数理解析研究所講究録
242巻 1975年 112-123

Spectra of Elliptic Operators in a Domain with Unbounded Boundary

Takao TAYOSHI

University of Electro-Communications*

0. Introduction

We shall study a spectral property of a 2nd order partial differential operator

$$L = -\Delta + \sum_{j,k=1}^{n} \alpha_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \beta_j \frac{\partial}{\partial x_j} + \gamma$$

in a domain $\Omega \subset R^n$ $(n \geqq 2)$.

In case that $\Omega = R^n$ or $\Omega =$ the exterior of a bounded hyper-surface, this subject have been discussed by many authors under various conditions on L. As for the case of unbounded boundary, we can refer to [1] [3,4] [6-11].

In §§ 1 and 2 of this note, we shall make the following conditions 1 and 2. (A list of notations will be given later.)

Condition 1. Ω is characterized by $x_n > \varphi(\tilde{x})$ $(\tilde{x} = (x_1, \cdots, x_{n-1}))$, where $\varphi(\tilde{x})$ is a single valued C^2-function on R^{n-1}, and there are positive numbers M_1 and M_2 such that $\varphi(\tilde{x}) > M_1(x_1^2 + \cdots + x_{n-1}^2)^{1/2}$, $|\partial\varphi/\partial x_j| < M_2$, $|\partial^2\varphi/\partial x_j \partial x_k| < M_2$ $(j,k = 1, \cdots, n-1)$.

Condition 2. Put

$$B = \sum_{j,k=1}^{n} \alpha_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \beta_j \frac{\partial}{\partial x_j} + \gamma$$

* 1-5-1, Chofugaoka, Chofu-shi, Tokyo 182.

then B is real and short range in the following sense.

(i) If we put $D(H) = \{u \in L^2(\Omega) \mid u \in H_1(\Omega),\ u|_\Gamma = 0,\ \Delta u \in L^2(\Omega)\}$, and $Hu = Lu$ for $u \in D(H)$, then H is a self-adjoint operator in $L^2(\Omega)$.

(ii) The inequality

$$\|u\|_2(\Omega_{R,R+1}) \leqq C_1\{\|(-\Delta + B)u\|_0(\Omega_{R-1,R+2}) + \|u\|_1(\Omega_{R-1,R+2})\}$$

holds for $\forall u \in H_{2,loc}(\bar{\Omega})$ with $u|_\Gamma = 0$, where C_1 is independent of R.

(iii) There are positive numbers h and C_2 such that

$$\|Bu\|_0(\Omega_{R,R+1}) \leqq C_2 R^{-1-h} \|u\|_2(\Omega_{R-1,R+2})$$

for $\forall R > 0$ and $\forall u \in H_{2,loc}(\bar{\Omega})$.

Under these conditions the discreteness of non-zero eigenvalues of H and the absolute continuity of the continuous spectrum of H will be shown. It should be noted that, under Condition 2, the coefficient γ may have a high singularity of some type.

In § 3, the conditions will be slightly weakened.

In § 4, a short remark on the absence of positive eigenvalues will be given.

There are some possibilities that we can extend the arguments in §§ 1-2 to include the perturbations which are not necessarily short range. For example, we can put

$B = B_1 + B_2$,

B_1 is real and short range in the above sense,

B_2 is multiplication by a real C^1-function q_2, and

$|q_2| + |\nabla q_2| \leqq C|x|^{-h}$, $\frac{\partial q_2}{\partial |x|} < C|x|^{-1-h}$ for some C, h >0.

But, in order to avoid some tedious culculations, we shall omit discussions about these possibilities. A full treatment will be given in [12].

List of notations

Ω: an unbounded domain in n-dimensional Euclidean space $R^n = \{x \mid x=(x_1,\cdots,x_n)\}$ $(n \geqq 2)$.

Γ: the boundary of Ω; $\Gamma = \partial\Omega$

$\bar{\Omega} = \Omega \cup \Gamma$, $\Omega_{R_1,R_2} = \Omega \cap \{x \mid R_1 < x_n < R_2\}$, $\Gamma_{R_1,R_2} = \Gamma \cap \{x \mid R_1 < x_n < R_2\}$,

$S_R = \Omega \cap \{x \mid x_n = R\}$.

Δ: the Laplace operator; $\Delta = \sum_{j=1}^{n} \partial^2/\partial x_j^2$.

ϕ: a smooth function which depends only on x_n,

$\phi' = d\phi/dx_n$, $\phi'' = d^2\phi/dx_n^2$, $\cdots$.

γ_n: the angle between x_n-axis and outer normal.

