Operator theoretical approach for transport equations

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§1. Introduction

The problem of neutron transport in an infinite slab leads, after an appropriate simplification, to the evolution equation

\[(1) \quad \frac{\partial}{\partial t} u(t, x, \mu) = -\mu \frac{\partial}{\partial x} u + \kappa \int_{-1}^{1} u(t, x, \mu') d\mu', \quad t > 0,\]

where \( u(t, x, \mu) \) is the density of neutrons at \( x \) (going in the direction \( \mu \) at time \( t \)), and \( \kappa \) is a positive parameter. If the slab is extended between the planes \( x = -a, x = a \) and the outside of the slab is a vacuum, we have the boundary conditions

\[(2) \quad u(t, \mp a, \mu) = 0, \quad \mu \geq 0, \quad t > 0.\]

Of course we have to add the initial condition

\[(3) \quad u(0, x, \mu) = u_0(x, \mu), \quad -a \leq x \leq a, \quad -1 \leq \mu \leq 1.\]

This equation was deeply studied by J. Lehner and G. M. Wing ([2] - [4]). In this lecture, a slight improvement will be done.
First we set the problem in an operator-theoretical framework. Put \( \mathcal{H} = L^2(-a,a) \), \( \mathcal{H} = L^2(-\infty,\infty) \), \( M = (-1,1) \), \( H = L^2(M;\mathcal{H}) \) and \( H_0 = L^2(M;\mathcal{H}_0) \). Define closed linear operators \( L \) in \( \mathcal{H} \) and \( A \) in \( H \) (similarly \( L_0 \) in \( \mathcal{H}_0 \) and \( A_0 \) in \( H_0 \) with \( (-a,a) \) replaced by \( (-\infty,\infty) \)) as follows:

\[
D(L) = \{ v(x) \in \mathcal{H} ; \quad \frac{d}{dx} v(x) \in \mathcal{H}, \quad v(-a) = 0 \},
\]

\[
(Lv)(x) = -\frac{d}{dx} v(x)
\]

\[
D(A) = \{ u(x,\mu) \in H ; \quad u(\cdot,\mu) \in D(L) \text{ for a.e. } \mu > 0 \},
\]

\[
\begin{align*}
\{ u(\cdot,\mu) \in & D(L^*) \text{ for a.e. } \mu < 0, \quad Au \in H \}, \\
(Au)(\cdot,\mu) = \begin{cases} \\
\mu Lu(\cdot,\mu), & \mu > 0, \\
-\mu L^* u(\cdot,\mu), & \mu < 0.
\end{cases}
\end{align*}
\]

Denote by \( J \) (resp. \( \tilde{J} \)) the projection from \( \mathcal{H}_0 \) to \( \mathcal{H} \) (resp. from \( H_0 \) to \( H \)), and by \( K \) the "integral operator":

\[
H \ni u(x,\mu) \mapsto \frac{1}{\sqrt{2}} \int_{-1}^{1} u(x,\mu) d\mu \in \mathcal{H}.
\]

If we put

\[(4) \quad B = A + \kappa K^* K, \quad D(B) = D(A), \]

\[(5) \quad B_0 = A_0 + \kappa \tilde{J}^* K^* K \tilde{J}, \quad D(B_0) = D(A_0), \]

\( \mathbb{Z} \)
then the problem (1)-(3) can be written in an evolution equation in $H$:

$$\frac{d}{dt}u = Bu, \quad u(0) = u_0.$$  

Simultaneously we consider the corresponding evolution equation in $H_0$:

$$\frac{d}{dt}v = B_0v, \quad v(0) = v_0.$$  

It is easy to see that $L$ (and hence $L^*$) generates a contraction semi-group $e^{tL}$ (resp. $e^{tL^*}$) in $\mathcal{H}$, and $L_0$ generates an unitary group $e^{tL_0}$ in $\mathcal{H}_0$. Hence $A$ generates a contraction group $e^{tA}$ in $H$, and $A_0$ generates an unitary group $e^{tA_0}$ in $H_0$. In addition, we obtain that

(6) \quad e^{tL} = Je^{tL_0}J^*, \quad e^{tL^*} = Je^{-tL_0}J^* \quad (t \geq 0), \\

