<table>
<thead>
<tr>
<th>Title</th>
<th>Asymptotic Expansions for the Joint and Marginal Distributions of the Latent Roots of the Covariance Matrix (多変量統計解析)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>CHIKUSE, YASUKO</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 1975, 231: 47-57</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1975-03</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/105450">http://hdl.handle.net/2433/105450</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
Asymptotic Expansions for the Joint and Marginal Distributions of the Latent Roots of the Covariance Matrix

Yale 大 筑 濑 靖 子

1. Introduction.

Let $nS$ be an $m \times m$ matrix having the Wishart distribution $W_m(n, \Sigma)$. Let $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$ and $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$ denote the latent roots of $S$ and $\Sigma$ respectively. For large $n$ and simple latent roots of $\Sigma$, it is known that the latent roots of $S$ are asymptotically independently normal. We assume throughout this paper that all the roots of $\Sigma$ are simple. In this paper an expansion, up to and including the term of order $n^{-1}$, is given for the joint density function of $\lambda_1, \ldots, \lambda_m$ in terms of normal density functions. Expansions for the marginal distributions of the roots are also given, valid when the corresponding roots of $\Sigma$ are simple.

2. Expansions for the extreme root distributions.

We consider first the largest root $\lambda_1$. From the exact
expression for the distribution function of $z_i$ by Sugiyama [15], [16], the distribution function of $X = (n/2) \sqrt{2(1/\lambda_i - 1)}$ can be written as

\begin{equation}
(2.1) \quad P(x_i < x) = \left[ \frac{\gamma_m (p)}{\Gamma_n (n + p)} \right] \left( \det R \right)^{n/2} \mathcal{F}_1 \left( \frac{1}{2} n ; \frac{1}{2} n + p ; -R \right),
\end{equation}

where $p = \frac{1}{2} (m + 1)$, $\gamma_m (a) = \pi^{m(m-1)/4} \prod_{i=1}^{m} \Gamma (a - (i-1)/2)$, $R = \text{diag} \{ r_1, r_2, \ldots, r_m \}$ with $r_i = \left[ n/2 + (n/2) \frac{y}{x} \right] z_i$, $z_i = \lambda_i / \lambda$, $i = 1, \ldots, m$ and $\mathcal{F}_1$ is a confluent hypergeometric function of matrix argument (see Herz [9], Constantine [5]). A system of partial differential equations (pde's) satisfied by the $\mathcal{F}_1$ function has been given by Muirhead [12]. Starting with this system it can be readily verified that $P = P(x < x)$ satisfies each of the $m$ pde's

\begin{equation}
(2.2) \quad \frac{\partial^2 P}{\partial x^2} + x \frac{\partial P}{\partial x} + \left( \frac{2}{n} \right) \left[ 2x \frac{\partial P}{\partial x} + (1 + x^2 - \frac{1}{2} A) \frac{\partial P}{\partial x} - x \sum_{k=1}^{m} z_k \frac{\partial P}{\partial z_k} \right.

- 2 \sum_{k=1}^{m} z_k \frac{\partial P}{\partial x \partial z_k} \left. \right] + \frac{2}{n} \left[ x^2 \frac{\partial P}{\partial x^2} + x (1 - \frac{1}{2} A) \frac{\partial P}{\partial x} \right.

+ \sum_{k=1}^{m} z_k \left( 1 - \frac{1}{2} A - \frac{1}{2} (1 - 2A) \right) \frac{\partial P}{\partial z_k} - 2x \sum_{k=1}^{m} z_k \frac{\partial P}{\partial x \partial z_k}

\left. + \sum_{k=1}^{m} \sum_{j=1}^{m} z_k \frac{\partial P}{\partial z_k \partial z_j} \right] = 0
\end{equation}

and

\begin{equation}
(2.3) \quad \frac{\partial P}{\partial z_i} + \left( \frac{2}{n} \right) \left[ \frac{1}{2(1 - z_i)} \frac{\partial P}{\partial x} + x \frac{\partial P}{\partial z_i} \right] + \frac{2}{n} \left[ z_i \frac{\partial P}{\partial z_i} \right]

- 2
\end{equation}
\[ + \frac{x}{2(1 - z_i)} \frac{\partial^2 P}{\partial z_i^2} + (1 - \frac{1}{2} A_i) \frac{\partial P}{\partial z_i} - \frac{1}{2(1 - z_i)} \sum_{k=2}^{m} z_k \frac{\partial P}{\partial z_k} - \frac{1}{2} \sum_{j \neq i} \frac{z_j}{z_i - z_j} \frac{\partial P}{\partial z_j} = 0 \quad (i = 2, 3, \ldots, m), \]

where

\[ A_i = \sum_{j \neq i} \frac{z_j}{z_i - z_j} \quad (i = 1, 2, \ldots, m). \]

