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On the local surjectivity of analytic partial differential operators in the space of distributions with given wave front sets

Kimimasa NISHIWADA

In this note we shall state some characterizations of C^∞ and analytic wave front sets for distributions introduced in Hörmander [6], [7] and as an application, an existence theorem of solutions of partial differential equations with analytic coefficients. In so doing our viewpoint is that distributions (and hyperfunctions in general) can be decomposed into sums of boundary values of analytic functions. It will be shown that a decomposition of analytic wave front set for a distribution f is equivalent to a decomposition of f into sum of boundary values of analytic functions. This result must be related to the notion of micro-analyticity for hyperfunctions (Sato [11]), however the corresponding argument is omitted here.

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1. Boundary values of analytic functions.

We identify $\mathbb{C}^n$ with the hermitian product $\langle z,\zeta\rangle=\sum z_i\bar{\zeta}_i$ and $\mathbb{R}^{2n}$ with the scalar product $\mathrm{Re}\langle z,\zeta\rangle$. For $\Omega,\Gamma\subset\mathbb{C}^n$, we denote by $\mathcal{O}(\Omega)$ the space of germs of analytic functions in a neiborhood of Ω and by $T(\Gamma)$ the tube domain $\mathbb{R}^n+i\Gamma$ in $\mathbb{C}^n$. For given two cones Γ_1 and Γ_2, the notation $\Gamma_1 \subset\subset \Gamma_2$ means Γ_1 is generated by a relatively compact set in Γ_2.

Definition 1.1. Let ω be an open set in $\mathbb{R}^n$, Ω a complex neighborhood of ω and Γ an open convex cone in $\mathbb{R}^n$. For a function $f \in \mathcal{O}(\Omega \cap T(\Gamma))$, we shall say that f admits a boundary value in ω if the limit $\lim\limits_{\substack{y\downarrow 0\\ y\in\Gamma'}} f(x+iy)$ exists in $\mathcal{D}'$ for every subcone $\Gamma' \subset\subset \Gamma$. The limit is an element in $\mathcal{D}'(\omega)$ and denoted by $f(x+i\Gamma 0)$.

Now we shall recall one of the fundamental results in Martineau [10].

Lemma 1.1. $f \in \mathcal{O}(\Omega \cap T(\Gamma))$ admits a boundary value $f(x+i\Gamma 0) \in \mathcal{D}'(\omega)$ if and only if for every compact set K in ω and every subcone $\Gamma' \subset\subset \Gamma$, there are constants C and M such that

$$(1.1) \qquad \sup_{x\in k} |f(x+iy)| \leqq C|y|^{-M}$$

if $y \in \Gamma'$ and $|y|$ is sufficiently small.

It is sufficient for the proof of the if part of the lemma to assume (1.1) only for a half line Γ', in virture of Prop. 11.6 in Komatsu [8]. Note that if (1.1) is valid for a function $f \in \mathcal{O}(\Omega \cap T(\Gamma))$, one has

$$(1.2) \qquad \sup_{x\in k} |D_x^\alpha f(x+iy)| \leqq C(C|\alpha|)^{|\alpha|}\ |y|^{-M-|\alpha|},$$

for every α and sufficiently small $y \in \Gamma'$. Moreover since $f(x+i\Gamma 0) \in \mathcal{D}'(\omega)$, the functional

$$\langle f(x+i\Gamma 0), \varphi\rangle = \lim_{y\downarrow 0} \int f(x+iy)\varphi(x)dx, \ \varphi \in C_0^\infty(\omega),$$

is defind. Let $\varphi(x+iy)$ be a C^{∞} extension of $\varphi(x)$ into the complex domain such that $\operatorname{supp}\varphi(x+iy) \subset \Omega$ and that

$$(1.3) \qquad |\bar{\partial}\varphi(x+iy)| \leqq C_N |y|^N , \quad N = 1, 2, \ldots, .$$

Then it follows that with $\theta \in \Gamma$

$$(1.4) \qquad \langle f(x+i\Gamma 0), \varphi\rangle = 2i \iint_{t>0} f(x+it\theta)\langle\bar{\partial}\varphi(x+it\theta), \theta\rangle dxdt.$$

This integral is absolutely convergent by (1.1) and (1.3).

