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Theory of differential equations with regular singularity and eigen-functions of Laplacian of symmetric spaces.

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The purpose of this paper is to develop the theory of differential equations with regular singularity. Our study is invoked by the requirement in the representation theory of groups.

General speaking, any irreducible representation can be realized in the subspace of eigen-functions of invariant differential operators on the symmetric space, and, at the same time, in the space of functions on the Martin boundary of the symmetric space.

It is natural to think that the boundary value problem gives the intertwining operator between these two realization. The invariant differential operators, however, are degenerate at the boundary of the symmetric space. Therefore, the boundary value problem in the usual sense does not work in this situation and one must study the boundary value problem when the differential equation is degenerate at the boundary. This paper is devoted to the analysis of such a problem.

In order to help readers' understanding, we will analyse the typical example : $G = SL(2 : R)$.

The maximal compact subgroup K of $G = SL(2 : R)$ is $SO(2)$ and the symmetric space $X = G/K$ can be identified with the upper half domain $\{z = x + iy \in C ; y > 0\}$, and its boundary is the real projective space $R \cup \{\infty\}$. The invariant differential operator Δ is $y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$.

Now we will consider the differential equation

$$(\Delta - s(s-1))u(x, y) = 0.$$

In order to see the behavior of the eigenfunction $u(x, y)$ near the boundary $y = 0$, assume $u(x, y)$ has a formal development

$$u(x, y) = \sum_{j=0} f_j(x)y^{\lambda+j}.$$

Then, we have the relation

$$[(j+\lambda)(j+\lambda-1)-s(s-1)]f_j+f''_{j-2} = 0.$$

Setting $j = 0$, we get $\lambda = s$ or $1-s$. If $\lambda = s$, we have

$$f_j(x) = \begin{cases} 0 & \text{for } j \text{ odd} \\ \dfrac{1}{k!(s+\frac{1}{2},k)} (-\dfrac{D_x^2}{4}) f_0(x) & \text{for } j = 2k, \end{cases}$$

where $(a, n) = a(a+1) \quad (a+n-1)$.

Thus,

$$u(x, y) = \sum_{k=0} \frac{1}{k!(s+\frac{1}{2},k)} (-\frac{D_x^2}{4})^k f_0(x)y^{s+2k}.$$

If $f_0(x)$ is a real analytic function, this series converges when y is sufficiently small. Set

$$A_s(y, x, D_x) = \sum_{k=0} \frac{(-1)^k}{k!(s+\frac{1}{2},k)} (\frac{yD_x}{2})^{2k}.$$

Then, if $\varphi_1(x)$ and $\varphi_2(x)$ are real analytic, then

(1) $u(x, y) = A_s(y, x, D_x)\varphi_1(x)y^s + A_{1-s}(y, x, D_x)\varphi_2(x)y^{1-s}$

is convergent when y is sufficiently small and eigenfunctions of Δ with eigenvalue $s(s-1)$. We call the solution of this type ideally analytic. Of course, when $\mathrm{Re}\, s > \frac{1}{2}$

$$\lim_{y \to +0} y^{s-1}u(x, y) = \varphi_1(x),$$

and when $\mathrm{Re}\, s < \frac{1}{2}$

$$\lim_{y \to +0} y^{-s}u(x, y) = \varphi_2(x).$$

This investigation permits us to call $\varphi_1(x)$ and $\varphi_2(x)$ the boundary value of ideally analytic solution $u(x, y)$.

When $\varphi_1(x)$ and $\varphi_2(x)$ are not real analytic the representation (1) loses his meaning in the first appearance. But idear lying beneath the theory of hyperfunction endows the formula (1) with an authentic meaning of it.

Since u is real analytic for $y > 0$, $u(x, y)$ has an analytic continuation to the space $\{(x, y)\ ;\ x \in C,\ y \in R\}$. The theory of bicharacteristies says $u(x, y)$ is defined on $y > |\mathrm{Im}\, x|$. Moreover, $u(x, y)$ is the sum of two solutions $u_+(x, y)$ and $u_-(x, y)$, where $u_\pm(x, y)$ is defined on $\{(x, y) \in C \times R\ ;\ y > 0,\ \ |\mathrm{Im}\, x| > y\}$

and $u_\pm(x, y)$ also the form

(2) $$u_{\pm}(x, y) = A_s(y, D_x)\varphi_{\pm}(x)y^s + A_s(y, D_x)\psi_{\pm}(x)y^{1-s},$$

where $\varphi_{\pm}(x)$ and $\psi_{\pm}(x)$ are holomorphic functions defined on $\{x \in \mathbb{C} ; \ \mathrm{Im}\, x \gtrless 0\}$. Remark that the right hand side of (2) has meaning because they converge. Thus, intuitively, we can write

(3) $$\begin{aligned} u(x, y) &= u_+(x, y) + u_-(x, y) \\ &= A_s(y_1 D_x)(\varphi_+(x+i0) + \varphi_-(x\_i0))y^s \\ &\quad + A_{1-s}(y, D_x)(\psi_+(x+i0) + \psi_-(x-i0))y^s \end{aligned}$$

where $\varphi(x_{\pm}i0)$ is the boundary value of holomorphic functions $\varphi(x)$ defined on $\{\mathrm{Im}\, x \gtrless 0\}$. So, we call $\varphi(x) = \varphi_+(x+i0) + \varphi_-(x-i0)$ and $\psi(x) = \psi_+(x+i0) + \psi_-(x-i0)$ the first and the second boundary values of $u(x, y)$.

