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On the propagation of support of solutions to general systems of partial differential equations

by

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§1. Introduction.

The purpose of the present article is to generalize a theorem of F. John [5] to general (overdetermined) systems of partial differential operators(Theorem 3.3). At the same time, we refine the results by giving rather precise estimates for the support of solutions to such systems of equations in terms of convex hulls which are new even for single equations (Theorems 3.4 and 3.5). These estimates lead us immediately to a characterization criterion for systems of equations to have solutions with supports in a certain kind of prescribed subsets of the euclidean space (Theorem 4.1). Our criterion theorem contains, as very special cases, the results of D.K.Cohoon [2] and of K.Horie [3].

§2 Notations.

Let us fix some notations. Let $X = \mathbb{R}^{\ell}$ be the ℓ-dimensional euclidean space and $\Xi (\cong \mathbb{R}^{\ell})$ be its dual. When we choose a coordinate wystem $x = (x_1, \ldots, x_{\ell})$ in X, the coordinates $\xi = (\xi_1, \ldots, \xi_{\ell})$ in Ξ should be so chosen that the duality bilinear form be written as $\langle x, \xi \rangle = x_1\xi_1 + \ldots + x_{\ell}\xi_{\ell}$. Let $\mathbb{C}[D_1, \ldots, D_{\ell}]$ (or simply $\mathbb{C}[D]$) be the polynomial ring of ℓ indeterminates $D = (D_1, \ldots, D_{\ell})$ with complex coefficients. $D = (D_1, \ldots, D_{\ell})$ will operate in the space X as the imaginary gradient, i.e. $D_j = -i\frac{\partial}{\partial x_j}$ $(j = 1, \ldots, \ell;\ i = \sqrt{-1})$. Thus, each polynomial $P(D) \in \mathbb{C}[D]$ is a partial differential operator with constant coefficients.

Now, let $\wp(D)$ be an $m \times n$ matrix of partial differential operaters

$$(2.1)\quad \wp(D) = \begin{pmatrix} P_{11}(D) \ldots P_{1n}(D) \\ \ldots \\ P_{m1}(D) \ldots P_{mn}(D) \end{pmatrix}$$

and consider the following system of equations

$$(2.2)\quad \wp(D)U = 0$$

where $U = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}$ is a column vector of unknown functions (or generalized functions). For simplicity we consider mainly C^{∞}-functions defined in an open (or on a regular closed) subset of X.

In discussing the equation (2.2), we may alway assume that

(2.3) $m \geqq n$

by augmenting , if necessary, the number of rows in (2.1) by addition of zero entries. Then, let

(2.4) $\{\Delta_1, \ldots, \Delta_n\}$

be the totality of $n \times n$ minors of the matrix (2.1). Properties of the operator (2.1) can often be requced to those of the set (2.4) or to those of the ideal

(2.5) $I(\wp) = (\Delta_1, \ldots, \Delta_N)$

of $\mathbb{C}[D]$ generated by the set of polynomials (2.4).

For two subsets A, B of $\mathbb{R}^\ell$, their vector sum (resp. their vector difference) will be denoted by A+B (resp. by A−B). Their set difference will be denoted by $A \setminus B$. The convex hull (resp. the closed convex hull) of a set A will be denoted by ch A (resp. by $\overline{\text{ch}}$ A).

Given a vector $\vartheta \in \Xi \setminus \{0\}$, a subset of X is called a ϑ-slab if it can be written in the form

$$\Omega(\vartheta; I) = \{x \in X \; ; \; \langle x, \vartheta \rangle \in I\}$$

for an interval I on the real line $\mathbb{R}$. We use also the simplified notations

$$\Omega(\vartheta) = \Omega(\vartheta;\ (0,\infty)),$$

$$\Omega(-\vartheta) = \Omega(\vartheta;\ (-\infty,0)).$$

Taking their closures, we get the closed half spaces

$$\overline{\Omega}(\vartheta) = \Omega(\vartheta;[0,\infty)),$$

$$\overline{\Omega}(-\vartheta) = \Omega(\vartheta;(-\infty,0]).$$

As for hyperplanes, we use the notation

$$H(\vartheta,c) = \{x\ ;\ \langle x,\vartheta\rangle = c\}$$

for a given $c \in \mathbb{R}$ and the simplified notation

$$H(\vartheta) = \{x\ ;\ \langle x,\vartheta\rangle = 0\}$$

for $c = 0$.

