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Spectra of Elliptic Operators in a Domain with Unbounded Boundary

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0. Introduction

We shall study a spectral property of a 2nd order partial differential operator

$$L = -\Delta + \sum_{j,k=1}^{n} \alpha_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \beta_j \frac{\partial}{\partial x_j} + \gamma$$

in a domain $$\Omega \subset \mathbb{R}^n (n \geq 2)$$. In case that $$\Omega = \mathbb{R}^n$$ or $$\Omega$$ is the exterior of a bounded hypersurface, this subject have been discussed by many authors under various conditions on $$L$$. As for the case of unbounded boundary, we can refer to [1] [3, 4] [6-11].

In §6 1 and 2 of this note, we shall make the following conditions 1 and 2. (A list of notations will be given later.)

Condition 1. $$\Omega$$ is characterized by $$x_n > \varphi(\vec{x})$$ ($$\vec{x} = (x_1, \ldots, x_{n-1})$$), where $$\varphi(\vec{x})$$ is a single valued $$C^2$$-function on $$\mathbb{R}^{n-1}$$, and there are positive numbers $$M_1$$ and $$M_2$$ such that

$$\varphi(\vec{x}) > M_1 (x_1^2 + \cdots + x_{n-1}^2)^{1/2}, \quad |\partial \varphi / \partial x_j| < M_2, \quad |\partial^2 \varphi / \partial x_j \partial x_k| < M_2$$

($$j, k = 1, \ldots, n-1$$).

Condition 2. Put

$$B = \sum_{j,k=1}^{n} \alpha_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \beta_j \frac{\partial}{\partial x_j} + \gamma$$

* 1-5-1, Chofugaoka, Chofu-shi, Tokyo 182.
then $B$ is real and short range in the following sense.

(1) If we put $D(H) = \{ u \in L^2(\Omega) \mid u \in H^1(\Omega), \ u|_{\Gamma} = 0, \ \Delta u \in L^2(\Omega) \}$, and $Hu = Lu$ for $u \in D(H)$, then $H$ is a self-adjoint operator in $L^2(\Omega)$.

(ii) The inequality

$$\|u\|_2(\Omega_{R,R+1}) \leq C_1 \|(-\Delta + B)u\|_0(\Omega_{R-1,R+2}) + \|u\|_1(\Omega_{R-1,R+2})$$

holds for $u \in H^2_{2,loc}(\Omega)$ with $u_{|\Gamma} = 0$, where $C_1$ is independent of $R$.

(iii) There are positive numbers $h$ and $C_2$ such that

$$\|Bu\|_0(\Omega_{R,R+1}) \leq C_2 R^{-1-h} \|u\|_2(\Omega_{R-1,R+2})$$

for $\forall R > 0$ and $\forall u \in H^2_{2,loc}(\Omega)$.

Under these conditions the discreteness of non-zero eigenvalues of $H$ and the absolute continuity of the continuous spectrum of $H$ will be shown. It should be noted that, under Condition 2, the coefficient $\gamma$ may have a high singularity of some type.

In §3, the conditions will be slightly weakened.

In §4, a short remark on the absence of positive eigenvalues will be given.

There are some possibilities that we can extend the arguments in §§1-2 to include the perturbations which are not necessarily short range. For example, we can put

$$B = B_1 + B_2,$$

$B_1$ is real and short range in the above sense,

$B_2$ is multiplication by a real $C^1$-function $q_2$, and

$$|q_2| + |\nabla q_2| \leq C|x|^{-h}, \quad \frac{\partial q_2}{\partial x_k} < C|x|^{-1-h}$$

for some $C$, $h > 0$.

But, in order to avoid some tedious calculations, we shall omit...
discussions about these possibilities. A full treatment will be given in \[12\].

**List of notations**

\( \Omega \): an unbounded domain in \( n \)-dimensional Euclidean space

\( \mathbb{R}^n = \{ x | x=(x_1, \ldots, x_n) \} \) (\( n \geq 2 \)).