For example, as for the Poisson kernel

$$P_s(x, y) = \left(\frac{y}{x^2+y^2}\right)^s,$$

we have

$$P_s(x, y) = \frac{\Gamma(s+\frac{1}{2})}{\Gamma(s+1)}\left\{\frac{y^{1-s}}{y+ix} F(1, 1-s, 1+s, \frac{y-ix}{y+ix}) + \frac{y^{1-s}}{y-ix} F(1, 1-s, 1+s, \frac{y+ix}{y-ix})\right\}$$

$$\text{on } \ y > |\mathrm{Im}\, x|$$

and

$$\frac{y^{1-s}}{y+ix} F(1, 1-s, 1+s, \frac{y\pm ix}{y\mp ix}) = \frac{\Gamma(1+s)e^{\pm\pi is}y^s}{2\ (s+\frac{1}{2})\cos\pi s\,(y^2+x^2)^s}$$

$$\left. + \frac{s}{2s-1}\frac{1}{y\mp ix} F(1, 1-s, 2-2s, \frac{2y}{y\mp ix})\right\} \ \text{on } \ 0 < y < |\mathrm{Im}\, x|.$$

Thus, the first boundary value is $|x|^{-2s} = \{e^{\pi is}(x+i0)^{-2s} + e^{-\pi is}(x-i0)^{-2s}\}/2\cos\pi s$, and the second is $\dfrac{\sqrt{\pi}\,\Gamma(s-\frac{1}{2})}{\Gamma(s)}\,\delta(x)$.

The representation (3), although, has no meaning, because $A_s(y, D_x)\varphi_+(x)y^s$ converges only on $\{(x, y)\ ;\ 0 < y < \operatorname{Im} x\}$ and $A_s(y, D_x)\varphi_-(x)y^s$ on $\{(x, y)\ ;\ 0 < y < -\operatorname{Im} x\}$.

Therefore, we will reformulate (3). We have

$$\begin{aligned} A_s(y, D_x)\varphi(x)y^s &= \sum_k \frac{(-1)^k}{k!\,(s+\frac{1}{2},k)}\left(\frac{D_x}{2}\right)^{2k}\varphi(x)y^{2k+s} \\ &= \sum_k \frac{(-1)^k}{k!\,(s+\frac{1}{2},k)}\left(\frac{D_x}{2}\right)^{2k}\varphi(x)\,(s+1,2k)D_y^{-2k}y^s \\ &= \sum_k \frac{(-1)^k(1+\frac{s}{2},k)\,(\frac{1}{2}+\frac{s}{2},k)}{k!\,(s+\frac{1}{2},k)}\left(\frac{D_x}{D_y}\right)^{2k}\varphi(x)y^s \\ &= F\left(1+\frac{s}{2},\ \frac{1}{2}+\frac{s}{2},\ s+\frac{1}{2},\ \left(\frac{D_x}{iD_y}\right)^2\right)\varphi(x)y^s. \end{aligned}$$

We remark that

$$Q_s(D_x, D_y) = F\left(1+\frac{s}{2},\ \frac{1}{2}+\frac{s}{2},\ s+\frac{1}{2},\ \left(\frac{D_x}{iD_y}\right)^2\right)$$

is a micro-differential operator defined on $\{(x, y\ ;\ (\xi dx + \eta dy)\ )\ ;\ |\eta| > |\xi|\}$. Therefore, $Q_s(D_x, D_y)\varphi(x)y^s$ has a meaning as micro-function. In this way, we reach the final formulation.

For an eigenfunction $u(x, y)$, let $\tilde{u}(x, y)$ be the hyperfunction solution of

$$\{y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) - s(s-1)\}\tilde{u}(x, y) = 0$$

such that $\tilde{u} = u$ on $y > 0$ and $u = 0$ on $y < 0$. ($\tilde{u}(x, y)$ makes, in the situation (3), to be $u_+(x+i0, y)Y(y) + u_-(x-i0, y)Y(y)$), Then, there are unique hyperfunctions $\varphi(x)$ and $\psi(x)$ such that

$$sp(\tilde{u}) = Q_s(D_x, D_y)(\varphi(x)y_+^s) + Q_{1-s}(D_x, D_y)(\psi(x)y_+^{1-s}).$$

We will call $\varphi(x)$, $\psi(x)$ the boundary values of $u(x, y)$.

Since the first boundary value of $\frac{\Gamma(1-s)}{\sqrt{\pi}\Gamma(\frac{1}{2} - s)} P_{1-s}(x, y)$ is $\delta(x)$, $\frac{\Gamma(1-s)}{\sqrt{\pi}\Gamma(\frac{1}{2} - s)} P_{1-s}(x-x', y)\varphi(x')dx'$ is the eigenfunction with the first boundary value $\int \delta(x-x')\ (x')dx' = \varphi(x)$.

Since it will be shown that the two eigenfunctions with the same first boundary value coincide using the theory of zonal spherical functions, we have

$$u(x, y) = \frac{\Gamma(1-s)}{\sqrt{\pi}\Gamma(\frac{1}{2} - s)} \int P_{1-s}(x-x', y)\varphi(x')dx'. \tag{4}$$

Thus, the space of eigenfunction of Δ defined on the upper half plane with the eigenvalue $s(s-1)$ and the space of hyperfunctions on the boundary are isomorphic by the map of taking the boundary value and by the Poisson integral (4).