2. Real analyticity.

We first reproduce the definition of analytic wave front set for a distribution.

Definition 2.1. (Hörmander [7]). Let ω be an open set in R^n, $(x_0, \xi_0) \in T^*(\omega)\setminus 0$ and $u \in \mathcal{D}'(\omega)$. Then we shall say that $(x_0,\xi_0) \notin WF_A(u)$ if and only if there is an open neighborhood U of x_0, an open conic neighborhood V of ξ_0 and a bounded sequence $u_N \in \mathcal{E}'(\omega)$ which is equal to u in U and satisfies the estimates

$$(2.1) \qquad |\hat{u}_N(\xi)| \leqq C(CN/|\xi|)^N, \quad N = 1, 2, \ldots,$$

when $\xi \in V$ for some constant C.

One may put $u_N=\phi_N u$ in the above definition where ϕ_N, $N=1,2,\ldots,$ is a sequence of functions in $C_0^\infty(\omega)$ which is equal to 1 on U, vanishes at all points with distance larger than r from U and satisfies

$$(2.2) \qquad |D^{\alpha+\beta}\phi_N| \leqq C_\alpha r^{-|\alpha|}(CN/r)^{|\beta|} \quad \text{if} \quad |\beta| \leqq N.$$

Here C depends only on n and C_α depends only on n and α. For the existence of such functions, c.f. Lemma 2.2 in [7]. We need to extend ϕ_N into the complex domain and more precise estimates than (1.3).

Lemma 2.1. For any $M>0$, there is an extension

$\phi_{2N}(x+iy) \in C_0^\infty(\Omega)$ of ϕ_{2N}, which satisfies the estimate

$$|D_x^\beta \bar\partial \phi_{2N}(x + iy)| \leqq C_M C^N N^{|\beta|} |y|^{M+k}, \tag{2.3}$$

if $|\beta|+k \leqq N$, for some constants C_M and C both independent of N and y.

In view of (1.2) and (2.3), one can easily estimate the Fourier transform of $\phi_{2N} f(x+i\Gamma 0)$ i.e. one has

$$|<f(x + i\Gamma 0), \phi_{2N} e^{-i<x,\xi>}>| \leqq C(CN/|\xi|)^N , \tag{2.4}$$

$N=1,2,\ldots,$ when $\xi \notin V' = \{\eta; <y,\eta> \geqq 0$ for all $y \in \Gamma'\}$ the dual cone of $\Gamma' \subset\subset \Gamma$. With a converse study, one can prove the following

<u>Theorem 2.1.</u> Let $\{V_\alpha\}$ be a finite family of open convex proper cones in $\mathbb{R}^n$ and $\{\Gamma_\alpha\}$ a family of dual cones of V_α. Then the following statements are equivalent for a distribution f defined near $x_0 \in \mathbb{R}^n$.

(i) The fibre $WF_A(f)|_{x_0}$ is contained in $\bigcup_\alpha V_\alpha$.

(ii) There is a neighborhood ω of x_0, its complex neighborhood Ω and are functions $f_\alpha \in \mathcal{O}(\Omega \cap T(\Gamma'_\alpha))$ for some open cones $\Gamma'_\alpha \supset\supset \Gamma_\alpha$ such that

$$f = \sum_\alpha f_\alpha(x + i\Gamma_\alpha 0) \quad \text{in} \quad \omega. \tag{2.5}$$

The decomposition (2.5) is carried out in the space of C^∞

functions, provided that f is C^∞.