\( \Gamma \): the boundary of \( \Omega \); \( \Gamma = \partial \Omega \)

\( \Omega^c = \Omega \cup \Gamma \), \( \Omega_{R_1,R_2} = \Omega \cap \{ x | R_1 < x_n < R_2 \} \), \( \Gamma_{R_1,R_2} = \Gamma \cap \{ x | R_1 < x_n < R_2 \} \)

\( S_{R} = \Omega_\gamma | x_n = R \} \)

\( \Delta \): the Laplace operator; \( \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x^2_j} \)

\( \phi \): a smooth function which depends only on \( x_n \)

\( \phi' = \frac{d\phi}{dx_n} \), \( \phi'' = \frac{d^2\phi}{dx^2_n} \), \( \cdots \)

\( \gamma_n \): the angle between \( x_n \)-axis and outer normal.

\( |x| = (x_1^2 + x_n^2)^{1/2} \)

\( |\nabla u| = (\sum_{j=1}^{n} (\frac{\partial}{\partial x_j} u)^2)^{1/2} \)

\( H_{m,s} \): the space of functions with the norm

\[ \| u \|_{m,s} = \left( \int_\Omega (1+|x|^2)^5 \sum_{|\alpha| \leq m} |D^{\alpha} u|^2 dx \right)^{1/2} \]

\( D^\alpha = \frac{\partial^{a_1}}{\partial x_1^{a_1}} \cdots \frac{\partial^{a_n}}{\partial x_n^{a_n}}, |\alpha| = a_1 + \cdots + a_n \)

\( H_m(G) \): the space of functions with the norm

\[ \| u \|_{m}(G) = \left( \int_G \sum_{|\alpha| \leq m} |D^{\alpha} u|^2 dx \right)^{1/2} \]

\( L^2(\Omega) = H_0^1(\Omega) = H^{1,0} \).

1. **Discreteness of non-zero eigenvalues**

   The next equality is important for our purpose (c.f. \[3\] chap.4).

Let \( u \in H^2_{\text{loc}}(\Omega) \) be a solution of the equation \((-\Delta - z)u = f\) with the boundary condition \( u|_{\Gamma} = 0 \) (\( z = \sigma + i\tau \), a complex number), then
\[ 2 \int_{\Omega_{\rho_1, \rho_2}} \phi' \left| \frac{\partial \phi}{\partial x_n} \right|^2 dx = 2 \mathcal{R} \int_{\Omega_{\rho_1, \rho_2}} \phi \frac{\partial \phi}{\partial x_n} \overline{\phi} dx + \frac{1}{2} \int_{\Omega_{\rho_1, \rho_2}} \phi'' |\phi|^2 dx \]

\[ + \mathcal{R} \int_{\Omega_{\rho_1, \rho_2}} \phi' \overline{\phi} dx + 2 \tau \text{Im} \int_{\Omega_{\rho_1, \rho_2}} \phi \frac{\partial \phi}{\partial x_n} \overline{\phi} dx \]

\[ + \int_{\Gamma_{\rho_1, \rho_2}} \phi |\nabla \phi|^2 \cos \gamma_n d\sigma + \int_{\Gamma_{\rho_1, \rho_2}} \left| \phi \right|^2 \left| \frac{\partial \phi}{\partial x_n} \right|^2 - |\nabla \phi|^2 |\phi|^2 d\sigma \]

\[ + \mathcal{R} \int_{\Sigma_{\rho_1, \rho_2}} \left\{ \phi' \frac{\partial \phi}{\partial x_n} \overline{\phi} d\sigma - \frac{1}{2} \left\{ \int_{\Sigma_{\rho_1, \rho_2}} \phi'' |\phi|^2 d\sigma, \right. \right\} \]

**Theorem 1.1.** Each non-zero eigenvalue of \( \mathcal{H} \) has a finite multiplicity. All of the non-zero eigenvalues of \( \mathcal{H} \) make a discrete set on the real axis.