We now look for a solution of these \( m \) p.d.e.'s (2.2) and (2.3) of the form

\[ P = \Phi(x) + \sum_{k=1}^{\infty} (2/\pi)^{k/2} Q_k, \]

where the \( Q_k \) are functions of \( x, z_2, \ldots, z_m \). (That \( P \) possesses such an expansion follows from results in the next section.)

We substitute the series (2.5) into (2.2) and (2.3) and equate coefficients of powers of \((2/\pi)^{1/2}\) on the L.H.S. of to zero. Equating the coefficient of \((2/\pi)^{1/2}\) in (2.2) and (2.3) to zero and using the boundary conditions \( P(x, < \infty) = 1 \) and \( P(x, <- \infty) = 0 \), we have

\[ Q = -\frac{1}{6} \Phi(x) \{ 2H_1(x) + A_1 H_0(x) \}, \]

where \( H_j(x) \) denotes the Hermite polynomial of degree \( j \) (see Kendall and Stuart [19], p. 155). Similarly, equating the coefficient of \( 2/\pi^j \) in (2.2) and (2.3) to zero and solving
the resulting equations gives
\[ Q_2 = -(1/\sqrt{2})g(x) \left[ 4h_2(x) + 18h_3(x) + 12A_1h_1(x) - 12h_5(x) + 9A_2^2h_3(x) \right], \]
where \( A_1 = \sum_{i=2}^{m} (z_i - 1)^{-2}, \quad B_1 = \sum_{i=2}^{m} (z_i - 1)^{-2}. \)

Coefficients of higher powers of \((2/\sqrt{n})Q_2\) in (2.5) may be obtained in a similar manner if required. The expansion is summarized in the following

**Theorem 2.1.** The distribution function of \( X = (2/\sqrt{n})Q_2 \) is \((1/\lambda, -1)\), when the latent roots of \( \Sigma \) are simple, can be expanded for large \( n \) as
\[ P(x, x) = \Phi(x) + (2/\sqrt{n})Q_1 + (2/\sqrt{n})Q_2 + O(n^{-3/2}), \]
where \( Q_1 \) and \( Q_2 \) are given by (2.6) and (2.7) respectively.

Consider now the distribution of the smallest root \( \lambda_m \).

Since \( nS \sim W_m(n, \Sigma) \) we have
\[ P(\lambda_m > y) = \left( \frac{1}{2\pi n} \right)^{\frac{m-1}{2}} \left( \frac{1}{\lambda_m (\lambda + n)} \right)^{\frac{1}{2}} \exp \left( -\frac{1}{2n} \text{tr} (\Sigma^2) \right) \text{det} \left( \frac{1}{\lambda_m (\lambda + n)} \right) \text{det} S \frac{d \lambda}{d \Sigma}, \]
Making the transformation \( T = \gamma^{-1} \Sigma - I \) it is easily seen that
(2.9) becomes
\[ P(\lambda_m > y) = \left[ \frac{1}{\lambda_m (\lambda + n)} \right] \text{det} \left( \frac{1}{2n} \Sigma \right) \exp \left( -\frac{1}{2n} \text{tr} \Sigma^2 \right) \cdot \Psi \left( \frac{1}{2} \lambda_m + \rho ; \frac{1}{2} n \Sigma \Sigma^2 \right), \]
where \( \Psi(a, c; R) \equiv \left[ \frac{1}{
abla_n(a)} \right] \exp \left( -\lambda R S \right) \left( \det \frac{a-p}{\det (I+S)^{c-a-p}} \right) dS. \quad S>0 \)

The function \( \Psi \) is another confluent hypergeometric function of matrix argument (see Muirhead [13]).