3. Smoothness.

In [6], $WF(f)$ for a distribution f is defined as the set of points in the cotangent space which must be characteristic for every pseudo-differential operator P such that $Pf \in C^\infty$. It is clear that $WF(f) \subset WF_A(f)$. By means of the results in the preceding section, another characterization of $WF(f)$ is obtained.

Theorem 3.1. Let ω be an open set in $\mathbb{R}^n$, $(x_0,\xi_0) \in T^*(\omega)\setminus 0$ and $f \in \mathcal{D}'(\omega)$. Then $(x_0,\xi_0) \notin WF(f)$ if and only if there exists a finite family $\{\Gamma_\alpha\}$ of open convex cones in $\mathbb{R}^n$, a complex neighborhood Ω of x_0 and a decomposition of f near x_0,

$$f = \sum_\alpha f_\alpha(x + i\Gamma_\alpha 0),$$

with $f_\alpha \in \mathcal{O}(\Omega \cap T(\Gamma_\alpha))$ such that $f_\alpha(x+i\Gamma_\alpha 0) \in C^\infty$ near x_0 for every α with $\Gamma_\alpha \subset \{y;\ \langle y,\xi_0\rangle \geqq 0\}$.

Finally we remark that a necessary and sufficient condition for the boundary value $f(x+i\Gamma 0)$ of a function $f \in \mathcal{O}(\Omega \cap T(\Gamma))$ to be C^∞ on $\omega = \Omega \cap \mathbb{R}^n$, is the following; there exists a non-empty subcone $\Gamma' \subset\subset \Gamma$ such that for every compact set K in ω and multi-index α, $D_x^\alpha f(x+iy)$ is bounded on $(K+i\Gamma') \cap \Omega$ when $|y|$ is sufficiently small.

4. An existence theorem.

In this section we shall give an application of Theorem 2.1. Let $P=P(x,D)$ be a linear partial differential operator with analytic coefficients in an open set $X \subset \mathbb{R}^n$ and with principal part $P_m(x,D)$.

Theorem 4.1. Let $x_0 \in X$ and $f \in \mathcal{D}'(X)$. Assume that $P_m(x_0,\xi) \not\equiv 0$ as a polynomial of ξ and that

$$P_m(x_0, \xi) \neq 0 \quad \text{for all} \quad \xi \in WF_A(f)|_{x_0} . \tag{4.1}$$

Then the system of equations,

$$P(x, D)u(x) = f(x) , \tag{4.2}$$

$$WF_A(u) = WF_A(f) , \tag{4.3}$$

admits a solution $u \in \mathcal{D}'$ in a neighborhood of x_0. Such solutions are uniquely determined except for analytic solutions of $Pu = 0$ near x_0 and C^∞ if so is f.

We first remark that the condition (4.1) is always satisfied if either f is analytic or P is elliptic. Thus our theorem partly extends the Cauchy-Kowalevsky theorem. It is also mentioned that it has a close connection with the fundamental theorem of Sato in hyperfunction theory. (Sato [11], Bony-Schapira [1])

The last statement follows from the regularity theorem of Hörmander [6], [7]. In construction of a distribution solution u for given f, we shall need the Cauchy-Kowalevsky theorem "with estimates". Let $B(z_0, r)$ be the open ball in $\mathbb{C}^n$ of centre z_0 and radius r,

$$D(z_0, r) = D^{\zeta}(z_0, r) = B(z_0, r) \cap \{z \; ; \; \langle z - z_0, \zeta\rangle = 0\}$$

for $\zeta \in \mathbb{C}^n \setminus 0$ and $N_{\varepsilon,\delta}(z_0, r) = N^{\zeta}_{\varepsilon,\delta}(z_0, r)$ be the $\varepsilon\delta r$ - neighborhood of $D^{\zeta}(z_0, (1 - \varepsilon)r)$ in $B(z_0, r)$. $(0<\varepsilon<1)$

<u>Lemma 4.1</u>. Assume that $P_m(z_0, \zeta) \neq 0$. Then there exists a neighborhood V of z_0 and a constant $\delta \nless 1$ such that for every ball $B(z_1, r)$ in V, any germ on $D(z_1, r)$ of solution of

$$(4.4) \qquad P\,u = f \; , \qquad f \in \mathcal{O}(\overline{B(z_1, r)})$$

can be prolonged to the solution $u \in \mathcal{O}(\overline{B(z_1, \delta r)})$.

