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Limits of tangents on a hypersurface

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Let $V := \{z \in \mathbb{C}^{n+1} \mid f(z) = 0\}$ be an analytic complex hypersurface. We shall suppose that $0 \in V$ and that 0 is a singular point of V. We want to know what is the set of all possible limits of sequences of tangent hyperplanes to V at sequences of smooth points of V tending to 0. More precisely, let us denote by $\check{\mathbb{P}}^n$ the space of hyperplane directions of $\mathbb{C}^{n+1}$, let Σ be the singular locus of V and let $\varphi : V - \Sigma \longrightarrow \check{\mathbb{P}}^n$ be the map which makes correspond to a smooth point of V the direction of its tangent hyperplane. Then let us call $\tilde{V}$ the closure of the graph of φ in $V \times \check{\mathbb{P}}^n$. The first projection $V \times \check{\mathbb{P}}^n \to V$ induces a mapping $\pi : \tilde{V} \longrightarrow V$ which is called the Jacobian blowing-up of V. The set we are interested about is $\pi^{-1}(0)$.

Notice that $\dim \pi^{-1}(\Sigma) = \dim V - 1$. Thus if $0 \in V$ is an isolated singularity $\dim \pi^{-1}(0) = \dim V - 1$.

First notice the following lemma:

<u>Lemma</u> 1 (cf [1]) Let (x_n) be a sequence of smooth points of V such that the lines $0x_n$ tend to l and the hyperplanes $T(x_n, V)$ tend to T, then l is contained in T.

Recall that all possible limits of lines $0x_n$ for sequences of points of V tending to 0 define the tangent cone of V at 0 and the corresponding set in the projective space $\mathbb{P}^n$ of lines through 0 is the Proj of this tangent cone.

Now we have the following lemma which was indicated to us by Professor O. Zariski:

Lemma 2 Let us suppose that the tangent cone of V at 0 is reduced. Then consider (x_n) a sequence of smooth points of V which tends to 0 and such that $T(x_n, V)$ tends to T and $0x_n$ tends to ℓ. Moreover suppose that ℓ gives a smooth point of the Proj of the tangent cone of V at 0, then T is tangent to the tangent cone of V at 0 along ℓ.

Such a lemma has led our interests to compare the limits of tangents ~~of~~ at 0, say $\pi^{-1}(0)$ defined above, and the limits of secants at 0, say the Proj of the tangent cone at 0.

We obtain the following result:

Theorem 3 Suppose $n = 2$ and $0 \in V$ is an isolated singularity, then the limits of tangents of V at 0 is the union of the dual curve of the curve, Proj of the reduced tangent cone at 0, and a finite number of lines which corresponds to the pencils of hyperplanes going through the singular lines of the reduced tangent cone and a finite number of non-singular lines of this reduced tangent cone.

Remark: The non-singular lines of the theorem are specified in the proof (of theorem 6)

The proof of the theorem 3 is based on some results of B. Teissier from [4] and a geometrical study involving some results about equisingularity of O. Zariski [5].

First let us define:

<u>Definition</u> We shall say that a hyperplane H cuts V generically at 0 if $V \cap H$ has an isolated singularity at 0 and the Milnor number of $V \cap H$ at 0 is minimum among all the hyperplane sections with an isolated singularity at 0.

Let us remind that such a hyperplane section has a well-defined topology (at least when $n \neq 3$) because of the results of [2].

The result of B. Teissier can be stated as it follows:

<u>Theorem</u> 4 (cf [4]) (If $0 \in V$ is an isolated singularity) The hyperplane H cuts V generically at 0 if and only if H is not a limit of tangents of V at 0, i.e. $H \notin \pi^{-1}(0)$.

Now let $n = 2$. Call $Z_1 \xrightarrow{p} \mathbb{C}^3$ the blowing-up of the point 0. Let V_1 be the strict transform of V by p. Let H be a hyperplane of $\mathbb{C}^3$ and H_1 be its strict transform by p. Call $C = H \cap V$ and C_1 the strict transform of C. Thus $C_1 = H_1 \cap V_1$.

Then C is reduced if and only if C_1 is reduced. Thus C is reduced if and only if $p^{-1}(0) \cap H_1$ cuts transversally the reduced curve $p^{-1}(0) \cap V_1$ and near each point of intersection H_1 cuts the smooth part of V_1 transversally

But we have the following lemma:

<u>Lemma 5</u> (cf [3]) Let C be a plane curve of multiplicity n at 0, let C_1 be the strict transform of C after blowing-up 0. Let $O_1, \dots, O_k$ be the points of C_1 over O.
Then:

$$\mu(C,0) = \mu(C_1,O_1) + \dots + \mu(C_1,O_k) + n(n-1) - (k-1)$$

Thus if we apply it to our above situation we find that at each point where $p^{-1}(0)\cap H_1$ cuts $p^{-1}(0)\cap V_1$ the local Milnor number of $H_1\cap V_1$ must be the minimum one to get the minimum one for $H\cap V$ at O.

Then consider the components $\Gamma_1, \dots, \Gamma_\ell$ of $p^{-1}(0)\cap V_1$. We have the following case:

a) V_1 is singular along Γ_i. Thus V_1 is equisingular along Γ_i outside a finite number of exceptional points defines lines we shall call exceptional secants of V at O. Remark that the singular points of Γ_i and points of $\Gamma_i \cap \Gamma_j$ $(j\neq i)$ are among these exceptional points;
b) V_1 is not singular along Γ_i except maybe at a finite number of points. The corresponding lines of these points will be exceptional secants of V at O, too.

Then using Zariski's theory of equisingularity we can prove:
<u>Theorem 6</u> A hyperplane H cuts V generically at O if and only if it does not contain any exceptional secants.

Bibliography

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