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KEY ASSERTIONS AND BACKWARD SUBSTITUTIONS.

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Definition: Let T be a mathematical theory and M a model of T. We assume that a correspondence between $m \in M$ and a term $\overline{m}$ in T is given (for example nutural number m and numeral $\overline{m} = 0^{(m)}$ where $0^{(m)}$ is the result of m fold applications of successor function to 0). A function $f(x_1,\ldots,x_n)$ defined within M is said to be representable [8] in T if there exists a well-formed formula $\Gamma(x_1,\ldots,x_n,x_{n+1})$ with n+1 free variables such that for any $(m_1,\ldots,m_n) \in M^n$,

(1) if $m_{n+1} = f(m_1,\ldots,m_{n+1})$ then $\vdash_T \Gamma(\overline{m}_1,\ldots,\overline{m}_{n+1})$, and

(2) $\vdash_T (\exists! x_{n+1}) \Gamma(x_1,\ldots,x_{n+1})$.

Example . We consider the flowchart of Fig.2.3 which computes gcd(m,n). To this flowchart we attach the formulas:

$$\varphi(m,n) \equiv m>0 \wedge n>0,$$

$$p(m,n,c,d,r) \equiv c=m \wedge d=n \wedge \varphi(m,n)$$

$$\psi(m,n,d) \equiv d=gcd(m,n)$$

and the predicate $q(m,n,c,d,r)$ is to be determined. ($s(m,n,c,d,r)$ will be used when we consider EB_P.) We further parametrize the predicates p and q to the predicates P and Q respectively, introducing a control variable i which expresses the number of visits to the loop.

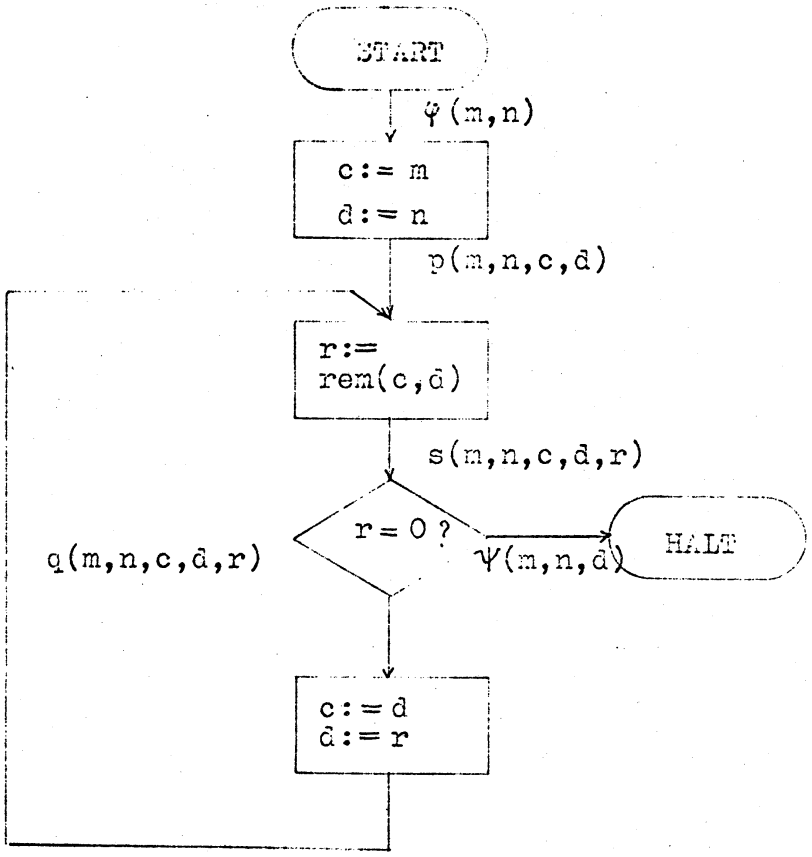

Fig.2.3. A flowchart to compute gcd(m,n)

We have the system EF_P as follows:

$$Q(x,y,0) \equiv c=m \wedge d=n \wedge i=0 \wedge \varphi(m,n) \quad ,$$

$$Q(x,y,\overline{i}) \equiv \exists c' \exists d' \exists r'.\ Q(x,c',d',r',\overline{i-1}) \wedge r=\mathrm{rem}(c,d) \wedge c=d' \wedge d=r \quad ,$$

$$(\exists r'.\ Q(m,n,c,d,r',\overline{i}) \wedge r=\mathrm{rem}(c,d) \wedge r=0) \supset d=\gcd(m,n)$$

where $x=(m,n)$ and $y=(c,d,r)$. For each numeral $\overline{i}$, we rewrite EF_P as follows:

$$Q(m,n,c^{(i)},d^{(i)},r^{(i)},\overline{i})$$
$$\equiv \exists c^{(i-1)} \exists d^{(i-1)} \exists r^{(i-1)}.\ Q(m,n,c^{(i-1)},d^{(i-1)},r^{(i-1)},\overline{i-1})$$
$$\wedge r^{(i)}=\mathrm{rem}(c^{(i-1)},d^{(i-1)}) \wedge r^{(i)} \neq 0$$
$$\wedge c^{(i)}=d^{(i-1)} \wedge d^{(i)}=r^{(i)},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots$$

$$Q(m,n,c^{(1)},d^{(1)},r^{(1)},1)$$
$$\equiv \exists c^{(0)} \exists d^{(0)} .\ c^{(0)} = m \wedge d^{(0)} = n \wedge m>0 \wedge n>0$$
$$\wedge\ r^{(1)} = \mathrm{rem}(c^{(0)},d^{(0)}) \wedge r^{(1)} \neq 0$$
$$\wedge\ c^{(1)} = d^{(0)} \wedge d^{(1)} = r^{(1)} .$$

If we set

$$a^{(0)} = 0,$$
$$b^{(0)} = 1,$$
$$\underline{a}^{(0)} = 1,$$
$$\underline{b}^{(0)} = 0,$$

and

$$a^{(i)} = \underline{a}^{(i-1)} - \mathrm{quo}(c^{(i-1)},d^{(i-1)}) \times a^{(i-1)} ,$$
$$b^{(i)} = \underline{b}^{(i-1)} - \mathrm{quo}(c^{(i-1)},d^{(i-1)}) \times b^{(i-1)} ,$$
$$\underline{a}^{(i)} = a^{(i-1)} ,$$
$$\underline{b}^{(i)} = b^{(i-1)} ,$$

then there hold

$$c^{(j)} = \underline{a}^{(j)} m + \underline{b}^{(j)} n \ , \text{and}$$
$$d^{(j)} = a^{(j)} m + b^{(j)} n, \text{ where } j = 0,1,2,\ldots .$$

Since $a^{(j)}$, $b^{(j)}$, $\underline{a}^{(j)}$ and $\underline{b}^{(j)}$ are recursively defined functions of m, n, and j, they are representable in the elementary number theory so that we have

$$Q(m,n,c,d,r,i) \equiv c = \underline{a}^{(i)} m + \underline{b}^{(i)} n \wedge d = a^{(i)} m + b^{(i)} n$$
$$\wedge\ r = d \wedge r \neq 0 \wedge m>0 \wedge n>0$$

where i is a proper variable of the predicate Q. Therefore we have

$$q(m,n,c,d,r) \equiv \overset{\infty}{\underset{i=1}{\vee}} Q(m,n,c,d,r,i)$$
$$\equiv \exists i .\ i \geq 1 \wedge Q(m,n,c,d,r,i) .$$

Now it is clear that the implication fomula of $EF_p(q)$ or $EF_p(Q)$ holds if one knows the theorem

$$\exists a \exists b \exists \underline{a} \exists \underline{b} .\ c = \underline{a}m + \underline{b}n \wedge d = am + bn \wedge r = 0$$
$$\supset d = gcd(m,n) .$$

We consider the system EB_p for the flowchart of Fig.2.3 where we use $s(m,n,c,d,r)$ instead of $q(m,n,c,d,r)$:

$$p(m,n,c,d,r) \supset s(m,n,c,d,rem(c,d)),$$

$$s(m,n,c,d,r) \wedge r \neq 0 \equiv s(m,n,c,d,rem(c,d)),$$

$$s(m,n,c,d,r) \wedge r = 0 \equiv \psi(m,n,d).$$

First we have the followings by the process of substitution:

$$s(m,n,c,d,r) \equiv s(m,n,c,d,r) \wedge r = 0 \vee s(m,n,c,d,r) \wedge r \neq 0,$$

$$\equiv \psi(m,n,d) \wedge r = 0 \vee s(m,n,f(c,d,r),g(c,d,r),h(c,d,r)),$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\equiv \bigvee_{i=0}^{k} \Big\{ \psi(m,n,g^{(i)}(c,d,r)) \wedge h^{(i)}(c,d,r) = 0 \wedge \Big(\bigwedge_{j=0}^{i-1} h^{(j)}(c,d,r) \neq 0 \Big) \Big\}$$

$$\vee \Big\{ s(m,n,f^{(k+1)}(c,d,r),g^{(k+1)}(c,d,r),h^{(k+1)}(c,d,r)) \wedge \Big(\bigwedge_{j=0}^{k} h^{(j)}(c,d,r) \neq 0 \Big) \Big\} \qquad \cdots\cdots (*)$$