The cones in X (or in Ξ) that we consider in this article are always convex cones with their vertices at the origin 0. Thus a subset Γ of X is called a cone if, for every $x \in \Gamma$ and for every $t > 0$, $tx \in \Gamma$. A closed convex cone Γ in X is called a ϑ-*proper* cone if it is contained in $\overline{\Omega}(\vartheta)$ and if its intersection with the hyperplane $H(\vartheta)$ is reduced to the one point set $\{0\}$. (i.e. $\Gamma \cap H(\vartheta) = \{0\}$). The last condition is clearly equivalent to say that $\Gamma \cap H(\vartheta, c)$ is always compact for any $c \in \mathbb{R}$.

3. Quasihyperbolicity, Main theorems.

For a polynomial $P \in \mathbb{C}[D]$, we denote by P^0 its principal part (i.e. its homogeneous part of the highest degree). Given a $\vartheta \in \Xi \setminus \{0\}$, we shall shortly say that a polynomial P is ϑ-hyperbolic if P is hyperbolic with respect to ϑ. A polynomial P is called weakly ϑ-hyperbolic if its principal part P^0 is ϑ-hyperbolic (c.f. [1],[4]). We define the dual cone $\Gamma(P,\vartheta)$ and the propagarion cone $\Gamma^*(P,\vartheta)$ by the identifications

$$\Gamma(P,\vartheta) = \Gamma(P^0,\vartheta),$$

$$\Gamma^*(P,\vartheta) = \Gamma^*(P^0,\vartheta) ,$$

when P is weakly ϑ-hyperbolic (even though P might not be ϑ-hyperbolic). We note here that the propagation of a weakly ϑ-hyperbolic polynomial is always ϑ-proper closed convex cone. If P is weakly ϑ-hyperbolic and if u is a C^∞-solution to the equation

$$P(D)u = 0 \tag{3.1}$$

in the closed half space $\Omega(\vartheta;[c,\infty))$, then for the support of u we have the inclusion relation

$$\operatorname{supp} u \subseteq K + \Gamma^*(P,\vartheta) \tag{3.2}$$

where K denotes a closed convex set containing the support of Cauchy data of u on $H(\vartheta,c)$. (Since $P^0(\vartheta) \neq 0$, the support of Cauchy data of u concides $H(\vartheta,c) \cap \operatorname{supp} u$.)

Now we introduce the following concept.

Definition 3.1. A system $\mathcal{P}$ of operators (2.1) is called ϑ-quasihyperbolic if there exists a non-constant weakly ϑ-hyperbolic polynomial P which divides all the elements of the ideal $I(\mathcal{P})$ defined by (2.5).

Remark 3.2. We say that a system $\mathcal{P}$ is degenerate if $I(\mathcal{P}) = \{0\}$, i.e. if the rank of the matrix $\mathcal{P}$ is $\leq n-1$. Degenerate systems are always ϑ-quasihyperbolic for any $\vartheta \in \Xi\setminus\{0\}$.

