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The traces of Hecke operators in the space of the 'Hilbert modular' type cusp forms of weight two.

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Introduction.

The purpose of the present note is to calculate the trace of Hecke operators acting in the space of the cusp forms of weight two belonging to a Hilbert modular group over a totally real algebraic number field. More generally, we carry it out for a discontinuous groups acting on $\mathfrak{Z}_n$, which consists of all $z=(z^{(1)},---,z^{(n)})$ with $z^{(i)}\in \mathbf{C}$, Im $z^{(i)}\neq 0$. Namely, let G be the product of n copies of $GL_2(\mathbf{R})$, considering of G as a group of transformations in $\mathfrak{Z}_n$. Let Γ be a subgroup of G operating on $\mathfrak{Z}_n$ discontinuously with a fundamental domain of finite volume. Let G^o be the connected component of the identity of G, and set $\Gamma^o=\Gamma\cap G^o$. We denote by Z(G) the center of G and by ι the canonical homomorphism of G onto G/Z(G). It is assumed through out this paper that

(G.1) $\lambda(\Gamma^{o})$ is an irreducible subgroup of $\lambda(G^{o})$ such that $\lambda(G^{o})/\lambda(\Gamma^{o})$ is non-compact and of finite measure,

(G.2) $\lambda(\Gamma^{o})$ satisfies the assumption (F) in [8] .

We fix once for all an element α in G^{o} such that Γ and $\alpha\Gamma\alpha^{-1}$ are commensurable, and denote by Γ' the subgroup of G generated by Γ and α. Let χ be a linear character of Γ'. We assume that χ satisfies

(C.1) the kernel of Γ_{χ} of χ in Γ is of finite index in Γ,

(C.2) $\chi(\varepsilon)=1$ for $\varepsilon \in Z(\Gamma)$ $(=\Gamma \cap Z(G))$.

Let k be an even integer. Let $T=T(\Gamma\alpha\Gamma)$ be the Hecke operator acting on the space of cusp forms of weight k with respect to Γ and χ; we denote above space by $S(\Gamma,k,\chi)$. We calculate the trace of T for the case k=2, $n>1$. For the case of $k>2$, the trace of T has been explicitly calculated in Shimizu [9]. Also for the case of k=2, the trace has been calculated in our previous papers [4], [5] under the condition of n=1 or the condition that Γ has a compact fundamental domain in $\mathfrak{H}_n$.

§1. A few facts from [5].

Let H be the direct product of n complex upper half planes. Let $S(\Gamma^o)$ be the set of all restrictions of the cusp forms in $S(\Gamma,2,\chi)$ to H. In this and next sections, from now on, we consider that T is restricted to $S(\Gamma^o)$. Let us recall a few facts from [5]. We fix once for all a fundamental domain D of Γ^o in H. Let $\kappa_1,---,\kappa_h$ be all Γ^o_χ-inequivalent cusps belonging to D. g_p denotes an element of G^o such that $g_p\infty=\kappa_p$. Set $B=\Gamma^o\alpha\Gamma^o$, $B^{(1)}_p=\{\gamma\in B;\gamma\kappa_p=\kappa_p\}$, $\Gamma^{o(1)}_p=\{\gamma\in\Gamma^o;\gamma\kappa_p=\kappa_p\}$ and $\Gamma^o_p=\{\gamma\in\Gamma^{o(1)}_p;\gamma$ is a parabolic$\}$. Let $\tilde{H}=H\times(R/2\pi Z)^n$, $\tilde{D}=D\times(R/2\pi Z)^n$ with elements (z,ϕ) $(\phi=(\phi^{(1)},---,\phi^{(n)}))$ and we identify $\phi^{(i)}$ and $\phi^{(i)}+2\pi$. Let $\tilde{G}^o=G^o\times(R/2\pi Z)^n$ with elements (g,θ), and it acts on the space (z,ϕ) as

$$(g,\theta)(z,\phi)=(gz,(\phi^{(i)}+\arg(c^{(i)}z^{(i)}+d^{(i)})-\theta^{(i)})),\quad g^{(i)}=\begin{pmatrix}a^{(i)}, & b^{(i)}\\ c^{(i)}, & d^{(i)}\end{pmatrix}.$$

Let $L^2_o(\tilde{D})$ be the space of measurable functions $F(z,\phi)$ on $\tilde{H}$ taking values in C and satisfying the following conditions:

(i) $F(\gamma(z,\phi))=\chi(\gamma)F(z,\phi)$ for $\gamma\in\Gamma^o$,

(ii) $\int_{\tilde{D}}F(z,\phi)\overline{F(z,\phi)}\,dz\,d\phi<\infty$, $\left(dz=\prod_{i=1}^n\frac{dx^{(i)}dy^{(i)}}{y^{(i)2}},\ d\phi=\prod_{i=1}^n d\phi^{(i)}\right)$,

(iii) $\int_{R^n/M_p}F(g_p(z,\phi))\,dx^{(1)}---dx^{(n)}=0\quad(1\leq p\leq h)$, where $M_p=\{\mu=(\mu^{(1)},---,\mu^{(n)});\ (g_p^{-1}\gamma g_p)^{(i)}=z^{(i)}+\mu^{(i)},\ \gamma\in\Gamma_p^o\}$. Let k_s be a $\tilde{G}^o$-invariant integral operator defined by a point pair invariant kernel: for $s>0$,

$$(1.1)\quad k_s(z,\phi,z',\phi')=\prod_{i=1}^{n}\Big\{(\exp(-2\sqrt{-1}(\phi^{(i)}-\phi'^{(i)})))$$

$$\times\left[\frac{(y^{(i)}y'^{(i)})^{\frac{1}{2}}}{(z^{(i)}-\bar{z}'^{(i)})/2\sqrt{-1}}\right]^2\frac{(y^{(i)}y'^{(i)})^{s/2}}{|(z^{(i)}-\bar{z}'^{(i)})/2\sqrt{-1}|^s}-\frac{s}{2+s}\,\frac{(y^{(i)}y'^{(i)})^{1+s/2}}{|(z^{(i)}-z'^{(i)})/2\sqrt{-1}|^{2+s}}\Big\}.$$

It is well known that the ring of all $\tilde{G}^o$-invariant differential operators is generated by

$$(1.2)\quad \frac{\partial}{\partial\phi^{(i)}},\ \tilde{\Delta}^{(i)}=y^{(i)2}\Big(\frac{\partial}{\partial x^{(i)2}}+\frac{\partial}{\partial y^{(i)2}}\Big)+y^{(i)}\frac{\partial}{\partial x^{(i)}}\frac{\partial}{\partial\phi^{(i)}},\ (1\leq i\leq n).$$

Denote by $M(m,\lambda)$ the subspace of $L_0^2(\tilde{D})$ consisting of φ satisfying the following conditions

$$\frac{\partial}{\partial\phi^{(i)}}\varphi=-\sqrt{-1}m^{(i)}\varphi,\ \tilde{\Delta}^{(i)}\varphi=\lambda^{(i)}\varphi\quad(1\leq i\leq n).$$

By the general theory, the eigenvalues of k_s only depend on (m,λ); so we write the eigenvalue of k_s with $h_s(m,\lambda)$. The following proposition comes from [5, Proposition 1 & 2].

PROPOSITION. 1 The eigenspace $M(m,\lambda)$ in which k_s does not vanish and its eigenvalue are in the following table. The notations are defined as follows. In the series C, J is denoted a proper subset of $[1,n]$ and $I\cup J=[1,n]$; $\lambda_\rho^{(i)}$ ranges

Series	m	λ	Isomorphic to $M(m,\lambda)$	Eigenvalue $h_s(m,\lambda)$ of k_s	Trace of T
B	$m^{(i)}=2$ $(1\leq i\leq n)$	$\lambda^{(i)}=0$	$S(\Gamma^0)$	$\left(8\pi 2^s\frac{\Gamma(1+\frac{s}{2})^2\Gamma(\frac{1}{2})\Gamma(\frac{1+s}{2})}{\Gamma(1+s)\Gamma(2+\frac{s}{2})}\right)^n$	t_0
C	$m^{(i)}=0,2$ $m^{(j)}=2$ $(i\in I)$ $(j\in J)$	$\lambda^{(i)}=\lambda_\rho^{(i)}$ $\lambda^{(j)}=0$	$M(\{0,2\},\{\lambda_\rho^{(i)},0\})$	$\prod_{i\in I}\{(-8\pi 2^s c(s)/\Gamma(1+s))$ $\Gamma(\frac{s}{2}+\delta_\rho^{(i)})\Gamma(\frac{s+2}{2}-\delta_\rho^{(i)})$ $\times\prod(-8\pi 2^s\frac{\Gamma(1+\frac{s}{2})^2\Gamma(\frac{1}{2})\Gamma(\frac{1+s}{2})}{\Gamma(1+s)\Gamma(2+\frac{s}{2})})$	$t_{m,\lambda}$
D	$m^{(i)}=0$ $(1\leq i\leq n)$	$\lambda^{(i)}=0$	C	$\left(-8\pi 2^s\frac{\Gamma(1+\frac{s}{2})^2\Gamma(\frac{1}{2})\Gamma(\frac{1+s}{2})}{\Gamma(1+s)\Gamma(2+\frac{s}{2})}\right)^n$	t_1

