Operator theoretical approach for transport equations

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§1. Introduction

The problem of neutron transport in an infinite slab leads, after an appropriate simplification, to the evolution equation

\[ \frac{\partial}{\partial t} u(t, x, \mu) = -\mu \frac{\partial}{\partial x} u + \frac{\kappa}{2} \int_{-1}^{1} u(t, x, \mu') d\mu', \quad t > 0, \]

where \( u(t, x, \mu) \) is the density of neutrons at \( x \) (going in the direction \( \mu \) at time \( t \)), and \( \kappa \) is a positive parameter. If the slab is extended between the planes \( x = -a, x = a \) and the outside of the slab is a vacuum, we have the boundary conditions

\[ u(t, \mp a, \mu) = 0, \quad \mu > 0, \quad t > 0. \]

Of course we have to add the initial condition

\[ u(0, x, \mu) = u_0(x, \mu), \quad -a \leq x \leq a, \quad -1 \leq \mu \leq 1. \]

This equation was deeply studied by J. Lehner and G. M. Wing ([2] - [4]). In this lecture, a slight improvement will be done.
First we set the problem in an operator-theoretical framework. Put $\mathcal{H} = L^2(-a,a)$, $\mathcal{K} = L^2(-\infty,\infty)$, $M = (-1,1)$, $H = L^2(M;\mathcal{K})$ and $H_0 = L^2(M;\mathcal{K}_0)$. Define closed linear operators $L$ in $\mathcal{K}$ and $A$ in $H$ (similarly $L_0$ in $\mathcal{K}_0$ and $A_0$ in $H_0$ with $(-a,a)$ replaced by $(-\infty,\infty)$) as follows:

$$D(L) = \{ v(x) \in \mathcal{K} ; \frac{d}{dx}v(x) \in \mathcal{K}, v(-a) = 0 \},$$

$$(Lv)(x) = -\frac{d}{dx}v(x)$$

$$D(A) = \{ u(x,\mu) \in H ; u(\cdot,\mu) \in D(L) \text{ for } a.e. \mu > 0, u(\cdot,\mu) \in D(L^*) \text{ for } a.e. \mu < 0, Au \in H \},$$

$$(Au)(\cdot,\mu) = \begin{cases} \mu Lu(\cdot,\mu), & \mu > 0, \\ -\mu L^* u(\cdot,\mu), & \mu < 0. \end{cases}$$

Denote by $J$ (resp. $\tilde{J}$) the projection from $\mathcal{K}_0$ to $\mathcal{K}$ (resp. from $H_0$ to $H$), and by $K$ the "integral operator":

$$H \ni u(x,\mu) \mapsto \sqrt{\frac{1}{2}} \int_{-1}^{1} u(x,\mu) d\mu \in \mathcal{K}.$$}

If we put

(4) $B = A + \kappa K^* K$, $D(B) = D(A)$,

(5) $B_0 = A_0 + \kappa \tilde{J}^* K^* K \tilde{J}$, $D(B_0) = D(A_0)$,
then the problem \((1)-(3)\) can be written in an evolution equation in \(H\):

\[
\frac{d}{dt} u = Bu, \quad u(0) = u_0.
\]

Simultaneously we consider the corresponding evolution equation in \(H_0\):

\[
\frac{d}{dt} v = B_0 v, \quad v(0) = v_0.
\]

It is easy to see that \(L\) (and hence \(L^*\)) generates a contraction semi-group \(e^{tL}\) (resp. \(e^{tL^*}\)) in \(\mathcal{H}\), and \(L_0\) generates an unitary group \(e^{tL_0}\) in \(\mathcal{H}_0\). Hence \(A\) generates a contraction group \(e^{tA}\) in \(H\), and \(A_0\) generates an unitary group \(e^{tA_0}\) in \(H_0\). In addition, we obtain that

\[
(6) \quad e^{tL} = J e^{tL_0} J^*, \quad e^{tL^*} = J e^{-tL_0} J^* \quad (t \geq 0),
\]

\[
(7) \quad e^{tA} = \tilde{J} e^{tA_0} J^*, \quad e^{tA^*} = \tilde{J} e^{-tA_0} J^* \quad (t \geq 0).
\]