**Proof.** Let \( u \in D(\mathcal{H}), \mathcal{H}u = \lambda u, \lambda \in [a, b], \text{ and } 0 \in [a, b]. \) It suffices to show that \( u \in \mathcal{H}_{2, \xi} \) for some \( \xi > 0, \) and

\[ (1.2) \quad \|u\|_{2, \xi} \leq C\|u\|_{0, 0}, \]

where \( \xi \) and \( C \) may depend on the interval \([a, b]\), but are independent of \( \lambda \) and \( u. \) Indeed, if we can show \((1.2)\), the assertion of the theorem follows from the Rellich's compactness theorem (see the proof of Th. 3.1 in [2]).

Note at first that

\[ (1.3) \quad \|u\|_{2, 0} \leq \text{const.} \|u\|_{0, 0}. \]

This follows from Condition 2 not directly, but easily.

For any \( w \in \mathcal{H}_{2, s} \) (\( s \) a real number),

\[ (1.4) \quad \|Bw\|_{0, 1+h+s} \leq \text{const.} \|w\|_{2, s}. \]
This follows from Condition 2-(iii).

Note that \((-Δ-\lambda)u=-Bu\), and \(u|_{R_\perp}=0\). Put \(\phi=1\) in (1.1) and let \(R_\perp \to \infty\) (along suitable sequence if necessary), then we have

\[
0 = -2 \operatorname{Re} \int_{\partial R_\perp, \infty} \frac{\partial u}{\partial \nu} \overline{Bu} \, d\nu + \int_{\partial R_\perp, \infty} |\nabla u|^2 \cos \gamma_n \, dS
- \int_{\partial R_\perp} (2|\frac{\partial u}{\partial \nu}|^2 - |\nabla u|^2 + \lambda |u|^2) \, dS
\]

for any \(R_\perp > 0\). In view of \(\cos \gamma_n < 0\) (Condition 1), we see

\[
\int_{\partial R_\perp} (2|\frac{\partial u}{\partial \nu}|^2 - |\nabla u|^2 + \lambda |u|^2) \, dS \leq -2 \operatorname{Re} \int_{\partial R_\perp, \infty} \frac{\partial u}{\partial \nu} \overline{Bu} \, d\nu,
\]

In consequence of \(Bu \in \mathbb{H}_0, l+h\),

\[
\lim_{R \to \infty} \left[ \int_{\partial R, \infty} \frac{\partial u}{\partial \nu} \overline{Bu} \, d\nu \right] = 0.
\]

From (1.6) and (1.7),

\[
\lim_{R \to \infty} \sup_{R > 0} \left( 2|\frac{\partial u}{\partial \nu}|^2 - |\nabla u|^2 + \lambda |u|^2 \right) \, dS \leq 0.
\]

Return to (1.1), and put \(\phi = x^{n+2\xi}_{\perp} - 0 < 2\xi \leq \min\{h, l\}, R_\perp = 0\).

(1.8) allows us to take the limit \(R_\perp \to \infty\), and we have

\[
2(1+2\xi) \int_{\partial R} x^{2\xi}_{\perp} \left| \frac{\partial u}{\partial \nu} \right|^2 \, d\nu \leq -2 \operatorname{Re} \int_{\partial R} x^{1+2\xi}_{\perp} \frac{\partial u}{\partial \nu} \overline{Bu} \, d\nu + \operatorname{Re} (1+2\xi) \int_{\partial R} x^{2\xi}_{\perp} |u|^2 \, d\nu
\]

(1.9)

\[
+ (1+2\xi) \xi (2\xi-1) \int_{\partial R} x^{-2+2\xi}_{\perp} |u|^2 \, d\nu
\]

From this we can show

\[
\int_{\partial R} x^{-2+2\xi}_{\perp} |u|^2 \, d\nu \leq \text{const.} \, Bu \, h_{\perp}^2, |\xi|
\]

Because of Condition 1, we have some latitude in choice of the direction of \(x_{\perp}\)-axis. So (1.10) means
(1.11) \[ \int_{\Omega} x_n^2 |\nabla u|^2 \, dx \leq \text{const.} \| Bu \|_{0, l+\varepsilon}^2 \]

Multiply the equation \((-\Delta - \lambda)u = Bu\) by \(x_n^2 u\) and integrate over \(\Omega\), then we have by partial integration and (1.11)