Moreover all such solutions satisfy the estimates,

$$(4.5) \qquad \sup_{B(z_1,\delta r)} |u(z)| \leqq C_{\varepsilon,\delta}\Big(r^m \sup_{B(z_1,r)}|f(z)| + \sup_{N_{\varepsilon,\delta}(z_1,r)}|u(z)|\Big) ,$$

where $0 < \varepsilon \nless 1$, $m = \deg P$ and the constant $C_{\varepsilon,\delta}$ is independent of u, f, z_1 and r.

The proof of Lemma 4.1 has a good linkage with Theorem 5.1.1 in [4]. The next lemma is crucial for our theorem. We set $\Gamma_\varepsilon = \{z \; ; \; \mathrm{Re}\langle z,\zeta\rangle < -\varepsilon|z|\}$, $x_0 = 0$ without loss of generality and $H(\zeta, a) = \{\zeta \; ; \; \mathrm{Re}\langle z,\zeta\rangle = a\}$. Denote ζ - slab domains by

$$\Omega(\zeta\ ;\ a,\ b) = \{z;\ a< \mathrm{Re}\ \langle z,s\rangle < b\}\ .$$

Lemma 4.2. Let $P_m(0,\ \zeta) \neq 0$. Then there exists $\varepsilon > 0$ so that for any neighborhood V of 0 there is a neighborhood W of 0 such that for any cone $\Gamma \supset\supset \Gamma_\varepsilon$, the equation.

$$(4.6) \qquad P(z,\ D)u(z) = f(z), \quad f \in \mathcal{O}(\Gamma \cap V)$$

admits a solution $u \in \mathcal{O}(W \cap \Gamma_\varepsilon)$ which satisfies

$$(4.7) \qquad \sup_{\Gamma_\varepsilon \cap \Omega(\zeta;-a,-t)} |u(z)| \leqq Ct^{-M}\Big(\sup_{\Gamma\cap\Omega(\zeta;-a,-bt)} |f(z)| + \sup_{\Gamma_\varepsilon\cap\Omega(\zeta;-a,-ab)} |u(z)|\Big)$$

Here $0 < t \leqq a$ and the positive constants C, M, a, b (<1) depend only on ε, V, Γ and δ in Lemma 4.1.

As for the existence of u in $\Gamma_\varepsilon \cap W$, our assertion is a special case of more gerneral result by Zerner,Bony-Schapira. ([1], [12], see Lemma 3.2 in [1].)

Proof of Theorem 4.1. We assume $x_0 = 0$ and set $I = WF_A(f)|_0$. Since each point $-i\xi$, $\xi \in I$, is non-characteristic with respect to P, one finds the constants δ_ξ and ε_ξ corresponding to $-i\xi$ in Lemma 4.1 and 4.2. To each $\xi \in I$, choose a small real conical neighborhood V_ξ of ξ with its dual cone Γ_ξ so that

$$C_\xi = \{y \in \mathbb{R}^n\ ;\ \langle y,\ \xi\rangle \geqq \varepsilon_\xi |y|\} \subset\subset \Gamma_\xi .$$

Then

$$\mathbb{R}^n + i\Gamma_\xi \supset \{z = x + iy\ ;\ \langle y, \xi\rangle \geqq \varepsilon_\xi |z|\}$$

$$= \{z\ ;\ \mathrm{Re}\ \langle z, -i\xi\rangle \leqq -\varepsilon_\xi |z|\}\ .$$

