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On the fundamental solution of partial differential operators of Schrodinger's type

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§ 1 Preliminaries

We shall construct the fundamental solution of partial differential operators of Schrodinger's type;

$$L = (h/i)\partial/\partial t + \frac{1}{2}\sum 1/\sqrt{g(x)}\,(h/i)\partial/\partial x_j\,(\sqrt{g(x)}g^{jk}(x)(h/i)\partial/\partial x^k + V(x),$$

where h is a positive constant and $(g^{jk}(x))_{jk}$ is a positive definite matrix-valued function of class $C^{\infty}(R^n)$. The notion of Feynman integral has been explained mathematically by several authors (for example [1] and [5][7] and their references.) as a limit of analytic continuation of Wiener integrals. Direct treatment of it was proposed by Ito (6) in introducing an " ideal uniform measure on $R^{[0,t]}$.
In this note we prove that the Riemannian sum approximation of Feynman's path integral that Feynman himself defined in [2] actually converges in the operator norm to fundamental solution ofthe operator L if the function exp (i/h)S, S being the classical action, oscillates rapidly. Note that our method enables one to treat the case that $g^{jk}(x)$ are not constant.

§ 2 assumptions

The Laglangean function is of the form

$$L(q, \dot{q}) = \tfrac{1}{2} \sum g_{jk}(q)\, \dot{q}_j \dot{q}_k - V(q),$$

where $(g_{jk}(\xi))$ is a positive definite matrix valued function, i.e, $ds^2 = \sum g_{jk}(x)\, dx^j dx^k$ is a Riemannian metric in R^n. The Hamiltonian function is $H(p, q) = \dot{q} \cdot p - L$, where $\dot{q} \cdot p = \sum_{j=1}^{n} \dot{q}_j p_j$. We denote by $q(t, y, \xi)$ and $p(t, y, \xi)$ the solution of Hamilton equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} \qquad \frac{dp}{dt} = -\frac{\partial H}{\partial q}$$

satisfying initial conditions $q = y$, $p = \xi$ at $t = 0$.

Our first assumption is

(A-I) There exists a constant $\delta > 0$ such that the canonical transformation $\chi_t ;\ (y, \xi) \longrightarrow (x, \eta) = (q(t, y, \xi),\ p(t, y, \xi))$ induces a diffeomorphism of the configuration space for any $t \in [0, \delta]$.

Let $x^0 = x^0(t, x, \xi)$ be the unique solution of $q(t, x^0, \xi) = x$. Then a generating function of χ_t is given by

$$S_0(t, x, \xi) = \int_0^t L(\dot{q}, q)\, ds + x^0 \cdot \xi ,$$

where the integral should be made along the classical orbit from x^0 to x. Denoting the Euclidean length of a vector x in R^n by $|x|$, we shall further make the following assumptions;

(A-II) $\Phi = \left| \operatorname{grad}_\xi \left(S(x, \xi) - S(z, \xi) \right) \right| \geq \Xi_1(x, z, \xi)\ \theta(|x - z|)$,

$\Psi = \left| \operatorname{grad}_x \left(S(x, \xi) - S(x, \eta) \right) \right| \geq \Xi_2(x, \xi, \eta)\ \theta(|\xi - \eta|)$,

where $\Xi_1(x, z, \xi)$ and $\Xi_2(x, \xi, \eta)$ are smooth functions with a positive lower bound and $\theta(t)$ is a function such that $\theta(t) = O(t)$ near $t = 0$ and $\theta(t) = t^\sigma$ for $t > 1$, with some $\sigma > 0$.

(A-III) For any multi-index α, there exists a constant $C > 0$

such that we have

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}(S(x,\xi) - S(z,\xi))\right| \le C\,\Phi,$$

$$\left|\left(\frac{\partial}{\partial\eta}\right)^{\alpha}(S(x,\xi) - S(x,\eta))\right| \le C\,\Psi.$$

Let $Y(t,x,\xi) = \det \dfrac{\partial q(t,y,\xi)}{\partial y}$. Then $Y \neq 0$ for $0 \le t \le \delta$ by (A-I). OUr last assumption is