where $f^{(i)}$, $g^{(i)}$ and $h^{(i)}$ are defined by setting

$$f(c,d,r) = d, \quad g(c,d,r) = r, \quad h(c,d,r) = rem(d,r),$$

and

$$f^{(0)}(c,d,r) = c, \quad g^{(0)}(c,d,r) = d, \quad h^{(0)}(c,d,r) = r,$$

$$f^{(i)}(c,d,r) = f(f^{(i-1)}(c,d,r),g^{(i-1)}(c,d,r),h^{(i-1)}(c,d,r)),$$

$$g^{(i)}(c,d,r) = g(f^{(i-1)}(c,d,r),g^{(i-1)}(c,d,r),h^{(i-1)}(c,d,r)),$$

$$h^{(i)}(c,d,r) = h(f^{(i-1)}(c,d,r),g^{(i-1)}(c,d,r),h^{(i-1)}(c,d,r)).$$

If we make explicit the operations, then the previously defined functions $a^{(j)}$, $b^{(j)}$, $\underline{a}^{(j)}$ and $\underline{b}^{(j)}$ where their arguments are c and d this time instead of m and n respectively, define the above functions, namely we have

$$f^{(j)}(c,d,r) = \underline{a}^{(j)}\, c + \underline{b}^{(j)}\, d\ ,$$
$$g^{(j)}(c,d,r) = a^{(j)}\, c + b^{(j)}\, d\ ,$$
$$h^{(j)}(c,d,r) = a^{(j+1)}\, c + b^{(j+1)}\, d.$$

Therefore the suffix j can be considered as a proper variable of predicates.

Now we set

$$K(m,n,c,d,r) \equiv \bigvee_{i=0}^{\infty} \Big\{ \psi(m,n,g^{(i)}(c,d,r)) \wedge h^{(i)}(c,d,r)=0 \wedge \Big(\bigwedge_{j=0}^{i-1} h^{(j)}(c,d,r) \neq 0 \Big) \Big\} .$$

Then from (*) we have $K(m,n,c,d,r) \supset s(m,n,c,d,r)$. On the other hand

$$\begin{aligned} g^{(1)} &= \mathrm{rem}(c,d) < g^{(0)} = d, \\ g^{(j+1)} &= a^{(j+1)}\, c + b^{(j+1)}\, d \\ &= (\underline{a}^{(j)} - \mathrm{quo}(f^{(j)},g^{(j)})a^{(j)})c \\ &\quad + (\underline{b}^{(j)} - \mathrm{quo}(f^{(j)},g^{(j)})b^{(j)})d \\ &= \underline{a}^{(j)}\, c + \underline{b}^{(j)} d - \mathrm{quo}(f^{(j)},g^{(j)})(a^{(j)}c+b^{(j)}d) \\ &= f^{(j)} - \mathrm{quo}(f^{(j)},g^{(j)})g^{(j)} \\ &= \mathrm{rem}(f^{(j)},g^{(j)}) < g^{(j)}, \end{aligned}$$

namely,

$$d = g^{(0)} > g^{(1)} > \ldots > g^{(j)} > \ldots \geq 0$$

and therefore

$$s(m,n,c,d,r) \supset \exists j.\ h^{(j)}(c,d,r) = 0$$

so that there holds

$$s(m,n,c,d,r) \supset K(m,n,c,d,r).$$

Hence we have

$$s(m,n,c,d,r) \equiv K(m,n,c,d,r)\ .$$

Using the properties of gcd we have

$$\Psi(m,n,g^{(i)}(c,d,r)) \wedge h^{(i)}(c,d,r)=0$$
$$\equiv \gcd(f^{(i)}(c,d,r),g^{(i)}(c,d,r)) = g^{(i)}(c,d,r)$$
$$\wedge\, g^{(i)}(c,d,r)=\gcd(m,n) \wedge h^{(i)}(c,d,r)=0$$
$$\equiv \gcd(c,d)=\gcd(m,n) \wedge g^{(i)}(c,d,r)=\gcd(m,n)$$
$$\wedge\, h^{(i)}(c,d,r)=0$$

and

$$\bigwedge_{j=0}^{i-1} h^{(j)}(c,d,r)\neq 0$$
$$\equiv \forall j.\ j<i \supset h^{(j)}(c,d,r)\neq 0$$

so that we conclude

$$s(m,n,c,d,r) \equiv \gcd(m,n)=\gcd(c,d)$$
$$\wedge\, \exists i.\ g^{(i)}(c,d,r)=\gcd(m,n) \wedge h^{(i)}(c,d,r)=0$$
$$\wedge\, \forall j.\ j<i \supset h^{(j)}(c,d,r)\neq 0\ .$$

Here we note that the implication formula of EB_P clearly describes the termination of P.

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