Putting \( x_m = (n/2)^{1/2} (l_m/l_m - 1) \) and using the system of pde's satisfied by the \( \Psi \) function given by Muirhead [13] it can readily be shown that the distribution function of \( x_m \), \( P = P(x_m < x) \), satisfies each of the \( m \) pde's (2.2) and (2.3). The only difference here is that now \( z_i = \lambda_m / \lambda_{m-i+1} \) instead of \( \lambda_i / \lambda_1 \) as it was in the largest root distribution. Hence

**Theorem 2.2.** The distribution function of \( x_m = (n/2)^{1/2} (l_m/l_m - 1) \), when the latent roots of \( \Sigma \) are simple, can be expanded for large \( n \) as

\[
P(x_m < x) = \Phi(x) + (2/n)^{1/2} \Theta_1 + (2/n) \Theta_2 + O(n^{-3/2}),
\]

where \( z_i = \lambda_m / \lambda_{m-i+1} \) in \( \Theta_1 \) and \( \Theta_2 \) given by (2.6) and (2.7) respectively.

3. Expansion for the joint distribution.

The joint density function of \( l_1, \ldots, l_m \) can be
expressed in the form (see James [8])

\[
(3.1) \quad \mathcal{F}_{\frac{m}{2}}(\pm n)^{-\frac{m}{2}} \prod_{m} \frac{1}{\mathcal{F}_{\pm m} \mathcal{F}_{\pm m} \Gamma_{\frac{m}{2}}(l_{\pm m})} \Pi_{m} \left\{ \left( l_{\pm m} - \frac{m}{2} \right) \right\} \mathcal{F}_{\pm m}(-\frac{1}{2}mL, \Sigma^{-1})
\]

where \( p = \frac{1}{2}(m+1) \), \( L = \text{diag} (l_{1}, \ldots, l_{m}) \), \( \Sigma = \text{diag} (\lambda_{1}, \ldots, \lambda_{m}) \) and \( \mathcal{F}_{\pm m} \) is a hypergeometric function with two argument matrices. The function in (3.1) has been expanded for large \( n \) by G. Anderson [11] by expressing it as an integral over the orthogonal group. In [11] it is shown that the joint density function can be expressed as

\[
(3.2) \quad \mathcal{F}_{\frac{m}{2}} \prod_{i=1}^{m} \left[ \lambda_{i}^{(m-n-1)/2} \frac{n}{2} \exp \left( -n\lambda_{i}^{2}/2 \right) \right] \prod_{i<j}^{m} \left[ (\lambda_{i} - \lambda_{j}) / (\lambda_{i} - \lambda_{j}) \right]^{\frac{m}{2}} \mathcal{F}_{\pm m}
\]

where \( \lambda_{i} = (n/2) \frac{m}{2} - \frac{m(m-1)}{4} / \prod_{i=1}^{m} \left( (n-i+1)/2 \right) \) and

\[
(3.3) \quad \mathcal{F} = 1 + (2n)^{-\frac{m}{2}} \sum_{i<j}^{m} \lambda_{i} \lambda_{j} (\lambda_{i} - \lambda_{j})^{*} (\lambda_{i} - \lambda_{j})^{*} + O(n^{-2})
\]

(Anderson did not show in general that the remainder term in (3.3) is of order \( n^{-2} \); this has been shown by the author in her Ph.D. thesis at Yale University in the more general case.) Now putting \( x_{i} = (n/2) \frac{m}{2} (\lambda_{i} - \lambda_{i})^{*} (\lambda_{i} - \lambda_{i})^{*} - 1 \)

\( i = 1, \ldots, m \), from (3.2) the joint density function of \( x_{1}, \ldots, x_{m} \) can be expressed as

\[
\cdots
\]
\[(3.4) \quad k_2 \left( \frac{1}{\pi} \right) \left[ 1 + \left( \frac{2n}{m} \right)^{m/2} \sum_{i \neq j} \frac{\lambda'_i \lambda'_j}{(\lambda'_i - \lambda'_j)^2} \right] + O \left( n^{-3/2} \right), \]

where

\[
k_2 = \left( \frac{n}{2} \right)^{\frac{m-1}{2} - \frac{m(m+1)}{4}} \exp \left( -\frac{mN}{2} \right) / \pi^{\frac{mN}{2}} \left( \frac{m}{2} + 1 \right) \frac{1}{\sqrt{2}},
\]

\[
F_1 = \prod_{i \neq j} \left[ 1 + \left( \frac{2n}{m} \right)^{\frac{m}{2}} \left( x_i \lambda_i - x_j \lambda_j \right) / (\lambda_i - \lambda_j) \right]^{\frac{1}{2}},
\]

and

\[
F_2 = \prod_{i \neq j} \left[ 1 + \left( \frac{2n}{m} \right)^{\frac{m}{2}} \left( x_i \lambda_i - x_j \lambda_j \right) / (\lambda_i - \lambda_j) \right]^{\frac{1}{2}}.
\]