(1.12) \[ \int_{\Omega} x_n^2 |u|^2 \, dx \leq \text{const.} \| Bu \|_{0, l+\varepsilon}^2 \]

From (1.11) and (1.12),

(1.13) \[ \| u \|_{1, \varepsilon} \leq \text{const.} \| u \|_{0, l+\varepsilon} \]

If we combine this with the inequality*

(1.14) \[ \| w \|_{2, s} \leq \text{const.} (\| Lw \|_{0, s} + \| w \|_{1, s}) \quad (s; \text{real}, w \in \mathcal{H}_{2,s}, w|_t = 0), \]

we have

(1.15) \[ \| u \|_{2, \varepsilon} \leq \text{const.} \| u \|_{0, l+\varepsilon} \]

(1.3), (1.4) and (1.5) give

\[ \| u \|_{2, \varepsilon} \leq \text{const.} \| u \|_{2, \varepsilon-h} \leq \text{const.} \| u \|_{0, 0} \]

Thus we can conclude (1.2). \((Q.E.D.)\)

**Theorem 1.2.** Let \(\lambda\) be a non-zero real number, \(Lu = \lambda u, \ u|_t = 0, \) and \(u \in \mathcal{H}_{2,s}\) for some \(s > -\frac{1-h}{2}\). Then \(u\) is in \(\mathcal{H}_{2, \infty} = \bigcap_{-\infty}^{+\infty} \mathcal{H}_{2, t}\).

The proof of this theorem is based on the facts that (1.5) is valid also for \(u \in \mathcal{H}_{2,s} (s > -\frac{1-h}{2})\), and that

\[ \lim_{R \to \infty} \int_{R^2 \times \Omega} \left| \frac{\partial u}{\partial n} \right| \, dx = 0. \]

* This inequality follows from Condition \(\mathcal{L}-(ii)\).
2. Absolute continuity of the continuous spectrum

Let $H_0$ be the self-adjoint realization of $-\Delta$ defined as follows:

$$D(H_0) = D(H) = \{ u \in L^2(\Omega) \mid u |_{\partial \Omega} = 0, \Delta u \in L^2(\Omega), H_0 u = -\Delta u \}$$

for $u \in D(H_0)$. And put $R_0(z) = (H_0 - z)^{-1} (\text{Im } z \neq 0)$.

**Theorem 2.1.** Let $\varepsilon > 0$ and $K$ be a compact set in the complex plane. If the origin is not in $K$, there exists a positive constant $C = C(K, \varepsilon)$ such that

$$\| R_0(z)f \|_2 \leq \frac{1 + \varepsilon}{\varepsilon} C \| f \|_0$$

for any $f \in H_0$, $\frac{1 + \varepsilon}{\varepsilon}$ and $z \in K$ with $\text{Im } z \neq 0$.

The proof also can be carried out by use of (1.1). But we omit it because it is somewhat lengthy and partly overlaps with the proof of Theorem 1.1.

**Lemma 2.2.** Let $0 < \varepsilon < h$ and $K$ be a compact set in the complex plane. If the origin is not in $K$, there are $r_0 = r_0(K, \varepsilon) > 0$ and $C = C(K, \varepsilon) > 0$ such that

$$\| BR(z)f \|_2 \leq \frac{1 + \varepsilon}{\varepsilon} C \left\{ \| f \|_0 + \| R(z)f \|_1 (r_0, r_0) \right\}$$

for any $f \in H_0$, $\frac{1 + \varepsilon}{\varepsilon}$ and $z \in K$ with $\text{Im } z \neq 0$. Here $R(z) = (H - z)^{-1}$.