over all eigenvalues of $\Delta^{(i)}=y^{(i)2}\left(\frac{\partial^2}{\partial x^{(i)2}}+\frac{\partial^2}{\partial y^{(i)2}}\right)$ satisfying $M(\{0,2\},\{\lambda_\rho^{(i)},0\})\neq\{0\}$ expect $\lambda_\rho^{(i)}\neq 0$; $\lambda_\rho^{(i)}=\delta_\rho^{(i)}(\delta_\rho^{(i)}-1)$. The series D appears only if χ is trivial. $c(s)=\frac{s}{2}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1+s}{2})}{\Gamma(2+\frac{s}{2})}$.

We shall carry the action of T to $M(2,0)$ by the isomorphism in the series B and extend it to $L_0^2(\tilde{D})$. We can expess T restricted to $M(m,\lambda)$ by k_s in the following way;

$$T(\Gamma\alpha\Gamma)=h_s(m,\lambda)^{-1}\int_{\tilde{D}}K_s(z,\phi,z',\phi')\,F(z',\phi')\,dz'd\phi',\tag{1.3}$$

$$K_s(z,\phi,z',\phi')=\sum_{g\in\Gamma^0\alpha\Gamma^0}\chi(g)\,k_s(z,\phi,g(z',\phi')).$$

But for $s>0$, the kernel k_s is of (a)-(b) type in the sense of [7], therefore K_s is absolutely convergent and uniformly, if $(z,\phi),(z',\phi')$ are contained in some compact subregion of $\tilde{H}$. But as the fundamental domain $\tilde{D}$ is non-compact, the operator K_s is not, generally, completely continuous.

§ 2. An operator H_s.

2.1. Now we shall define a series M. Put $h_s(\delta)=h_s(2,\lambda)$ for simplicity. $e=(e^{(1)},---,e^{(n)})$ denotes a combination of $e^{(i)}=0$ or 2 $(1\leq i\leq n)$. For a complex number σ with $\mathrm{Re}(\sigma)>1$, we set

$$M_p^e(z,\phi,z',\phi';\sigma)=(2\pi)^{-n}\sum_{\{g\}\in\Gamma_p^o\backslash\Gamma^o}\chi(g)^{-1}\int_{\mathrm{Re}(\delta^{(1)})}---\int_{\mathrm{Re}(\delta^{(n)})}h_s(\delta)$$

$$\times\prod_{i=1}^{n}\{b_p^{e^{(i)}}(g^{(i)}(z^{(i)},\phi^{(i)});\delta^{(i)}+\sigma-\tfrac{1}{2})\,\overline{b}_p^{e^{(i)}}(z'^{(i)},\phi'^{(i)};\delta^{(i)})\,d\delta^{(i)}\},\qquad(2.1)$$

$$M'^e_p(z,\phi,z',\phi';\sigma)=\sum_{\nu}\chi(\beta_\nu)^{-1}M_p^e(\beta_\nu(z,\phi),z',\phi';\sigma),$$

where $b^{e^{(i)}}(z^{(i)},\phi^{(i)};u)=\exp(-\sqrt{-1}e^{(i)}(\phi^{(i)}+\arg(c^{(i)}_{g_p^{-1}}z^{(i)}+d^{(i)}_{g_p^{-1}})))(\mathrm{Im}\,g_p^{(i)-1}z^{(i)})^u$, and that $\Gamma^o\alpha\Gamma^o=\bigcup_\nu\Gamma^o\beta_\nu$ (disjoint union).