(7) \quad e^{tA} = \tilde{J}e^{tA_0}J^*, \quad e^{tA^*} = \tilde{J}e^{-tA_0}J^* \quad (t \geq 0). \\

Since $C = \hat{K}^*\hat{K}$ (resp. $C_0 = \tilde{J}\hat{K}^*\hat{K}\tilde{J}$) is a bounded linear operator in $H$ (resp. $H_0$), $B$ (resp. $B_0$) generates a semi-group $e^{tB}$ in $H$ (resp. a group $e^{tB_0}$ in $H_0$). Furthermore we have

(8) \quad e^{tB} = \tilde{J}e^{tB_0}J^*, \quad t \geq 0.$$

Following Lehner and Wing, we are concerned with spectral
properties of \( B \) and \( B_0 \), and asymptotic properties of \( e^{tB} \) and \( e^{tB_0} \). However the relation (8) implies that there are no essential differences between \( e^{tB} \) and \( e^{tB_0} \) in the physical meaning. Thus we treat only \( B_0 \) and \( e^{tB_0} \) in this lecture.

Our main result is as follows:

The continuous spectrum of \( B_0 \), which is the whole imaginary axis, is similar to the spectrum of \( A_0 \) except for the discrete values of \( \kappa \).
§2. The spectrum of $B_0$

Put $\widetilde{K} = KJ$. Then the second resolvent equation for $A_0$ and $B_0$:

\[(9) \quad (\lambda - B_0)^{-1} = (\lambda - A_0)^{-1} + \kappa (\lambda - A_0)^{-1}\widetilde{K}\widetilde{K}(\lambda - B_0)^{-1}\]

gives the following

\[(10) \quad (\lambda - B_0)^{-1} = (\lambda - A_0)^{-1} + \kappa (\lambda - A_0)^{-1}\widetilde{K} (1 - \kappa G(\lambda))^{-1}\widetilde{K}(\lambda - A_0)^{-1},\]

where

\[G(\lambda) = \widetilde{K}(\lambda - A_0)^{-1}\widetilde{K} = KJ(\lambda - A_0)^{-1}J^*K^* .\]

Thus the study of $G(\lambda)$ is essential for our purpose. Denoting by $\mathcal{B}(\mathcal{H})$ (resp. $C_\infty(\mathcal{H})$) the set of all bounded (resp. compact) linear operators in $\mathcal{H}$, and by $\|T\|$ the operator norm of $T \in \mathcal{B}(\mathcal{H})$, we summarize some properties of $G(\lambda)$.

Lemma 2.1. (i) $G(\lambda)$ is a $C_\infty(\mathcal{H})$-valued analytic function in $\mathcal{C}_\pm = \{ \lambda : \text{Re} \lambda \neq 0 \}$ and satisfies

\[G(\lambda) = G(\lambda)^*, \quad G(-\lambda) = -G(\lambda)^*.\]

(ii) Let $\lambda \in \mathcal{C}_\pm$. $\lambda$ belongs to the resolvent set $\rho(B_0)$ of $B_0$ (i.e., there exists $(\lambda - B_0)^{-1} \in \mathcal{B}(\mathcal{H}_0)$) if and only if there exists $(1 - \kappa G(\lambda))^{-1} \in \mathcal{B}(\mathcal{H})$.

(iii) For $\lambda \in \mathcal{C}_+$, $G(\lambda)$ satisfies
\[ 0 < \text{Re} G(\lambda) = \frac{1}{2}(G(\lambda) + G(\lambda)^*) \leq \frac{1}{\text{Re} \lambda}, \]

\[ \text{Im} G(\lambda) = \frac{1}{2i} \{G(\lambda) - G(\lambda)^*\} \leq 0 \quad (\text{Im} \lambda \geq 0). \]

(iv) For \( 0 < \beta < \beta' \), \( G(\beta) > G(\beta') > G(+\infty) = 0 \).