This lemma is obtained from Theorem 2.1 and the inequality

$$\| Bw \|_0, 1 + h + s \leq \text{const.} (\| (L - z)w \|_0, s + \| w \|_1, s) (w \in \mathcal{H}_2, s, w|_p = 0)$$

by estimating

$$\int_{\Omega} |u|^{2-1+\varepsilon} \frac{h}{2} \sum_{|\omega| \leq 1} |\nabla u|^2 \, dx$$

($u = R(z)f = R_0(z)f - R_0(z)Bu$).
In consequence of the above lemma and Theorem 2.1 we have

Lemma 2.3. Let $\xi, K$ as above, then there are $r_0 = r_0(K, \varepsilon)$ and $C = C(K, \varepsilon)$ such that
\[
(2.4) \quad \| R(z) f \|_{L^2, \frac{1+\varepsilon}{2}} \leq C \left\{ \| f \|_{0, \frac{1+\varepsilon}{2}} + \| R(z) f \|_{1, (Z_0, r_0)} \right\}
\]
for any $f \in H_{0, \frac{1+\varepsilon}{2}}$ and $z \notin K$ with $\text{Im } z \neq 0$.

Theorem 2.4. Let $K$ be a compact set in the complex plane. If $K$ contains neither the origin nor eigenvalues of $H$, then, for any $\varepsilon > 0$, one can take a constant $C = C(K, \varepsilon) > 0$ such that the inequality
\[
(2.5) \quad \| R(z) f \|_{L^2, \frac{1+\varepsilon}{2}} \leq C \| f \|_{0, \frac{1+\varepsilon}{2}}
\]
holds for any $f \in H_{0, \frac{1+\varepsilon}{2}}$ and $z \notin K$ with $\text{Im } z \neq 0$.

Proof. Without loss of generality, we may assume $0 < \varepsilon < \text{ch}$. If $|\text{Im } z| > \delta > 0$, there is a constant $C_0$ which may depend on $\delta$, and
\[
\| R(z) f \|_{1, (Z_0, r_0)} \leq C_0 \| f \|_{0, \frac{1+\varepsilon}{2}}
\]
According to Lemma 2.3, to prove the theorem, it suffices to show the assertion: if $z \notin K$ and $\text{Im } z \neq 0$, $C_0$ actually can be taken independently of $\delta > 0$.

Suppose that this is not true, then there would be sequences $\{ z_N \}$ and $\{ u_N \}$ which satisfy the following $(2.6)$.
\[
(2.6) \quad \begin{cases} 
\text{Im } z_N \neq 0, \quad \{ z_N \} \subset K, \quad \{ z_N \} \text{ converges to a real number } \xi \notin K. \\
\{ u_N \} \subset D(H), \quad \| u_N \|_{1, (Z_0, r_0)} = 1, \quad f_N = (H - z) u_N \text{ converges strongly to } 0 \text{ in } H_{0, \frac{1+\varepsilon}{2}}.
\end{cases}
\]
We must show that $(2.6)$ leads to a contradiction. Note that
\[
(-\Delta - z_N) u_N = f_N - B_N \quad \| Bu_N \|_{0, \frac{1+\varepsilon}{2}} \leq \text{const.} \| f_N \|_{0, \frac{1+\varepsilon}{2}} + 1 \text{ by}
\]
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Lemma 2.2, so \( \{ (-\Delta - z_N^*) u_N^* \} \) is bounded in \( H_{0, \frac{(1+\varepsilon)}{2}} \). Consequently, \( \{ u_N \} \) is bounded in \( H_{\frac{1}{2}, -(1+\varepsilon)/2} \) (Theorem 2.1). Let us take \( \varepsilon' \) such that \( \varepsilon < \varepsilon' < \delta \). Then, taking a subsequence if necessary, we may assume that \( \{ u_N \} \) converges strongly to a function \( u_0 \) in \( H_{1, -(1+\varepsilon')/2} \) (Rellich's compactness theorem). Moreover we can see

\[
\begin{align*}
\begin{cases}
\| u_0 \|_1(\Omega_0, r_0) &= 1 \\
(L-\sigma_0) u_0 &= 0 \\
u_0 |_{\partial \Omega} &= 0
\end{cases}
\end{align*}
\tag{2.7}
\]

In view of (1.14), \( u_0 \in H_{2, -(1+\varepsilon')}, \) and so \( u_0 \in H_{2, \infty} \) (Theorem 1.2). This contradicts the assumption that \( K \) contains no eigenvalues of \( H \).