For simplicity, we may assume that $\kappa_1=\infty, g_1=1$, $e=(0,---,0)$ in this and next paragrapg and treat M_1^0 mainly; we shall M, $\Gamma_\infty^{(1)}$, Γ_∞ instead of M_1, $\Gamma_1^{o(1)}$, Γ_1^o. By a simple calculation, we get

$$M(z,z';\sigma)=\sum_{\{g\}\in\Gamma_\infty^{(1)}\backslash\Gamma^o}\chi(g)^{-1}\prod_{i=1}^{n}(\mathrm{Im}\,gz)^{(i)\sigma}\alpha(\mathrm{Im}\,gz,y'),$$

where $\alpha(y,y')=\sum\prod_{i=1}^{n}((\lambda yy'^{-1})^{\frac{1}{2}}+(\lambda yy'^{-1})^{-\frac{1}{2}})^{(i)-(1+s)}$,

and $\Lambda_\infty=\{(\lambda^{(i)})=(a^{(i)}d^{(i)-1});\ g\in\Gamma_\infty^{(1)}\}$. It follows from [8, No.17] that $\alpha(y,y')\leq k$, k being a constant independent of y,y' and s. Thus, by the same way as in [8, Lemma 12], M converges absolutely and is holomorphic respect to σ for $\mathrm{Re}(\sigma)>1$. Further, from the definition, it follows immediately that

$M(gz,z';\sigma)=\chi(g)\,M(z,z';\sigma)$ for $g\in\Gamma^{0}$.

2.2. In this paragraph, we shall obtain the analytic continuation of M to the domain $\mathrm{Re}(\sigma)>\frac{1}{2}$, minus the interval $(\frac{1}{2},1]$, Now, we need some notations and propositions. We define Eisenstein series attached to the cusp κ_p by

$$E_p(z,\sigma)=\sum_{\{g\}\in\Gamma_p^{0(1)}\backslash\Gamma^{0}}\chi(g)^{-1}\prod_{i=1}^{n}(\mathrm{Im}\ g_p^{-1}z)^{(i)\sigma}. \tag{2.2}$$

By a simple calculation, the constant term of the Fourier expansion of $E_p(z,\sigma)$ is given

$$\delta_{pq}\prod_{i=1}^{n}y^{(i)\sigma}+\prod_{i=1}^{n}y^{(i)1-\sigma}\varphi_{pq}(\sigma), \tag{2.3}$$

where $\varphi_{pq}(\sigma)=\left(\dfrac{\Gamma(\sigma-\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\sigma)}\right)^{n}\displaystyle\sum_{\{g\}\in g_p^{-1}\Gamma_p^{0(1)}g_p\backslash g_p^{-1}(\Gamma^{0}-\Gamma_p^{0(1)})g_q/g_q^{-1}\Gamma_q^{0}g_q}\left(\prod^{n}(\det g^{(i)}/|c^{(i)}|^{2})\right)^{\sigma}\chi(g)^{-1}$,

and $\delta_{pq}=1$ or 0 according as $p=q$ or not.

PROPOSITION 2. $\varphi_{pq}(\sigma)$ may be continued holomorphically to the domain $\mathrm{Re}(\sigma)>\frac{1}{2}$, $\sigma\notin(\frac{1}{2},1]$.

This proof comes from [6, Theorem 3.1.1] with a little modification. Let $F(z,\sigma)$ be an analytic function of z,σ which is automorphic with respect to Γ^{0}, whose constant term of the Fourier expansion at κ_p $(1\leq p\leq h)$ has the form :

$$c_p(\sigma)\prod_{i=1}^{n}(\mathrm{Im}\ g_p^{-1}z)^{(i)\sigma}+d_p(\sigma)\prod_{i=1}^{n}(\mathrm{Im}\ g_p^{-1}z)^{(i)1-\sigma}.$$

For $Y>0$, we define the function $F^Y(z,\sigma)$ by

$$F^Y(z,\sigma)=\begin{cases} F(z,\sigma)-(c_p(\sigma)\prod_{i=1}^{n}(\mathrm{Im}\ g_p^{-1}z)^{(i)\sigma}+d_p(\sigma)\prod_{i=1}^{n}(\mathrm{Im}\ g_p^{-1}z)^{(i)1-\sigma}, & \text{if } \prod_{i=1}^{n}(\mathrm{Im}\ g_p^{-1}z)^{(i)}>Y \\ F(z,\sigma) & \text{otherwise.}\end{cases}$$