(v) \( G(\lambda) \) is continuous in \( \mathbb{T}_+ - \{0\} = \{\lambda \in \mathbb{C} ; \text{Re} \lambda > 0, \lambda \neq 0\} \)
with respect to the norm of \( \mathfrak{B}(\mathcal{H}) \) and satisfies

\[ 0 < \text{Re} G(\beta + i\gamma) \leq \frac{1}{|\gamma|} (1 + \pi), \]

\[ \text{Im} G(\beta + i\gamma) \geq 0 \quad \text{for} \quad \gamma \geq 0 \quad \text{and} \quad \beta \geq 0. \]

(vi) For \( \lambda \in \mathbb{T}_+ - [0, \infty) \), there exists \( (1 - \kappa G(\lambda))^{-1} \in \mathfrak{B}(\mathcal{H}) \).
For any \( \delta > 0 \), there exists a constant \( c_{\kappa, \delta} > 0 \) such that

\[ \| (1 - \kappa G(\lambda))^{-1} \| \leq c_{\kappa, \delta} \quad (\text{Re} \lambda > 0, \, |\text{Im} \lambda| \geq \delta). \]

For \( \lambda \in \mathbb{T}_- - \{0\} \), there holds

\[ \| (1 - \kappa G(\lambda))^{-1} \| \leq 1. \]

For \( \beta > 0 \), there exists \( (1 - \kappa G(\beta))^{-1} \in \mathfrak{B}(\mathcal{H}) \) except for the finite set of \( \beta \) which depends on \( \kappa \).

Carrying out simple calculations we obtain

\[ G(\lambda) = \int_0^\infty \frac{1}{2}(e^{tL} + e^{tL^*})dt \int_0^1 \frac{1}{\mu} e^{\frac{\lambda t}{\mu}} d\mu. \]
Using the equality

\[ \int_0^{\infty} \frac{z}{\mu} e^{-\mu z} d\mu = \int_1^{\infty} \frac{z}{\mu} e^{-\mu z} d\mu = -\log z - b + E_0(z), \]

where \( b \) is Euler number and \( E_0(z) \) is an entire analytic function of \( z \) which satisfies \(|E_0(z)| \leq |z|\) for \( z \in \mathbb{C}_+ \), we have

(11) \( G(\lambda) = \int_0^{\infty} \text{Re } e^{tL}(-\log \lambda t - b - E_0(\lambda t)) dt \).

We put

\[ K(\lambda) = -\int_0^{\infty} \text{Re } e^{tL} dt (\log \lambda + b) + \int_0^{\infty} \text{Re } e^{tL} (-\log t) dt, \]

\[ G_0(\lambda) = \int_0^{\infty} \text{Re } e^{tL} E_0(\lambda t) dt. \]

Since \( \int_0^{\infty} \text{Re } e^{tL} dt = \text{Re } L^{-1} \) reduces to the 1-dimensional operator:

\[ \mathcal{H} \ni u(x) \mapsto \frac{1}{2} \int_a^a u(x) dx = \frac{1}{2a}(u,1) \in \mathcal{H}, \]

we have

(12) \( K(\lambda) = -aN \log \lambda - baN + K_0 \)

where \( N \) is the orthogonal projection \( \frac{1}{2a}(\cdot,1)1 \) in \( \mathcal{H} \) and

\[ K_0 = \int_0^{\infty} \text{Re } e^{tL} (-\log t) dt \in C_0(\mathcal{H}). \]
The inequality \( |E_0(z)| \leq |z| \quad (z \in \mathbb{C}_+) \) implies that
\[
||G_0(\lambda)|| \leq \int_0^\infty |\lambda t|dt = \frac{\alpha^2}{2}||\lambda||.
\]

This implies that the spectrum \( \sigma(G(\beta)) \) of \( G(\beta) \) converges to the spectrum \( \sigma(K(\beta)) \) of \( K(\beta) \) as \( \beta \to 0 \). Thus we have the following

**Lemma 2.2.** Let \( \{\rho_n(\beta)\} \) be the set of (positive) eigen values of \( G(\beta) \) (counted as many times as multiplicities). We can arrange \( \{\rho_n(\beta)\} \) in the following way;

- \( \rho_n(\beta) \) is monotone decreasing in \( \beta \in (0,\infty) \),
- \( \rho_n(\beta) \to 0 \) as \( \beta \to \infty \),
- \( \rho_n(\beta) \to \rho_n^* \) as \( \beta \to 0 \),
- \( \rho_n(\beta) \) is real analytic in \( \beta \in (0,\infty) \).