Since \(C = K^* K\) (resp. \(C_0 = \tilde{J}^* \tilde{J}^* K \tilde{J}\)) is a bounded linear operator in \(H\) (resp. \(H_0\)), \(B\) (resp. \(B_0\)) generates a semi-group \(e^{tB}\) in \(H\) (resp. a group \(e^{tB_0}\) in \(H_0\)). Furthermore we have

\[
(8) \quad e^{tB} = \tilde{J} e^{tB_0} J^*, \quad t \geq 0.
\]

Following Lehner and Wing, we are concerned with spectral
properties of $B$ and $B_0$, and asymptotic properties of $e^{tB}$ and $e^{tB_0}$. However the relation (8) implies that there are no essential differences between $e^{tB}$ and $e^{tB_0}$ in the physical meaning. Thus we treat only $B_0$ and $e^{tB_0}$ in this lecture.

Our main result is as follows:

The continuous spectrum of $B_0$, which is the whole imaginary axis, is similar to the spectrum of $A_0$ except for the discrete values of $\kappa$. 
§2. The spectrum of $B_0$

Put $\tilde{K} = K\tilde{J}$. Then the second resolvent equation for $A_0$ and $B_0$:

$$(9) \quad (\lambda - B_0)^{-1} = (\lambda - A_0)^{-1} + \kappa(\lambda - A_0)^{-1}\tilde{K}^* \tilde{K}(\lambda - B_0)^{-1}$$

gives the following

$$(10) \quad (\lambda - B_0)^{-1} = (\lambda - A_0)^{-1} + \kappa(\lambda - A_0)^{-1}\tilde{K}^* (1 - \kappa G(\lambda))^{-1}\tilde{K}(\lambda - A_0)^{-1},$$

where

$$G(\lambda) = \tilde{K}(\lambda - A_0)^{-1}\tilde{K}^* = \tilde{K}(\lambda - A_0)^{-1}\tilde{J}^* K^*.$$ 

Thus the study of $G(\lambda)$ is essential for our purpose. Denoting by $\mathcal{B}(\mathcal{H})$ (resp. $C_\infty(\mathcal{H})$) the set of all bounded (resp. compact) linear operators in $\mathcal{H}$, and by $\|T\|$ the operator norm of $T \in \mathcal{B}(\mathcal{H})$, we summarize some properties of $G(\lambda)$.

**Lemma 2.1.**

(i) $G(\lambda)$ is a $C_\infty(\mathcal{H})$-valued analytic function in $\mathbb{C}_\pm = \{\lambda \; ; \; \text{Re} \lambda \neq 0\}$ and satisfies

$$G(\lambda^*) = G(\lambda)^*, \quad G(-\lambda) = -G(\lambda)^*.$$ 

(ii) Let $\lambda \in \mathbb{C}_\pm$. $\lambda$ belongs to the resolvent set $\rho(B_0)$ of $B_0$ (i.e., there exists $(\lambda - B_0)^{-1} \in \mathcal{B}(\mathcal{H}_0)$) if and only if there exists $(1 - \kappa G(\lambda))^{-1} \in \mathcal{B}(\mathcal{H})$.

(iii) For $\lambda \in \mathbb{C}_+$, $G(\lambda)$ satisfies
\[ 0 < \text{Re}G(\lambda) = \frac{1}{2}(G(\lambda) + G(\lambda)^*) \leq \frac{1}{\text{Re}\lambda}, \]

\[ \text{Im}G(\lambda) = \frac{1}{2i}\{G(\lambda) - G(\lambda)^*\} \leq 0 \quad (\text{Im}\lambda \geq 0). \]

(iv) For \( 0 < \beta < \beta' \), \( \text{G(\beta)} > \text{G(\beta')} > \text{G(+\infty)} = 0 \).