(Q.E.D.)

**Theorem 2.5.** The continuous spectrum of \( H \) is absolutely continuous.

This is a consequence of Theorem 2.4. See, for example, [3] Chap.1 or [8] Appendix C.

3. **An extension**

Let us suppose \( L' \) to be a differential operator in a domain \( \Omega' \subseteq \mathbb{R}^N \). We make the following conditions.

(i) There is a diffeomorphism \( \Omega' \to \Omega : x_1 = x_1(X_1, \ldots, X_n), \ldots, x_n = x_n(X_1, \ldots, X_n) \), and \( \Omega \) satisfies Condition 1.
(ii) If we define the transform of functions as follows;

$$(uu)(X)=u(x(X))|J(X)|^{1/2},$$

where $J(X)$ is the Jacobian

$$\Delta(x_1,\cdots,x_n)/\Delta(x_1,\cdots,x_n).$$

Then $L=\mu L^U$ satisfies Condition $\xi$.

In this situation, $H=UH,U$ is a self-adjoint realization of $L'$

in $L^2(\Omega')$ under the Dirichlet boundary condition, and we can replace

$H$ by $H'$ in Theorem 1.1 and 2.5. This follows from the fact that $U$

defines a unitary transform $L^2(\mathbb{R})\to L^2(\Omega')$.

4. A remark on the absence of positive eigenvalues

In this section, we write $L$ in the form

$$L = \sum_{j,k=1}^n \left( \frac{1}{2} \frac{\partial^2}{\partial x_j^2} + b_j \right) a_{jk} \left( \frac{1}{2} \frac{\partial^2}{\partial x_k^2} + b_k \right) + q$$

Let us assume that there exists a positive constant $r_0$ such that

the following conditions are satisfied for $|x| \geq r_0$.

(a) $a_{jk}$

and $b_j$ are real valued $C^1$-functions and $a_{jk}=a_{kj}$.

(b) $\exists \epsilon > 0$ such that $c |\xi|^2 \leq \sum a_{jk} \xi_j \xi_k \leq \frac{1}{c} |\xi|^2$ for any complex vector $(\xi_1,\cdots,\xi_n)$.

(c) $a_{jk}/\partial x_k = o(|x|^{-1})$. (d) $a_{jk} \to \delta_{jk}$ (as $|x| \to \infty$). (e) $\frac{\partial^2}{\partial x_j^2} - \frac{3 \xi_j}{\partial x_k} = o(|x|^{-1})$

(f) $q(x) = o(|x|^{-1})$. (g) The unique continuation property holds.

(h) $\sum a_{jk} x_j \xi_k \leq 0$ for each boundary point $x$ ($|x| > r_0$), where

$(\nu_1,\cdots,\nu_n)$ is the outer normal to $\Gamma$ at $x$.

The conditions (a)-(g) are nothing but those which are proposed by Ikebe and Uchiyama in [5].

**Theorem 4.1.** If $u$ is a not identically vanishing solution of

$$Lu=\lambda u \quad (\lambda > 0)$$

in $\Omega \cap \{ |x| > r \}$ and $\lim_{r \to \infty} \int_{\Omega \cap \{ |x| > r \}} |x|^2 \partial \Omega \, dS = \infty$,

then we have for any $\epsilon > 0$

$$\lim_{r \to \infty} \int_{\Omega \cap \{ |x| > r \}} \left( \sum a_{jk} \hat{x}_j \hat{x}_k \right)^2 \partial \Omega \, dS = \infty,$$
where \( S(r) = \Omega \cap \{x \mid |x| = r\} \), \( \hat{x}_j = \frac{x_j}{|x|} \), and \( D_j = \left[ -1 - \frac{\partial}{\partial x_j} \right] + b_j \).

The proof can be carried out in the same way of [5]. Because of the condition (h), the unboundedness of \( \Gamma \) brings about no essential difficulties.

If \( L \) enjoys the unique continuation property on the whole of \( \Omega \), the absence of positive eigenvalues of \( H \) follows directly from this theorem.

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