Then the Fourier expansion gives

LEMMA. If F ia a function as above, we have

$$(2.4)\quad d(\Lambda_p)^{-2}(E_p^Y(z,\sigma),F^Y(z,\sigma'))=\frac{\bar{c}_p(\sigma')Y^{\sigma+\bar{\sigma}'-1}}{\sigma+\bar{\sigma}'-1}+\frac{\bar{d}_p(\sigma')Y^{\sigma-\bar{\sigma}'}}{\sigma-\bar{\sigma}'}$$
$$-\sum_{q=1}^{h}\left(\frac{\varphi_{pq}(\sigma)\bar{d}_q(\sigma')Y^{-(\sigma+\bar{\sigma}'-1)}}{\sigma+\bar{\sigma}'-1}+\frac{\varphi_{pq}(\sigma)\bar{c}_q(\sigma')Y^{\bar{\sigma}'-\sigma}}{\sigma-\bar{\sigma}'}\right),$$

where $d(\Lambda_q)=\det(l_j^{(i)})$, $\lambda_1,---,\lambda_{n-1}$ being generators of Λ_q, and $l_j^{(i)}=\log\lambda_j^{(i)}$ $(1\leq j<n)$, $l_n^{(i)}=1/n$. Using above formula, we get the following proposition by same arguments as in [6, Theorem 3.2.2, 4.2.1.-4.2.3,& 4.3.1.-4.3.5].

PROPOSITION 3. $E_p(z,\sigma)$ is holomorphic in the domain $\mathrm{Re}(\sigma)>\frac{1}{2}$ expect at point of finite number which are simple poles of $\varphi_{pq}(\sigma)$ on $(\frac{1}{2},1]$, Moreover E_p and φ_{pq} have a unique and finite limit σ tending to a point on the line $\mathrm{Re}(\sigma)=\frac{1}{2}$.

Now we come back to $M(z,z';\sigma)$. The constant term of this Fourier expansion at ∞ is given by

$$(2.5)\quad \prod_{i=1}^{n}(y^{(i)})^{\sigma}\alpha(y,y')+\prod_{i=1}^{n}(y^{(i)})^{1-\sigma}\varphi_{11}(\sigma)\beta(y,y';\sigma),$$
$$\beta(y,y';\sigma)=\left(\frac{\Gamma(\sigma)}{\Gamma(\sigma-\frac{1}{2})\Gamma(\frac{1}{2})}\right)^n\int_{R^n}\prod_{i=1}^{n}\frac{du^{(i)}}{(u^{(i)2}+1)^{\sigma}}\alpha\left(\left(\frac{y^{(i)}}{(u^{(i)2}+1)}\right),y'\right).$$

Using the Fourier expansion of M, we get

PROPOSITION 4. $M(z,z';\sigma)$ can be continued holomorphically to the domain $\mathrm{Re}(\sigma) > \frac{1}{2}$ minus points which are poles of $\varphi(\delta)$ belonging to $(\frac{1}{2},1]$. Moreover $M(z,z';\sigma)$ has a unique and finite limit for any sequence $\{\sigma_n\}$ of complex numbers such that $\mathrm{Re}(\sigma_n) > \frac{1}{2}$, $\lim \mathrm{Re}(\sigma_n)=\frac{1}{2}$.

2.3. Now we shall construct an operator H_s. Let $\{\mu_1,---,\mu_n\}$ be a basis of M_p and $d(M_p)= \det(\mu_j^{(i)})$. The kernel of H_s will be defined by

$$(2.6)\quad H_s(z,\phi,z',\phi')=\left(\frac{2^s c(s)}{2\pi}\right)^n \sum_{p=1}^{h} d(M_p)^{-1}\sum_{e}(-1)^{n-\Sigma\frac{e^{(i)}}{2}} \times M'^{e}_{p}(z,\phi,z',\phi';\tfrac{1}{2}),$$

where e runs over all conbination of $e^{(i)}=0$ or 2 $(1\leq i\leq n)$. By the direct calculation, when z and z' tend simultaneously towards the cusp ζ_p, the kernel $H_s(z,\phi,z',\phi')$ is approximately equal to $\sum_{g\in B_p^{(1)}} \chi(g)\, k_s(z,\phi,g(z',\phi'))$. It follows that