(v) \( \text{G(\lambda)} \) is continuous in \( \mathbb{T}_+ - \{0\} = \{\lambda ; \text{Re}\lambda > 0, \lambda \neq 0\} \) with respect to the norm of \( \mathcal{B}(\mathcal{H}) \) and satisfies

\[ 0 < \text{Re}G(\beta + i\gamma) \leq \frac{1}{|\gamma|}(1 + \pi), \]

\[ \text{Im}G(\beta + i\gamma) \geq 0 \text{ for } \gamma \geq 0 \text{ and } \beta \geq 0. \]

(vi) For \( \lambda \in \mathbb{T}_+-[0,\infty) \), there exists \( (1 - \kappa G(\lambda))^{-1} \in \mathcal{B}(\mathcal{H}) \). For any \( \delta > 0 \), there exists a constant \( c_{\kappa,\delta} > 0 \) such that

\[ \| (1 - \kappa G(\lambda))^{-1} \| \leq c_{\kappa,\delta} \quad (\text{Re}\lambda > 0, |\text{Im}\lambda| \geq \delta). \]

For \( \lambda \in \mathbb{T}_+-\{0\} \), there holds

\[ \| (1 - \kappa G(\lambda))^{-1} \| \leq 1. \]

For \( \beta > 0 \), there exists \( (1 - \kappa G(\beta))^{-1} \in \mathcal{B}(\mathcal{H}) \) except for the finite set of \( \beta \) which depends on \( \kappa \).

Carrying out simple calculations we obtain

\[ G(\lambda) = \int_0^\infty \frac{1}{2}(e^{tL} + e^{tL^*})dt \int_0^1 \frac{1}{\mu} e^{\frac{\lambda t}{\mu}} d\mu. \]
Using the equality

\[ \int_0^1 \frac{1}{\mu} e^{\frac{-z}{\mu}} d\mu = \int_1^\infty \frac{1}{\mu} e^{-\mu z} d\mu. \]

\[ = -\log z - b + E_0(z), \]

where \( b \) is Euler number and \( E_0(z) \) is an entire analytic function of \( z \) which satisfies \( |E_0(z)| \leq |z| \) for \( z \in \mathbb{C}_+ \), we have

(11) \( G(\lambda) = \int_0^\infty \text{Re} e^{tL} \{ -\log \lambda t - b - E_0(\lambda t) \} dt. \)

We put

\[ K(\lambda) = -\int_0^\infty \text{Re} e^{tL} dt (\log \lambda + b) + \int_0^\infty \text{Re} e^{tL} (-\log t) dt, \]

\[ G_0(\lambda) = \int_0^\infty \text{Re} e^{tL} E_0(\lambda t) dt. \]

Since \( \int_0^\infty \text{Re} e^{tL} dt = \text{Re} L^{-1} \) reduces to the 1-dimensional operator:

\[ \mathcal{H} \ni u(x) \mapsto \frac{1}{2} \int_{-a}^a u(x) dx = a \frac{1}{2a} (u,1) \in \mathcal{H}, \]

we have

(12) \( K(\lambda) = -aN \log \lambda - baN + K_0 \)

where \( N \) is the orthogonal projection \( \frac{1}{2a} (\cdot,1) 1 \) in \( \mathcal{H} \) and

\[ K_0 = \int_0^\infty \text{Re} e^{tL} (-\log t) dt \in \mathcal{C}_0(\mathcal{H}). \]
The inequality \( |E_0(z)| \leq |z| \) (\( z \in \mathbb{C}_+ \)) implies that
\[
\|G_0(\lambda)\| \leq \int_0^\infty |\lambda t| \, dt = \frac{\lambda^2}{2} |\lambda|.
\]

This implies that the spectrum \( \sigma(G(\beta)) \) of \( G(\beta) \) converges to the spectrum \( \sigma(K(\beta)) \) of \( K(\beta) \) as \( \beta \to 0 \). Thus we have the following

Lemma 2.2. Let \( \{\rho_n(\beta)\} \) be the set of (positive) eigen values of \( G(\beta) \) (counted as many times as multiplicities). We can arrange \( \{\rho_n(\beta)\} \) in the following way;

\[\begin{align*}
\rho_n(\beta) & \text{ is monotone decreasing in } \beta \in (0, \infty), \\
\rho_n(\beta) & \to 0 \text{ as } \beta \to \infty, \\
\rho_n(\beta) & \to \rho^*_n \text{ as } \beta \to 0, \\
\rho_n(\beta) & \text{ is real analytic in } \beta \in (0, \infty).
\end{align*}\]

Here \( \rho^*_1 = \infty \) and \( \rho^*_2 \geq \rho^*_3 \geq \cdots \) are the eigen values of \( N'K_0N' \) arranged in the decreasing order. (In above we have put \( N' = 1 - N \). Note that \( N'K_0N' > 0 \) on the range \( R(N') \) of \( N' \).)