Since I is compactly generated, a finite family $\{V_\alpha\}$ of such neighborhoods covers I. Denoting the interiors of dual cones of V_α by Γ_α and decomposing

$$f = \sum_\alpha f_\alpha(x + i\Gamma_\alpha 0),\ f_\alpha \in \mathcal{O}(V \cap T(\Gamma_\alpha)),$$

for a neighborhood V of 0, we consider each equation

$$\text{(4.8)} \qquad P(z, D)\, u_\alpha = f_\alpha \quad \text{in} \quad V \cap T(\Gamma_\alpha)\ .$$

Let $V_\alpha = V_\xi$ with $\xi \in I$ and $\varepsilon = \varepsilon_\xi$. There exists a sufficiently small real neighborhood ω of 0, a complex one W and $a > 0$ such that from Lemma 4.2 (4.8) has a solution u_α in $(x + \Gamma_\varepsilon) \cap W$ for every $x \in \omega$ with initial data 0 on the complex hyperplane $\langle z, -i\xi\rangle = -a$. Hence we obtain an analytic solution u_α of (4.8) defined in $(\omega + iC_\xi) \cap W$. Since $f_\alpha(x + iy)$ is locally uniformly temperate with respect to y, (4.7) implies

$$|u_\alpha(x + iy)| \leqq C|\langle y, \xi\rangle|^{-M}$$

$$\leqq C|\varepsilon y|^{-M}\ ,$$

when $y \in C_\xi$ is sufficiently small. This means $u_\alpha(x + iC_\xi 0) \in \mathcal{D}'$, $WF_A(u_\alpha)|_x \subset$ dual cone of C_ξ for $x \in \omega$ and then that $WF_A(u)|_x$ is contained in a small neighborhood of I on which $P_m(x, \xi) \neq 0$. Here we have defined $u(x) = \sum_\alpha u_\alpha(x + iC_\xi 0)$, a distribution solution of the equation (4.2). Applying the regularity theorem of Hörmander [7], we conclude that $WF_A(u)|_x = WF_A(f)|_x$, $x \in \omega$.

References

[1] Bony, J-M. and Schapira, P., Existence et prolongement des solutions holomorphes des équations aux derivées partielles, Inventiones math., 17 (1972), pp.95-105.

[2] Bontet de Monvel, L. and Krée, P., Psendodifferential operators and Gevrey classes, Ann. Inst. Fourier Grenoble, 17 (1967), pp.295-323.

[3] Bros, J and Inagornitzer, D., Ann. Inst. Poincaré, 18, n°2 (1973), p.147.

[4] Hörmander, L., Linear Partial Differential Operators, Springer-Verlag, Berlin, Göttingen, Heidelberg, (1963).

[5] Hörmander, L., On the singularities of solutions of partial differential equations, Comm. Pure Appl.Math., 23 (1970), pp.329-358.

[6] Hörmander, L., Fourier Integral Operators I, Acta Math., 127 (1971), pp.79-183.

[7] Hörmander, L., Uniqueness theorems and wave front sets for solutions of linear differential equations with analytic coefficients, Comm. Pure Appl. Math., 24 (1971), pp.671-704.

[8] Komatsu, H., Ultradistributions I, Structure theorems and a characterization, J. Fac. Sci. Univ. Tokyo, Sec. IA, 20 (1973), pp.25-105.

[9] Leray, J., Uniformisation de la solution du problème linéair analytique de Cauchy près de la variété qui porte les données de Cauchy. Bull. Soc. Math. Fr., 85 (1957), pp.389-429.

[10] Martineau A., Distributions et valeurs au bord des fonctions holomorphes, Proc. Intern. Summer Course on the theory of Distributions, Lisbon, (1964), pp.195-326.

[11] Sato, M., Regularity of hyperfunction solutions of partial differential equations, Actes Congrès Intern. Math., (1970) Tome 2, pp.785-794.

[12] Zerner, M., Domaine d'holomorphie des fonctions vérifiant une équation aux derivées partielles, C. R. Acad. Sci. Paris 272 (1971), 1646.