

RESUME

(A Note on Isolated Singularity)

Isao NARUKI

0. Notation

(X, x) : pair of analytic space and $x \in X$ such that $X \setminus x$ smooth.

$\iota: X \setminus x \hookrightarrow X$ (inclusion).

Ω_X^p : sheaf of (analytic) p -forms on X ($\mathcal{O}_X = \Omega_X^0$).

1. Duality

Lemma 1. $R^q \iota_* \iota^* \Omega_X^p \xleftrightarrow{\text{dual}} R^{n-q-1} \iota_* \iota^* \Omega_X^{n-p}$.

($1 \leq q \leq n-2$, $n = \dim X$).

Proof. Andreotti-Grauert [1].

2. Coherence of Local Cohomology

Lemma 2. S : coherent \mathcal{O}_X -Module, $S|_{X \setminus x}$: locally free

$\Rightarrow \mathcal{H}_x^q(S)$: coherent for $q < \dim X$

Proof. Siu [7].

3. Condition (L), Poincaré Lemma

Definition. (X, x) satisfies (L) $\Leftrightarrow \mathcal{H}_x^q(\Omega_X^p) = 0$ if $p+q < \dim X$.

Lemma 3. (Partial Poincaré Lemma). (X, x) satisfies (L)

$\Rightarrow H^p(\Omega_{X,x}^q) = 0$ ($0 < p < \dim X$).

Proof. Bloom-Herrera [2].

4. Nice Function, Hypersurface section analytic

Definition. $f: X \xrightarrow{\text{analytic}} \mathbb{C}$ is nice on (X, x)

$$\iff f(x)=0, \quad df_z \neq 0 \quad (\forall z \in X \setminus x).$$

(Y, y) : hypersurface section of (X, x)

$$\iff \exists f: \text{ nice on } (X, x) \text{ such that } (Y, y) \stackrel{\text{iso.}}{\cong} (f^{-1}(0), x).$$

Lemma 4 (de Rham Lemma). $f: \text{ nice on } (X, x)$, (X, x) satisfies

$$(L) \implies 0 \rightarrow \Omega_X^0 \xrightarrow{df} \Omega_X^1 \rightarrow \dots \rightarrow \Omega_X^{\dim X} \quad \text{exact.}$$

5. Conservation of (L).

Let (Y, y) : hypersurface section of (X, x)

Theorem 1. (X, x) satisfies (L)

$$\iff \begin{cases} \text{i) } (Y, y) \text{ satisfies (L)} \\ \text{ii) } \dim \mathcal{H}_Y^0(\Omega_Y^n) = \dim \mathcal{H}_Y^1(\Omega_Y^n) \end{cases}$$

$$(n = \dim Y \geq 2)$$

Corollary. (X, x) : a complete intersection of hypersurfaces

$$\implies (X, x) \text{ satisfies (L)}$$

Proof. Hamm [4].

6. Milnor Fiberings

Let $f: \text{ nice on } (X, x)$, (Y, y) : hypersurface section defined by f . We can assume by Milnor [6]

- a) $(Y, y) \hookrightarrow (X, x) \xrightarrow{\text{closed}} (B, 0)$ (B : open ball in $\mathbb{C}^N: (z_1, z_2, \dots, z_N)$)
 b) $r|_{X \setminus x}, r|_{Y \setminus y}$ have no critical point ($r(z) = \sum_{i=1}^N |z_i|^2$)

Theorem 2. Under the above assumption, $\exists S$: a neighborhood of 0 in \mathbb{C} such that i) $R^p f_*(\Omega_f^\bullet)|_S$ are coherent \mathcal{O}_S -Modules

$$\text{ii) } H^p(\Omega_{f,x}^\bullet) \cong R^p f_*(\Omega_f^\bullet)_0$$

where we have set $\Omega_f^p = \Omega_X^p / df \wedge \Omega_X^{p-1}$ ($p=0,1,2,\dots$).

Proof. Brieskorn [3] and Kiehl-Verdier [5].

7. Main Theorem

Theorem 3. $(X,x), f, (Y,y), \Omega_f^\bullet$ as in §6, (X,x) satisfies (L)

\Rightarrow

- i) $H^p(\Omega_{f,x}^\bullet) = 0 \quad 1 \leq p \leq n-1 \quad (n = \dim Y).$
- ii) $H^n(\Omega_{f,x}^\bullet)$: free $\mathcal{O}_{\mathbb{C},0}$ -module of finite rank.
- iii) $\mu = \text{rank}_{\mathcal{O}_{\mathbb{C},0}} H^n(\Omega_{f,x}^\bullet) = \dim_{\mathbb{C}} H^n(\Omega_{Y,y}^\bullet)$
 $(\Omega_Y^\bullet = 0 \rightarrow \Omega_Y^0 \xrightarrow{d} \Omega_Y^1 \xrightarrow{d} \dots \xrightarrow{d} \Omega_Y^n \rightarrow 0).$
- iv) $\mu = \dim R^1 i_* i^* \Omega_Y^{n-1} + \dim H^n(i_* i^* \Omega_f^\bullet) - \dim H^{n-1}(i_* i^* \Omega_f^\bullet)$
- v) In case (X,x) is smooth, there are isomorphisms i

$$\mathcal{H}_Y^0(\Omega_Y^{n+1}) \xrightarrow{i} \mathcal{H}_Y^1(\Omega_Y^n) \xrightarrow{i} \dots \xrightarrow{i} \mathcal{H}_Y^{n-1}(\Omega_Y^2)$$

$$\mathcal{H}_Y^1(\Omega_Y^{n-1}) \xrightarrow{i} \mathcal{H}_Y^2(\Omega_Y^{n-2}) \xrightarrow{i} \dots \xrightarrow{i} \mathcal{H}_Y^{n-1}(\Omega_Y^1).$$

Reference

- [1] Andreotti-Grauert: Théorèmes de finitude pour la cohomologie des espaces complexes, Bull. Soc. Math. France 90(1972), 193-259.
- [2] Bloom-Herrera: De Rham cohomology of an analytic space, Inventiones Math. 7(1969), 275-296.
- [3] Brieskorn: Die Monodromie der isolierten Singularitäten von Hyperflächen, Manuscripta Math. 2(1970), 103-161.
- [4] Hamm: Locale topologische Eigenschaften komplexer Räume, Math. Ann. 191(1971), 235-252.
- [5] Kiehl-Verdier: Ein einfacher Beweis des Kohärenzsatzes von Grauert, Math. Ann. 195(1971).
- [6] Milnor: Singular points of complex hypersurfaces, Ann. Math. Studies, No.51, Princeton Univ.
- [7] Siu: Analytic sheaves of local cohomology, Trans. Amer. Math. Soc. 148(1970), 347-366.