When $\mathcal{P}$ is non-degenerate, it is ϑ-quasihyperbolic if and only if the greatest common divisor G of the elements of the ideal is a ϑ-quasihyperbolic polynomial (i.e. G is divisible by a non-constant weakly ϑ-hyperbolic polynomial). Thus G can be factored in the form

$$G(D) = P(D)R(D)$$

where P is a non-constant weakly ϑ-hyperbolic polynomial and R is a polynomial having no irreducible weakly ϑ-hyperbolic factor in $\mathbb{C}[D]$. We denote this factor P(D) by $\chi(\mathcal{P})$. $\chi(\mathcal{P})$ and its factorization into irreducible components

$$(3.3) \qquad \chi(\mathcal{P}) = P_1(D)^{r_1} \cdots P_s(D)^{r_s}$$

will play a central role in our present work. Each irreducible factor P_j is of course weakly -hyperbolic.

Now we state our main results .

Theorem 3.3. Given a slab domain $\Omega = \Omega(\vartheta;(c_1,c_2))$, the equation (2.2) has a non-trivial solution $U \in [\mathcal{D}'(\Omega)]^n$ such that the closed set $H(\vartheta,c) \cap \text{supp } U$ has a compact connected component for some $c \in (c_1,c_2)$, then the system $\mathcal{P}$ is ϑ-quasihyperbolic.

Theorem 3.4. Suppose that a system of operators $\mathcal{P}$ is ϑ-quasihyperbolic and non-degenerate. $P_1,\cdots,P_s$ be the collection of all the distinct irreducible components in the factorizetion of $\chi(\mathcal{P})$. Then, for any non-empty subset Λ of $\{1,2,\cdots,s\}$, for any $c \in \mathbb{R}$ and for any compact convex subset K of $H(\vartheta,c)$ with non-empty interior (in $H(\vartheta,c)$), there exists a solution $U \in [C^\infty(\Omega)]^n$ to the equation (2.2) such that the following simultaneous set equalities hold:

(3.4) $$\overline{\text{ch}}(\overline{\Omega}(\vartheta;[c,\infty)) \cap \text{supp } U) = K + \sum_{\lambda\in\Lambda} \Gamma^*(P_\lambda,\vartheta) ,$$

(3.5) $$\overline{\text{ch}}(\overline{\Omega}(\vartheta;(-\infty,c]) \cap \text{supp } U) = K - \sum_{\lambda\in\Lambda} \Gamma^*(P_\lambda,\vartheta) .$$

In particular, since propagation cones are ϑ-proper, $H(\vartheta,c') \cap \text{supp } U$ is compact for all $c' \in \mathbb{R}$.

Theorem 3.5. Suppose $\mathcal{P}$ be a non-degenerate system.

1) If the equation (2.2) in $\Omega(\vartheta;[a,\infty))$ admits a non-trivial solution $U \in [C^\infty(\Omega;[a,\infty))]^n$ such that $H(\vartheta,c) \cap \text{supp } U$ is compact for all $c \geq a$, then the system $\mathcal{P}$ is ϑ-quasihyperbolic and the set equality (3.4) holds for some non-emply subset Λ of $\{1,2,\cdots,s\}$ and for all $c \geq a$ with $K = \text{ch}(H(\vartheta,c) \cap \text{supp } U)$.

2) If the equation (2.2) in $\mathbb{R}^{\ell}$ admits a non-trivial solution $U \in [C^{\infty}(\mathbb{R}^{\ell})]^{n}$ such that $H(\vartheta, c) \cap \operatorname{supp} U$ is compact for all $c \in \mathbb{R}$, then the system $\mathcal{P}$ is ϑ-quasihyperbolic and the simultaneous set equalities (3.4) and (3.5) hold for some non-empty Λ and for all $c \in \mathbb{R}$ with $K = \operatorname{ch}(H(\vartheta, c) \cap \operatorname{supp} U)$.

As for degenerate systems, we have

Theorem 3.6. The equation (2.2) admits a non-trivial solution $U \in [\mathcal{D}'(\mathbb{R}^{\ell})]^{n}$ with compact support if and only if the system $\mathcal{P}$ is degenerate. More precicely, if $\mathcal{P}$ is degenerate, then for any open set ω and for any $\varepsilon > 0$, there exists a solution $U \in [C^{\infty}(\mathbb{R}^{\ell})]^{n}$ such that

$$\overline{\omega} \subseteq \operatorname{supp} U \subseteq \omega_{\varepsilon}$$

where $\omega_{\varepsilon} = \{x \in \mathbb{R}^{\ell};\ d(x, \omega) < \varepsilon\}$ (d being a metric on $\mathbb{R}^{\ell}$).