Here \( \rho_1^* = \infty \) and \( \rho_2^* \geq \rho_3^* \geq \cdots \) are the eigen values of \( N'K_0N' \) arranged in the decreasing order. (In above we have put \( N' = 1 - N \). Note that \( N'K_0N' > 0 \) on the range \( R(N') \) of \( N' \).)

For \( \kappa > 0 \), denote by \( N(\kappa) \) the number of \( \rho_n^* \) such that \( \kappa \rho_n^* > 1 \). Let \( \beta_n = \beta_n(\kappa) \) be the root of \( \kappa \rho_n(\beta) = 1 \) for \( n = 1, \cdots, N(\kappa) \). Then \( (1-\kappa G(\lambda))^{-1} \in \mathcal{B}(\mathcal{H}) \) exists for \( \lambda \in \mathbb{C}_- \cup \mathbb{C}_+ - \{0, \beta_1(\kappa), \cdots, \beta_N(\kappa)(\kappa)\} \). The \( \beta_n(\kappa)'s \) are simple roots of \( (1-\kappa G(\lambda))^{-1} \). Hence \( (\lambda - B_0)^{-1} \in \mathcal{B}(\mathcal{H}) \) exists for \( \lambda \in \mathbb{C}_- \cup \mathbb{C}_+ - \{\beta_1(\kappa), \cdots, \beta_N(\kappa)(\kappa)\} \) and has simple poles at \( \{\beta_1(\kappa), \cdots, \beta_N(\kappa)(\kappa)\} \). A simple argument connected with Lemma 2.1 shows
that the there is not the point spectrum $\sigma_p(B_0)$ of $B_0$ on the imaginary axis $i\mathbb{R}$. Hence $\sigma_p(B_0)$ coincides with the discrete spectrum $\sigma_d(B_0)$ of $B_0$, i.e. $\sigma_p(B_0) = \sigma_d(B_0) = \{\beta_n(\kappa)\}$.

Similarly $\sigma_p(B_0^*) = \sigma_d(B_0^*) = \{\beta_n(\kappa)\}$. Furthermore the inequality (proved by Ukai)

$$\text{Re}(\tilde{K}^* u, (\lambda - A_0)^{-1}\tilde{K}^* u) \geq \text{Re}((\lambda - A_0)(\lambda - A_0)^{-1}\tilde{K}^* u, (\lambda - A_0)^{-1}\tilde{K}^* u)$$

$$= \text{Re}(\lambda \| (\lambda - A_0)^{-1}\tilde{K}^* u \|^2)$$

shows that for $\lambda \in C_+$

$$\| (\lambda - A_0)^{-1}\tilde{K}^* u \|^2 \leq \frac{1}{\text{Re} \lambda} \text{Re}(u, G(\lambda)u)$$

$$\leq \frac{1}{\text{Re} \lambda} \| u \| \| G(\lambda)u \| .$$

Thus the compactness of $G(\lambda)$ implies that of $(\lambda - A_0)^{-1}\tilde{K}^*$.

This implies that the essential spectrum of $B_0$ coincides with that of $A_0$, which is the whole imaginary axis. All these arguments show that the continuous spectrum $\sigma_0(B_0)$ of $B_0$ is the imaginary axis $i\mathbb{R}$, and the residual spectrum $\sigma_r(B_0)$ of $B_0$ is empty. Thus we have the following theorem due to Lehner.

**Theorem 1.** Let $\kappa > 0$ and $B_0$ be defined by (5). Then

$$\rho(B_0) = C_- \cup C_+ = \{\beta_1(\kappa), \cdots, \beta_N(\kappa)(\kappa)\}$$

$$\sigma_p(B_0) = \sigma_d(B_0) = \{\beta_1(\kappa), \cdots, \beta_N(\kappa)(\kappa)\}$$
\sigma_c(B_0) = i \mathbb{R}, \quad \sigma_r(B_0) = \phi

(\lambda - B_0)^{-1} \text{ has simple poles at } \{\beta_1(\kappa), \ldots, \beta_N(\kappa)(\kappa)\}.
§3. The similarity of the continuous spectra of $A_0$ and $B_0$