(A-IV) For any multi-index α there exists a constant $C > 0$ such that we have estimates;

$$\left|\left(\frac{\partial}{\partial x}\right)^{\alpha}\left(Y(t,x,\xi)^{-1}\ Y(t,x,\eta)^{-1}\right)\right| \le C\,\Xi_2(x,\xi,\eta)$$

$$\left|\left(\frac{\partial}{\partial\xi}\right)^{\alpha}\left(Y(t,x,\xi)^{-1}\ Y(t,y,\xi)^{-1}\right)\right| \le C\,\Xi_1(x,y,\xi).$$

§ 3 Results

We set $E_0(t,x,\xi,y) = \left(\dfrac{g(y)}{g(x)}\right)^{\frac14} Y(t,x,\xi)^{-\frac12} \exp i h^{-1} S(t,x,\xi,y)$, where $S(t,x,\xi,y) = S_0(t,x,\xi) - y\xi$. Then E_0 satisfies $(h/i)\,\partial/\partial t + H(h/i\,\partial/\partial x,\ x))\,E_0(t,x,\xi,y) = h^2 F(t,x,\xi,y)$, where $F(t,x,\xi) = \frac12 \triangle Y(t,x,\xi)^{-\frac12} \exp i h^{-1} S(tx,\xi)$.

We define two operators $E(t)$ and $F(t)$ by

(1) $E(t)\,f(x) = (2\pi h)^{-n} \iint E(t,x,\xi,y)\,f(y)\,dy\,d\xi,$

$F(t)\,f(x) = (2\pi h)^{-n}\,h^2 \iint F(t,x,\xi,y)\,f(y)\,dy\,d\xi.$

These are defined at least for any f in $C_0^{\infty}(R^n)$. It is easy to see $E(0)\,f(x) = f(x)$. Moreover we have

Lemma We have the following estimate

$$\|E(t)\| \leqq C \quad \text{and} \quad \|F(t)\| \leqq C\,h^2, \quad \text{for any } t \text{ in } [0,\ \delta].$$

~~This~~ This is an immediate consequence of our previous work [4].

We set $E(t) = E_0(t) + (i/h) \int_0^t E_0(t-s)\, F(s)\, ds.$

Let T_t be the one parameter unitary groupp generated by $H(-ih\,\partial/\partial x, x)$.
OUr main result is

THEOREM (Feynman [2])

$$\lim_{k \to \infty} \| E(t/k)\, E(t/k) \ldots E(t/k) - T_t \| = 0.$$

Proof

~~First note~~ Note that $-ih\,\partial/\partial t + H(-ih\,\partial/\partial x, x)E(t)$

$$= -ih^{-1} \int_0^t F(t-s)\, F(s)\, ds$$ and that

$$h^{-1} \left\| \int_0^t F(t-s)\, F(s)\, ds \right\| \leqq C|t|\, h^3 \text{ for } t \in [0, \delta].$$

Hence the difference $R(t) = T_t - E(t)$ is estimated as $\|R(t)\| \leqq C\,|t|^2\, h^4$.
Now let t be any positive number. Take k so large as t/k belongs to $[0, \delta]$ ~~Then~~ Then $\|R(t/k)\| \leqq C\, h^4\, (t/k)^2$. This implies that

$$\| E(t/k)\, E(t/k) \ldots E(t/k) - T_t \| \leqq (1 + \|R(t/k)\|)^k - 1,$$

and this converges to 0.

§ 4 Space time Approach

If we integrate first by ξ in (1) and use stationary phase method, we can prove that

$$E(t)\ f(x) = \int a(t,x,y) \exp i h^{-1} \varphi(x, y)\ f(y)\, dy,$$

where $a(t,x,y) = (hi/2\pi)^{-\frac{1}{2}n} (\det \mathrm{Hess}_{\xi} S)^{-\frac{1}{2}} \left(\dfrac{g(y)}{g(x)} \right)^{\frac{1}{4}} Y(tx\, \xi(x,y\ t))^{\frac{1}{2}}$,

where $\varphi(x,y)$ is the classical action $\int_x^y L\, ds$ along classical path from y to x. $\xi(x,y,\ t)$ is the solution of $q(t,x,\xi) = y$ and $\mathrm{Hess}_{\xi} S$ = the Hessian matrix of $S_0(t,x,\xi)$ with respect to ξ variables at $\xi = \xi(x,y,t,\ )$. Starting with this expression of $E(t)$, we can discuss everything and prove our result in the configuration space and time.

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