$$K_s^*(z,\phi,z',\phi')= K_s(z,\phi,z',\phi') - H_s(z,\phi,z',\phi')$$

is bounded for all (z,ϕ), $(z',\phi') \in \tilde{H}$; therefore an integral operator K_s^* turns to be completely continuous. Moreover, by the same way as [4, §§ 4.3-4.4], we see that, for $F\in L_0^2(\tilde{D})$ which is an eigenfunction of $\frac{\partial}{\partial\phi^{(i)}}$ and $\tilde{\Delta}^{(i)}$, an eigenvalue of F for K_s^* is equal to that for K_s, and that the image of K_s^* is contained in $L_0^2(\tilde{D})$. Considering the trace K_s^* in $L_0^2(\tilde{D})$ with the same argument as [5, §3], we obtain

(2.7) $$t_0 = -(-1)^n t_1 + \lim_{s\to 0}\int_{\tilde{D}} K_s^*(z,\phi,z,\phi)\, dz\, d\phi.$$

Define the equivalence relation of elements of B by

(2.8) $$g \sim g' \iff g' = \varepsilon\gamma g\gamma^{-1} \text{ for } \gamma\in\Gamma^0,\ \varepsilon\in Z(\Gamma^0).$$

Let $[g]$ denote an equivalence class in B containing g. Let $\Gamma^0(g)$ be the group of all $\gamma\in\Gamma^0$ such that $\gamma g\gamma^{-1} = \varepsilon g$ for some $\varepsilon\in Z(\Gamma)$ and F_g (resp. F_g^* ; $\tilde{D}^*$) a fundamental domain of $\Gamma^0(g)$ in H (resp. $\Gamma^0(g)$ in H^* ; Γ^0 in $\tilde{H}^*$) (H^* being a subregion of H obtained by substracting the neibourhood of each parabolic point of Γ^0 from H, and $\tilde{H}^* = H^*\times(R/2\pi Z)^n$). We can rewrite

$$\operatorname{tr}\int_{\tilde{D}^*} K_s(z,\phi,z,\phi)\,dz\,d\phi = (2\pi)^n \sum_{[g],\,g\in B} \chi(g)\int_{F_g^*} k_s(z,0,z,0)\, dz.$$

For simplicity, we denote by $A(g,s;H^*)$ each term of the right hand side of above formula.

§3. An explicit formula for trace of $T(\Gamma\alpha\Gamma)$.

3.1. In this section, we shall calculate the trace of

$T(\Gamma\alpha\Gamma)$ in $S(\Gamma,2,\chi)$. Firstly, we classify an element in B. $g\in B$ is of one of the following types; (i) $g\in B\cap Z(G^o)$, (ii) g is elliptic, (iii) g is hyperbolic and no fixed point of g is a parabolic point of Γ^o, (iv) g is hyperbolic and one of the fixed points of g is a parabolic point, (v) g is parabolic, (vi) g is mixed.

When g is of type (i),(ii),(iii) or (vi), $A(g,s,H^*)$ has been calculated in [5, § 4].

3.2. Case iv). We may assume that g leaves each of ∞ and 0 fixed. For Y, $Y'>0$, put $F_g^*=\{z=(r^{(i)}\exp(\sqrt{-1}\theta^{(i)}))$; $\log r^{(i)}=\sum u^{(i)}l_j^{(i)}$, for $0<u_j<1$ $(1\leq j<n)$, $\log(Y'^{-1}\prod_{i=1}^{n}|\sin\theta^{(i)}|)<u_n<\log(Y\prod_{i=1}^{n}|\sin\theta^{i}|^{-1})$, $0<\theta^{(i)}<\pi$ $(1\leq i\leq n)\}$. Writing $g^{(i)}z^{(i)}=\lambda^{(i)}z^{(i)}$, and $\rho^{(i)}=\left|\frac{\lambda^{i}+1}{\lambda^{i}-1}\right|$, we have

$$A(g,s;H^*)=(-8\pi 2^{s})^{n}\chi(g)\,|\det(l_j^{(i)})|\,\log(YY'\prod|\sin\theta^{(i)}|^{-2})\prod_{i=1}^{n}|\lambda^{(i)}-1|^{-(2+s)}$$
$$\times\int_0^{\pi}\cdots\int_0^{\pi}\prod_{i=1}^{n}\frac{|\tan\theta^{(i)}|^{s}\{1+\hat{c}(s)+(\hat{c}(s)-1)\rho^{(i)}\tan^2\theta^{(i)}}{(1+\rho^{(i)2}\tan^2\theta^{(i)})^{2+s/2}\cos^2\theta^{(i)}}\,d\theta^{(i)},$$

$(\hat{c}(s)=s/(2+s))$. Therefore, if $n>1$, $A(g,s;H^*)$ is vanishes.