Denote by $P_j = P_j(\kappa)$ the residue of $(\lambda - B_0)^{-1}$ at $\lambda = \beta_j(\kappa)$, that is the eigen projection of $B_0$ belonging to $\beta_j(\kappa)$, $j = 1, \ldots, N(\kappa)$. Put $Q_1 = \Sigma P_j$, $Q_2 = 1 - Q_1$, $B_1 = B_0 Q_1$ and $B_2 = B_0 Q_2$. Then $(\lambda - B_0)^{-1} Q_2 = (\lambda - B_2)^{-1} Q_2$ is analytic in $\mathbb{C}_\pm$ and there hold

$$(\lambda - B_0)^{-1} = (\lambda - B_0)^{-1} Q_2 + \sum_{j=1}^{N(\kappa)} \frac{1}{\lambda - \beta_j} P_j,$$

e^{t B_0} = e^{t B_0 Q_2} + \Sigma e^{t \beta_j P_j}.

In order to study the spectral property of $B_2$, we use the method of $A_0$-smooth perturbation developed by Kato [1]. In what follows, we put for a fixed $\alpha \in (0, 1)$

$$\alpha_1(s) = \begin{cases} 2^\alpha - \log|s|, & |s| < 1, \\ (1 + |s|)^\alpha, & |s| \geq 1, \end{cases}$$

$$\alpha_2(s) = (1 + |s|)^\alpha,$$

and for later conveniences $N_1 = N$ and $N_2 = N'$. From Lemma 2.1, (11) and (12), we obtain for some constant $a_0$

$$\|\Re N_j G(\pm \sigma + i\gamma) N_j\| \leq \frac{1}{2} a_0 \alpha_j(\gamma)^{-1}, \quad j = 1, 2.$$

Let $\{B_0(s)\}$ be the spectral resolution of $-iA_0$ and put $R(\lambda)$
\[ = (\lambda - A_0)^{-1} = \int (\lambda - is)^{-1} dE_0(s) \]. Following Kato [1], we have

\[ \| N_j \tilde{K}(\lambda - A_0)^{-1} u - N_j \tilde{K}(\lambda^* - A_0)^{-1} u \|^2 \]

\[ \leq 2 \| \text{Re} N_j G(\lambda) N_j \| \left( \| (\lambda - A_0)^{-1} - (\lambda^* - A_0)^{-1} \| u, u \right) \]

\[ \leq a_0 a_j(\gamma)^{-1} \int_{-\infty}^{\infty} \frac{2\sigma}{\sigma^2 + (\gamma - \xi)^2} d\| E_0(s) \|^2, \quad \lambda = \sigma + i\gamma. \]

This implies

\[ \int_{-\infty}^{\infty} a_j(\gamma) \| N_j \tilde{K}R(\sigma + i\gamma) u - N_j \tilde{K}R(\sigma - i\gamma) u \|^2 d\gamma \]

\[ \leq 2\pi a_0 \| u \|^2, \quad j = 1, 2. \]

Using estimates for Hilbert transforms with weighted norms, we have

\[ \int_{-\infty}^{\infty} a_j(\gamma) \| N_j \tilde{K}R(\sigma + i\gamma) u \|^2 d\gamma \]

\[ \leq C_0 \int_{-\infty}^{\infty} a_j(\gamma) \| N_j \tilde{K}R(\sigma + i\gamma) u - N_j \tilde{K}R(\sigma - i\gamma) u \|^2 d\gamma \]

\[ \leq 2\pi a_0 C_0 \| u \|^2, \]

Hence \( N_j \tilde{K}R(\sigma + i\gamma) u \) is an element of a \( \mathcal{H} \)-valued Hardy class with a weighted norm, and is a continuous function of \( \sigma \geq 0 \) and \( \sigma \leq 0 \) with values in \( L^2(\mathbb{R}, a_j(\gamma)^2 d\gamma; \mathcal{H}) \).