It remains to expand \(k_2, F_1, \) and \(F_2\) in (3.4) for large \(n\). For example, by expanding the gamma functions for large \(n\) it follows that

\[
k_2 = (2\pi)^{-m/2} \left[ 1 - \left( \frac{2m}{n} \right) \frac{m}{2} \frac{2m+3m-1}{2m+3m-1} + O \left( n^{-2} \right) \right].
\]

The functions \(F_1\) and \(F_2\) can be easily expanded in terms of powers of \(n^{-1/2}\); however, these expansions, up to and including the terms of order \(n^{-1}\), are quite lengthy and are omitted here. Substituting these expansions in (3.4) gives an expansion of the joint density function of \(x_1, \ldots, x_m\).

The final result is summarized in the following Theorem 3.1. The joint density function of \(x_i = (n/2)^{1/2}(\lambda_i, \lambda_i - 1)\) \((i = 1, \ldots, m)\), where \(\lambda_1, \ldots, \lambda_m\) are simple roots of \(\Sigma\), may be expanded for large \(n\) as
\[(3.5)\quad \prod_{i=1}^{m} g(x_i) \cdot \left\{ 1 + \left(\frac{2}{n}\right)^{\frac{1}{2}} \sum_{j=1}^{m} P_{1i}(x_i) + \left(\frac{2}{n}\right) \left( \sum_{i,j} \lambda_{ij} \right) \right\} \]
\[+ \sum_{i,j} \lambda_{ij} P_{ij}(x_i) P_{ij}(x_j) + \frac{1}{2} \sum_{i,j} \lambda_{ij} A_i A_j \right\} + O\left(n^{-3/2}\right) \}
\]

where

\[(3.6)\quad P_{1i}(x) = (1/6) \left\{ 2 H_3(x) + 3 A_i H_1(x) \right\},
\]
\[(3.7)\quad P_{2i}(x) = (1/12) \left\{ 4 H_4(x) + 4 H_2(x) + 12 A_i H_2(x) - 18 A_i H_2(x) + 9 A_i^2 H_2(x) \right\},
\]

\(H_j(x)\) is the Hermite polynomial of degree \(j\), and

\[(3.8)\quad A_i = \sum_{j \neq i} \lambda_{ji} / (\lambda_i - \lambda_j), \quad B_i = \sum_{j \neq i} \lambda_{ij} / (\lambda_i - \lambda_j)^2.
\]

Note that \(A_i\) is the same as in (2.4).

By integrating out the other variables in (3.5) an expansion of the marginal density function of \(X_i\) can be obtained.

**Corollary.** The marginal density function of \(X_i = (n/2) \left( 1; A_i^2 \right)\),

where \(A_i\) is a simple root of \(Z_i\), may be expanded for large \(n\) as

\[(3.9)\quad g(x_i) \cdot \left\{ 1 + \left(\frac{2}{n}\right)^{\frac{1}{2}} P_{1i}(x_i) + \left(\frac{2}{n}\right) P_{2i}(x_i) + O\left(n^{-3/2}\right) \right\},
\]

where \(P_{1i}(x_i)\) and \(P_{2i}(x_i)\) are given by (3.6) and (3.7) respectively.
The expansion (3.9), in the cases $i=1$ and $m$, agrees with the expansions for the extreme root distributions given in the previous section. Sugiyura [14] has also obtained (3.9) using another method.

Asymptotic moments of $l_i$ can be obtained from (3.9); we obtain

\[ E(l_i) = \lambda_i + A_i \lambda_i / n + O(n^{-2}), \]

\[ \text{Var}(l_i) = 2\lambda_i^2 / n - 2\lambda_i B_i / n^2 + O(n^{-3}), \]

\[ \kappa_3(l_i) = 8 \lambda_i^3 / n^2 + O(n^{-3}), \quad \kappa_4(l_i) = 48 \lambda_i^4 / n^3 + O(n^{-4}), \]

where $\kappa_3(l_i)$ and $\kappa_4(l_i)$ denote the third and fourth cumulants of $l_i$ and $\lambda_i$, $B_i$ are given by (3.8).

From (3.5) we obtain

\[ \text{Cov}(l_i, l_j) = 2 \left( \lambda_i \lambda_j / (\lambda_i - \lambda_j) \right)^2 / n^2 + O(n^{-3}). \]

These expansions agree with results obtained by Lawley [11] without using the asymptotic normality.

Acknowledgment: This is a portion of the author's Ph. D. thesis written at Yale University and supervised by Professor Robb J. Muirhead.
REFERENCES