§4 Application.

We say that a pair $T=(T^{(+)},T^{(-)})$ of closed convex subsets of $\mathbb{R}^{\ell}$ is ϑ-proper, if it satisfies the conditions

i) $T^{(+)} \subseteq \bar{\Omega}(\vartheta)$, $T^{(-)} \subseteq \bar{\Omega}(-\vartheta)$;

ii) $T^{(+)} \cap H(\vartheta)$, $T^{(-)} \cap H(\vartheta)$ are both compact;

iii) $T^{(+)} \cap T^{(-)}$ has non-empty interior in $H(\vartheta)$.

For a ϑ-proper pair of closed convex sets $T=(T^{(+)}, T^{(-)})$, we put $|T|=T^{(+)} \cup T^{(-)}$ and define the direction cone $\Gamma(T)$ of T by putting

(4.1) $$\Gamma(T) = \Gamma(T^{(+)}) \cap (-\Gamma(T^{(-)})).$$

Here, in the right hand side, Γ stands for the usual direction cones. It is clear that $\Gamma(T)$ is always a ϑ-proper closed convex cone.

We say that a subset S of $\mathbb{R}^{\ell}$ is a ϑ-proper set if there exist two ϑ-proper pairs $T_j=(T_j^{(+)},T_j^{(-)})$, $j=1,2$ with common direction cone $\Gamma(T_1)=\Gamma(T_2)$ and a vector $x_0 \in \mathbb{R}^{\ell}$ such that the inclusion relations

(4.2) $$|T_1| \subseteq S + x_0 \subseteq |T_2|$$

hold. For such a set S, we define its direction cone $\Gamma(S)$ by putting

(4.3) $$\Gamma(S) = \Gamma(T_1) = \Gamma(T_2).$$

It is easy to see that $\Gamma(S)$ does not depend on the choice of T_j $(j=1,2)$ and x_0.

Now we can state

<u>Theorem 4.1.</u> Given a non-degenerate system of operators $\mathcal{P}$ and a ϑ-proper subset S of $\mathbb{R}^\ell$. For the system of equations (2.2) to have a non-trivial solution $U \in [C^\infty(\mathbb{R}^\ell)]^n$ with supp $U \subseteq S$, it is necessary and sufficient that $\chi(\mathcal{P})$ has an irreducible factor P_λ such that

(4.4) $$\Gamma^*(P_\lambda, \vartheta) \subseteq \Gamma(S).$$

If dim $\Gamma(S)=k$ then it is easy to see that, for a suitable coordinate system and for a number $b \geq 0$, we have $\vartheta=(1,0,\ldots,0)$ and $\Gamma(S) \subseteq \{x;\ x_1 \geq 0,\ |x_j| \leq bx_1,\ j=2,\ldots,k,\ x_{k+1}=\ldots=x_\ell=0\}$ since $\Gamma(S)$ is a ϑ-proper cone. Then, the principal part $P_\lambda^0(\xi)$ of the polynomial $P_\lambda(\xi)$ satisfying (4.4) depends only on $\xi_1,\ldots,\xi_k$ and all the solutions $(\xi_1,\ldots,\xi_k) \in \mathbb{R}^k$ of the algebraic equation $P_\lambda^0(\xi) = 0$ should satisfy the inequality : $|\xi_1| \leq b(\xi_2^2+\ldots+\xi_k^2)$. With these remarks in mind, we get directly from the above theorem 4.1 the results in [2] by putting $m=n$, $k=1$ and those in [3] by putting $m=n=1$, $k=2$.

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