3.3. Case v). Consider the contribution of the parabolic classes in $\Gamma\alpha\Gamma$ on $\mathfrak{H}^n$. We may assume $\zeta_p=\infty$, $g_p=1$. In this paragraph, let us use the notations in [9, §3.4]. For $Y>0$, we put

$$F_g^* = \{ z=(x^{(i)}+\sqrt{-1}y^{(i)});\ x^{(i)}=\sum_{j=1}^{n} v_j \mu(\gamma_j^{(i)}) \text{ for } 0<v_j<1,\ 0<\prod_{i=1}^{n}|y^{(i)}|<Y\}.$$

Then we have

$$\begin{aligned} w &= \sum_{g\in L_p} (2\pi)^n \chi(g) \int_{F_g^*} k_s(z,0,z,0)\, dz \\ &= \lim_{\varepsilon\to 0} (-4\pi 2^{\varepsilon})^n \sum_{g\in L_p} \frac{d(g)\chi(g)}{m(g)^{1+\varepsilon}} \varepsilon^n \left(\frac{\Gamma(\frac{1+\varepsilon}{2})\Gamma(\frac{s-\varepsilon+1}{2})}{\Gamma(2+\frac{s}{2})}\right)^n + o(Y^{-1}). \end{aligned}$$

By [9, Lemma 3.2], the series has at most a pole of order 1 at $\varepsilon=0$. By the assumption of $n>1$, it follows that $w=0$.

3.4. By a simple calculation, $\lim_{s\to 0} \operatorname{tr}\int_{\tilde{D}^*} H_s(z,\phi,z,\phi)\,dz\,d\phi = 0$. Summing up the above results, we obtain

THEOREM 1. If $n>1$, the trace of $T(\Gamma a\Gamma)$ in $S(\Gamma,2,\chi)$ is given by the following formula :

$$\text{(3.1)}\qquad \operatorname{Tr} T(\Gamma a\Gamma) = \delta_1 (4\pi)^{-n} v(\Gamma\backslash\mathfrak{H}_n)\chi(g_o) + \sum_{\{g\}\in\mathfrak{S}} \frac{(-1)^n \chi(g)}{(\Gamma(g):Z(\Gamma))} - \delta_2(-1)^n \frac{2^n d}{(\Gamma:\Gamma^o)}.$$

The notations used in this formula are defined as follows :

$$\delta_1=\begin{cases}1 & \text{--- if } \Gamma a\Gamma\cap Z(G)\neq\phi \\ 0 & \text{--- otherwise}\end{cases},\qquad \delta_2=\begin{cases}1 & \text{--- if } \chi \text{ is trivial} \\ 0 & \text{--- otherwise}\end{cases},$$

$g_o\in\Gamma a\Gamma\cap Z(G)$,

$v(\Gamma\backslash\mathfrak{H}_n)$; the volume of a fundamental domain of Γ in $\mathfrak{H}_n$ relative to the invariant measure dz,

$\mathfrak{S}$; a complete system of inequivalent elliptic elements in $\Gamma a\Gamma$ with respect to the equivalence relation (2.8), d ; the number of right Γ-cosets in $\Gamma\alpha\Gamma$.

§4. The Hilbert modular groups.