For \( \kappa > 0 \), denote by \( N(\kappa) \) the number of \( \rho^*_n \) such that \( \kappa \rho^*_n > 1 \). Let \( \beta_n = \beta_n(\kappa) \) be the root of \( \kappa \rho_n(\beta) = 1 \) for \( n = 1, \cdots, N(\kappa) \). Then \( (1-\kappa G(\lambda))^{-1} \in \mathbb{B}(\mathcal{H}) \) exists for \( \lambda \in \mathbb{C}_- \cup \mathbb{C}_+ \setminus \{0, \beta_1(\kappa), \cdots, \beta_{N(\kappa)}(\kappa)\} \). The \( \beta_n(\kappa)'s \) are simple roots of \( (1-\kappa G(\lambda))^{-1} \). Hence \( (\lambda - B_0)^{-1} \in \mathbb{B}(\mathcal{H}) \) exists for \( \lambda \in \mathbb{C}_- \cup \mathbb{C}_+ \setminus \{\beta_1(\kappa), \cdots, \beta_{N(\kappa)}(\kappa)\} \) and has simple poles at \{\( \beta_1(\kappa), \cdots, \beta_{N(\kappa)}(\kappa)\)\}. A simple argument connected with Lemma 2.1 shows
that there is not the point spectrum $\sigma_p(B_0)$ of $B_0$ on the imaginary axis $i\mathbb{R}$. Hence $\sigma_p(B_0)$ coincides with the discrete spectrum $\sigma_d(B_0)$ of $B_0$, i.e. $\sigma_p(B_0) = \sigma_d(B_0) = \{\beta_n(\kappa)\}$.

Similarly $\sigma_p(B_0^*) = \sigma_d(B_0^*) = \{\beta_n(\kappa)\}$. Furthermore the inequality (proved by Ukai)

$$\text{Re} (\tilde{K}^* u, (\lambda - A_0)^{-1} \tilde{K}^* u) \geq \text{Re} ((\lambda - A_0)(\lambda - A_0)^{-1} \tilde{K}^* u, (\lambda - A_0)^{-1} \tilde{K}^* u)$$

$$= \text{Re} \lambda \| (\lambda - A_0)^{-1} \tilde{K}^* u \|^2$$

shows that for $\lambda \in \mathbb{C}_+$

$$\| (\lambda - A_0)^{-1} \tilde{K}^* u \|^2 \leq \frac{1}{\text{Re} \lambda} \text{Re} (u, G(\lambda) u)$$

$$\leq \frac{1}{\text{Re} \lambda} \| u \| \| G(\lambda) u \|.$$

Thus the compactness of $G(\lambda)$ implies that of $(\lambda - A_0)^{-1} \tilde{K}^* u$.

This implies that the essential spectrum of $B_0$ coincides with that of $A_0$, which is the whole imaginary axis. All these arguments show that the continuous spectrum $\sigma_c(B_0)$ of $B_0$ is the imaginary axis $i\mathbb{R}$, and the residual spectrum $\sigma_r(B_0)$ of $B_0$ is empty. Thus we have the following theorem due to Lehner.

Theorem 1. Let $\kappa > 0$ and $B_0$ be defined by (5). Then

$$\rho(B_0) = \mathbb{C}_- \cup \mathbb{C}_+ - \{\beta_1(\kappa), \ldots, \beta_N(\kappa)\}(\kappa)$$

$$\sigma_p(B_0) = \sigma_d(B_0) = \{\beta_1(\kappa), \ldots, \beta_N(\kappa)\}(\kappa)$$
\( \sigma_c(B_0) = i\mathbb{R} \), \( \sigma_r(B_0) = \phi \)

\((\lambda - B_0)^{-1}\) has simple poles at \( \{ \beta_1(\kappa), \cdots, \beta_N(\kappa)(\kappa) \} \).
§3. The similarity of the continuous spectra of $A_0$ and $B_0$