Putting \( R_1(\lambda) = (\lambda - B_0)^{-1} \) and recalling that

\[ \tilde{K}(\lambda - B_0)^{-1} = (1 - \kappa G(\lambda))^{-1} \tilde{K}(\lambda - A_0)^{-1}, \]
we define so called wave operators $W_\pm$ and $Z_\pm$ as follows:

$$(W_{\pm}u,v) = (u,v) \pm \frac{\kappa}{2\pi i} \int_{-\infty}^{\infty} (\tilde{KR}(\pm0+i\gamma)u, \tilde{KR}_1(\mp0+i\gamma)^*v) d\gamma,$$

$$(Z_{\pm}u,v) = (Q_2u,v) \pm \frac{\kappa}{2\pi i} \int_{-\infty}^{\infty} (\tilde{KR}_1(\pm0+i\gamma)Q_2u, \tilde{KR}(\mp0+i\gamma)^*v) d\gamma.$$

To see the convergence of these integrals, we have to investigate the behavior of $(1-\kappa G(\lambda))^{-1}$ near $\lambda = \pm0 \in \mathbb{C}_\pm$.

We put $N_iG_{ij}(\lambda)N_j = G_{ij}(\lambda)$, $i = 1,2$. Then $G_{ij}(\lambda)$'s have the following forms:

$$G_{11}(\lambda) = \{-\alpha \log \lambda - ab - g_1(\lambda)\}N_1,$$

$$G_{12}(\lambda) = G_{21}(\lambda)^* = N_1K_0N_2 + N_1G_0(\lambda)N_2,$$

$$G_{22}(\lambda) = N_2K_0N_2 + N_2G_0(\lambda)N_2,$$

$$|g_1(\lambda)| \leq \frac{1}{2}a^2|\lambda|, \quad \|N_1G_0(\lambda)N_j\| \leq \frac{1}{2}a^2|\lambda|.$$

Let us assume that $\kappa > 0$ and $\kappa^{-1} \notin \sigma(N_2K_0N_2)$. Then for sufficiently small $\lambda \in \mathbb{C}_+$, there exists $(1 - \kappa G_{22}(\lambda))^{-1} \in \mathcal{B}(\mathcal{H})$ with uniformly bounded norm. Hence we have

$$\|(1 - \kappa G(\lambda))^{-1}u\| \leq \frac{c_1}{2 - 10\log |\lambda|} \|N_1u\| + c_2 \|N_2u\|$$

for sufficiently small $\lambda \in \mathbb{C}_+$ (and hence for small $\lambda \in \mathbb{C}_-$). This implies

$$\|	ilde{KR}_1(\lambda)u\| \leq \frac{c_1}{2 - 10\log |\lambda|} \|N_1\tilde{KR}(\lambda)u\| + c_2 \|N_2\tilde{KR}(\lambda)u\|$$
for sufficiently small $\lambda \in \mathcal{C}_\pm$. Thus the above integrals converge absolutely, and $W_\pm, Z_\pm \in \mathcal{B}(H_0)$. Following Kato's argument, we can easily see that

\begin{equation}
(13) \quad Z_\pm W_\pm = 1, \quad W_\pm Z_\pm = Q_2
\end{equation}

\begin{equation}
(\lambda - E_2)W_\pm = W_\pm(\lambda - A_0)^{-1} \quad \text{i.e.} \quad B_2 = W_\pm A_0 Z_\pm .
\end{equation}

\begin{equation}
(14) \quad e^{tB_2} = W_\pm e^{tA_0} Z_\pm .
\end{equation}

Thus we have

Theorem 2. Let $\kappa > 0$ and $\kappa^{-1} \notin \sigma(N_2 K_0 N_2)$. Then $A_0$ and $B_2 = B_0 Q_2$ are similar to each other. That is, $W_\pm$ and $Z_\pm \in \mathcal{B}(H_0)$ exist and satisfy (13) and (14). Furthermore we have

\begin{equation}
W_\pm = \lim_{t \to \pm \infty} Q_2 e^{tB_0} e^{-tA_0} ,
\end{equation}

\begin{equation}
Z_\pm = \lim_{t \to \pm \infty} e^{tA_0} e^{-tB_0} Q_2 .
\end{equation}

If we put $F(\Delta) = W_\pm(\Delta)E_0(\Delta)Z_\pm(\Delta), \Delta = (a,b)$, then $F(\Delta)$ is the "spectral resolution" of $B_2$, i.e.,

\begin{equation}
B_0 = i \int_{-\infty}^{\infty} \lambda dF(\lambda) + \sum_j \beta_j P_j .
\end{equation}
References