Let Φ be a totally real algebraic number field of degree n over Q, and $A=M_2(\Phi)$. We denote by $\mathfrak{g}$, E_0, E_0^+, $\mathfrak{O}$, $\mathfrak{I}$ and U the ring of integers in Φ, the group of units in $\mathfrak{g}$, the subgroup of E_0 containing of all totally positive units, a maximal order in A, the idèle group of A and the idèle x such that $x_{\mathfrak{p}}$ is a unit of $\mathfrak{O}_{\mathfrak{p}}$ for all finite prime $\mathfrak{p}$, respectively. Writing $\Phi^{(i)}$ $(1\leq i\leq n)$ for the completion of Φ with respect to the infinite valuation of Φ and $A^{(i)}=A\otimes_{\Phi}\Phi^{(i)}$, every $x\in\mathfrak{I}$ is made to act on $\mathfrak{H}_n$ by $x(z)=(x^{(1)}(z^{(1)}),\text{---},x^{(n)}(z^{(n)}))$ $(x^{(i)}\in A^{(i)})$. Then Γ satisfies our assumptions (G.1) and (G.2), and $\Gamma\backslash\mathfrak{H}_n$ is not compact. Let $\mathfrak{M}$ be an integral two-sided ideal in $\mathfrak{O}$ of norm $\mathfrak{a}$, and χ a linear character of $(\mathfrak{O}/\mathfrak{M})^*$; we consider χ as a character of $V_{\mathfrak{a}}(=\{x\in\mathfrak{I};\ x_{\mathfrak{p}}\in U_{\mathfrak{p}}$ for all $\mathfrak{p}|\mathfrak{a}\}$) by means of a natural homomorphism of $V_{\mathfrak{a}}$ onto $(\mathfrak{O}/\mathfrak{M})^*$. Then χ satisfies (C.1). We assume that χ satisfies (C.2). $\mathfrak{I}$ is a finite union of double cosets of U and A^* in the following way :

$\mathfrak{J} = \bigcup_{\lambda=1}^{h} Ux_\lambda A^*$ ($x_\lambda \in V_{\mathfrak{a}}$, h is the class number of $\mathfrak{O}$).

Put $U_\lambda = x_\lambda^{-1} U x_\lambda$ and $\Gamma_\lambda = A^* \cap U_\lambda$ ($1 \leq \lambda \leq h$). Let S_λ be the space of all cusp forms on $\mathfrak{H}_n$ of type $(\Gamma_\lambda, 2, \chi)$ and S the direct produ. t of $S_1, ---, S_h$. For an integral ideal $\mathfrak{q}$ in $\mathfrak{g}$, wedenote by $T(\mathfrak{q})$ the linear operator in S defined in [10, § 3.4]. Note that $T(\mathfrak{q}) \neq 0$ only if $\mathfrak{q}$ is a principal ideal and only if we can write $\mathfrak{q} = q\mathfrak{g}$ such that q is a totally positive element in $\mathfrak{g}$.

Combining Theorem 1 with [5, §5.1 & 8, §4], we obtain

THEOREM 2. Let $\mathfrak{q} = q\mathfrak{g}$ be a principal ideal in Φ with a totally positive element q. The trace of $T(\mathfrak{q})$ is given by

$$(4.1) \qquad \mathrm{Tr}.T(\mathfrak{q}) = \delta(\mathfrak{q})\,(2\pi)^{-2n} 2h_0 D_0^{3/2} \zeta_0(2)\,\chi(q_0)$$
$$- \delta_2 (-1)^n h_0 (2^n/(E_0 : E_0^+)) \sum_{\mathfrak{n} | \mathfrak{q}} N(\mathfrak{n})$$
$$+ (-1)^n \tfrac{1}{2} \sum_{\sigma \in \Omega} \frac{h(\sigma)}{w(\sigma)} \sum_{a \in J(\sigma),\, a \bmod E_0} \chi(a) \quad .$$

The notations are as follows. h_0, D_0 and ζ_0 are the class number of Φ, the discriminant of Φ over Q and the zeta function of Φ, respectively. $\delta(\mathfrak{q}) = 1$ if $\mathfrak{q} = q_0^2 \mathfrak{g}$ for some $q_0 \in \mathfrak{g}$ and otherwise $\delta(\mathfrak{q}) = 0$. $\delta_2 = 1$ if χ is a trivial and otherwise $\delta_2 = 0$. $\mathfrak{n}$ runs over all divisors of $\mathfrak{q}$. Ω is the set of all orders σ(taken up isomorphism) in totally imaginary quadratic extensions of Φ.

$h(\mathfrak{O})$ is the class number of $\mathfrak{O}$, and $w(\mathfrak{O})$ is the index of E_0 in the group of units in $\mathfrak{O}$. $J(\mathfrak{O})$ is the set of all $\alpha\in\mathfrak{O}$ such that $\alpha\notin\mathfrak{P}$, $N(\alpha)\mathfrak{J}=\mathfrak{q}$.

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