Denote by $P_j = P_j(\kappa)$ the residue of $(\lambda - B_0)^{-1}$ at $\lambda = \beta_j(\kappa)$, that is the eigen projection of $B_0$ belonging to $\beta_j(\kappa)$, $j = 1, \ldots, N(\kappa)$. Put $Q_1 = \Sigma P_j$, $Q_2 = 1 - Q_1$, $B_1 = B_0 Q_1$ and $B_2 = B_0 Q_2$. Then $(\lambda - B_0)^{-1} Q_2 = (\lambda - B_2)^{-1} Q_2$ is analytic in $\mathbb{C}_\pm$ and there hold

$$(\lambda - B_0)^{-1} = (\lambda - B_0)^{-1} Q_2 + \sum_{j=1}^{N(\kappa)} \frac{1}{\lambda - \beta_j} P_j,$$

$$e^{tB_0} = e^{tB_0} Q_2 + \sum e^{t\beta_j} P_j.$$

In order to study the spectral property of $B_2$, we use the method of $A_0$-smooth perturbation developed by Kato [1]. In what follows, we put for a fixed $\alpha \in (0, 1)$

$$\alpha_1(s) = \begin{cases} 2^\alpha \log|s|, & |s| < 1, \\ (1+|s|)^\alpha, & |s| \geq 1, \end{cases}$$

$$\alpha_2(s) = (1+|s|)^\alpha,$$

and for later conveniency $N_1 = N$ and $N_2 = N'$. From Lemma 2.1, (11) and (12), we obtain for some constant $a_0$

$$\| \text{Re} N_j G(\pm \sigma + i\gamma) N_j \| \leq \frac{1}{2} a_0 \alpha_j(\gamma)^{-1}, \quad j = 1, 2.$$

Let $\{E_0(s)\}$ be the spectral resolution of $-iA_0$ and put $R(\lambda)$
\( (\lambda - A_0)^{-1} = \int (\lambda - is)^{-1}dE_0(s) \). Following Kato [1] , we have

\[
\| N_j \tilde{K}(\lambda-A_0)^{-1}u - N_j \tilde{K}(-\lambda-A_0)^{-1}u \|^2 \\
\leq 2\| \text{Re} N_j \tilde{G}(\lambda)N_j \| \left\{ (\lambda-A_0)^{-1} - (-\lambda-A_0)^{-1} \right\} u, u \\
\leq a_0\alpha \int_{-\infty}^{\infty} \frac{2\sigma}{\sigma^2 + (\gamma - \delta)^2} d\| E_0(s) \|^2, \ \lambda = \sigma + i\gamma.
\]

This implies

\[
\int_{-\infty}^{\infty} a_j(\gamma) \| N_j \tilde{K}R(\sigma+i\gamma)u - N_j \tilde{K}R(-\sigma+i\gamma)u \|^2 d\gamma \\
\leq 2\pi a_j \| u \|^2, \ j = 1, 2.
\]

Using estimates for Hilbert transforms with weighted norms, we have

\[
\int_{-\infty}^{\infty} a_j(\gamma) \| N_j \tilde{K}R(\sigma+i\gamma)u \|^2 d\gamma \\
\leq C_0 \int_{-\infty}^{\infty} a_j(\gamma) \| N_j \tilde{K}R(\sigma+i\gamma)u - N_j \tilde{K}R(-\sigma+i\gamma)u \|^2 d\gamma \\
\leq 2\pi a_j C_0 \| u \|^2,
\]

Hence \( N_j \tilde{K}R(\sigma+i\gamma)u \) is an element of a \( \mathcal{H} \)-valued Hardy class with a weighted norm, and is a continuous function of \( \sigma \geq 0 \) and \( \sigma \leq 0 \) with values in \( L^2(\mathbb{R}, a_j(\gamma)^2 d\gamma ; \mathcal{H}) \).

