longrightarrow \mathbb{C},$$

$$E_{2_0}(x, \xi): L^2(\mathbb{R}) \longrightarrow D_{\xi}^k(\mathbb{R}),$$

by

$$R^-(x, \xi)z = \Psi(x, \xi, t)z,$$

$$E_0^-(x, \xi)f = \int_{-\infty}^{\infty} f(t) \overline{\Psi(x, \xi, t)} dt,$$

$$E_{2_0}(x, \xi)f = \int_{-\infty}^{\infty} K_2(x, t, s, \xi) (f(s) - R^-E_0^-f(s)) ds,$$

respectively, where

$$\Psi(x, \xi, t) = \exp \left[\int_0^t \theta^k \overline{r(x, \theta, \xi)} d\theta \right] / \left(\int_{-\infty}^{\infty} \exp \left[2 \int_0^t \theta^k \operatorname{Re} \overline{r(x, \theta, \xi)} d\theta \right] dt \right)^{\frac{1}{2}}.$$

Then, after having been suitably modified for small ξ ,

$$\mathcal{L}_{2_0}(x, \xi) = (L_0(x, \xi), R^-(x, \xi)) \in S^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; D_{\xi}^k(\mathbb{R}) \oplus C, L^2(\mathbb{R})),$$

$$\mathcal{E}_{2_0}(x, \xi) = \begin{pmatrix} E_{2_0}(x, \xi) \\ E_0^-(x, \xi) \end{pmatrix} \in S^0(\mathbb{R}^{n-1} \times \mathbb{R}^{n-1}; L^2(\mathbb{R}), D_{\xi}^k(\mathbb{R}) \oplus C)$$

and $\mathcal{E}_{2_0}(x, \xi)$ is the inverse of $\mathcal{L}_{2_0}(x, \xi)$ for $|\xi| \geq 1$.

By Lemma 3 and Lemma 4 the construction of $\mathcal{E}_j(x, D_x)$ ($j=1,2$) in Proposition is formally the same as the construction of a parametrix of an elliptic operator in the scalar case.

References

- [1] Hörmander, L.: Pseudodifferential operators and hypoelliptic equations, Amer. Math. Soc. Symp. on Singular Integral Operators, 138-183 (1966).
- [2] Sjöstrand, J.: Operators of principal type with interior boundary conditions, Acta Math., 130, 1-51 (1973).
- [3] Sjöstrand, J.: Parametrices for pseudodifferential operators with multiple characteristics (to appear).
- [4] Treves, F.: A new method of proof of the subelliptic estimates, Comm. Pure Appl. Math., 24, 71-115 (1971).