Putting \( R_1(\lambda) = (\lambda - B_0)^{-1} \) and recalling that

\[
\tilde{K}(\lambda-B_0)^{-1} = (1-\kappa G(\lambda))^{-1} \tilde{K}(\lambda-A_0)^{-1},
\]
we define so called wave operators $W_\pm$ and $Z_\pm$ as follows:

$$
(W_{\pm}u, v) = (u, v) \pm \frac{k}{2\pi i} \int_{-\infty}^{\infty} (\mathcal{K}R(\pm 0 + i\gamma)u, \mathcal{K}R(\mp 0 + i\gamma)^* v) d\gamma,
$$

$$
(Z_{\pm}u, v) = (Q_2 u, v) \mp \frac{k}{2\pi i} \int_{-\infty}^{\infty} (\mathcal{K}R(\pm 0 + i\gamma)Q_2 u, \mathcal{K}R(\mp 0 + i\gamma)^* v) d\gamma.
$$

To see the convergence of these integrals, we have to investigate the behavior of $(1 - \kappa G(\lambda))^{-1}$ near $\lambda = \pm 0 \in \mathbb{C}_\pm$.

We put $N_i G_{ij}(\lambda) N_j = G_{ij}(\lambda)$, $i = 1, 2$. Then $G_{ij}(\lambda)$'s have the following forms:

$$
G_{11}(\lambda) = (-\log \lambda - ab - g_1(\lambda)) N_1,
$$

$$
G_{12}(\lambda) = G_{21}(\lambda)^* = N_1 K_0 N_2 + N_1 G_0(\lambda) N_2,
$$

$$
G_{22}(\lambda) = N_2 K_0 N_2 + N_2 G_0(\lambda) N_2,
$$

$$
|g_1(\lambda)| \leq \frac{1}{2} a^2 |\lambda|, \quad \|N_i G_0(\lambda) N_j\| \leq \frac{1}{2} a^2 |\lambda|.
$$

Let us assume that $\kappa > 0$ and $\kappa^{-1} \not\in \sigma(N_2 K_0 N_2)$. Then for sufficiently small $\lambda \in \mathbb{C}_+$, there exists $(1 - \kappa G_{22}(\lambda))^{-1} \in \mathcal{B}(\mathcal{H})$ with uniformly bounded norm. Hence we have

$$
\| (1 - \kappa G(\lambda))^{-1} u \| \leq \frac{c_1}{2 - \log |\lambda|} \|N_1 u\| + c_2 \|N_2 u\|
$$

for sufficiently small $\lambda \in \mathbb{C}_+$ (and hence for small $\lambda \in \mathbb{C}_-$).

This implies

$$
\| \mathcal{K}R_1(\lambda) u \| \leq \frac{c_1}{2 - \log |\lambda|} \|N_1 \mathcal{K}R(\lambda) u\| + c_2 \|N_2 \mathcal{K}R(\lambda) u\|
$$
for sufficiently small $\lambda \in \mathbb{C}_\pm$. Thus the above integrals converge absolutely, and $W_\pm, Z_\pm \in \mathcal{B}(\mathcal{H}_0)$. Following Kato's argument, we can easily see that

\begin{equation}
(13) \quad Z_\pm W_\pm = 1, \quad W_\pm Z_\pm = Q_2 = (\lambda - E_2) W_\pm = W_\pm (\lambda - A_0)^{-1}
\end{equation}

i.e. $B_2 = W_\pm A_0 Z_\pm$.

\begin{equation}
(14) \quad e^{tB_2} = W_\pm e^{tA_0} Z_\pm.
\end{equation}

Thus we have

Theorem 2. Let $\kappa > 0$ and $\kappa^{-1} \notin \sigma(N_2 K_0 N_2)$. Then $A_0$ and $B_2 = B_0 Q_2$ are similar to each other. That is, $W_\pm$ and $Z_\pm \in \mathcal{B}(\mathcal{H}_0)$ exist and satisfy (13) and (14). Furthermore we have

\[
W_\pm = s - \lim_{t \to \mp \infty} Q_2 e^{tB_0} e^{-tA_0},
\]

\[
Z_\pm = s - \lim_{t \to \mp \infty} e^{tA_0} e^{-tB_0} Q_2.
\]

If we put $F(\Delta) = W_\pm (\Delta) E_0 (\Delta) Z_\pm (\Delta)$, $\Delta = (a, b)$, then $F(\Delta)$ is the "spectral resolution" of $B_2$, i.e.,

\[
B_0 = i \int_{-\infty}^\infty \lambda dF(\lambda) + \sum_j E_0 P_j.
\]
References