$|x| = (x_1^2 + \ \ + x_n^2)^{1/2}$

$|\nabla u| = (\sum_{j=1}^{n} |\partial u/\partial x_j|^2)^{1/2}$

$H_{m,s}$: the space of functions with the norm

$$\|u\|_{m,s} = \left(\int_\Omega (1+|x|^2)^s \sum_{|\alpha| \leqq m} |D^\alpha u|^2 dx \right)^{\frac{1}{2}}$$

$(D^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}, \quad |\alpha| = \alpha_1 + \ \ + \alpha_n)$.

$H_m(G)$: the space of functions with the norm

$$\|u\|_m(G) = \left(\int_G \sum_{|\alpha| \leqq m} |D^\alpha u|^2 dx \right)^{\frac{1}{2}}.$$

$L^2(\Omega) = H_0(\Omega) = H_{0,0}$.

1. Discreteness of non-zero eigenvalues

The next equality is important for our purpose (c.f. [3] chap.4). Let $u \in H_{2,loc}(\bar{\Omega})$ be a solution of the equation $(-\Delta - z)u = f$ with the boundary condition $u_{|\Gamma} = 0$ $(z = \sigma + i\tau$, a complex number), then

$$2\int_{\Omega_{R_1,R_2}} \phi' \left|\frac{\partial u}{\partial x_n}\right|^2 dx = 2\,\mathrm{Re}\int_{\Omega_{R_1,R_2}} \phi \frac{\partial u}{\partial x_n}\bar{f}\,dx + \frac{1}{2}\int_{\Omega_{R_1,R_2}} \phi'''|u|^2 dx$$

$$(1.1) \qquad + \mathrm{Re}\int_{\Omega_{R_1,R_2}} \phi' f \bar{u}\,dx + 2\tau\,\mathrm{Im}\int_{\Omega_{R_1,R_2}} \phi \frac{\partial u}{\partial x_n}\bar{u}\,dx$$

$$+ \int_{\Gamma_{R_1,R_2}} \phi\,|\nabla u|^2 \cos\gamma_n\,dS + \left\{\int_{S_{R_2}} - \int_{S_{R_1}}\right\} \phi\left(2\left|\frac{\partial u}{\partial x_n}\right|^2 - |\nabla u|^2 + \sigma|u|^2\right) dS$$

$$+ \mathrm{Re}\left\{\int_{S_{R_2}} - \int_{S_{R_1}}\right\} \phi' \frac{\partial u}{\partial x_n}\bar{u}\,dS - \frac{1}{2}\left\{\int_{S_{R_2}} - \int_{S_{R_1}}\right\} \phi''|u|^2 dS.$$

Theorem 1.1. Each non-zero eigenvalue of H has a finite multiplicity. All of the non-zero eigenvalues of H make a discrete set on the real axis.

Proof. Let $u\in D(H)$, $Hu=\lambda u$, $\lambda\in[a,b]$, and $0\notin[a,b]$. It suffices to show that $u\in H_{2,\varepsilon}$ for some $\varepsilon>0$, and

$$(1.2) \qquad \|u\|_{2,\varepsilon} \leqq C\|u\|_{0,0} ,$$

where ε and C may depend on the interval $[a,b]$, but are independent of λ and u. Indeed, if we can show (1.2), the assertion of the theorem follows from the Rellich's compactness theorem (see the proof of Th. 3.1 in [2]).

Note at first that

$$(1.3) \qquad \|u\|_{2,0} \leqq \text{const.}\|u\|_{0,0} .$$

This follows from Condition 2 not directly, but easily.

For any $w\in H_{2,s}$ (s= a real number),

$$(1.4) \qquad \|Bw\|_{0,1+h+s} \leqq \text{const.}\|w\|_{2,s} .$$

This follows from Condition 2-(iii).

Note that $(-\Delta-\lambda)u=-Bu$, and $u|_{\Gamma}=0$. Put $\phi\equiv 1$ in (1.1) and let $R_2\to\infty$ (along suitable seqence if necessary), then we have

$$0=-2\,\mathrm{Re}\int_{\Omega_{R_1,\infty}}\frac{\partial u}{\partial x_n}\overline{Bu}\,dx+\int_{\Gamma_{R_1,\infty}}|\nabla u|^2\cos\gamma_n\,dS-\int_{S_{R_1}}\left(2\left|\frac{\partial u}{\partial x_n}\right|^2-|\nabla u|^2+\lambda|u|^2\right)dS \tag{1.5}$$

for any $R_1>0$. In view of $\cos\gamma_n<0$ (Condition 1), we see

$$\int_{S_{R_1}}\left(2\left|\frac{\partial u}{\partial x_n}\right|^2-|\nabla u|^2+\lambda|u|^2\right)dS\leqq -2\,\mathrm{Re}\int_{\Omega_{R_1,\infty}}\frac{\partial u}{\partial x_n}\overline{Bu}\,dx. \tag{1.6}$$

In consequence of $Bu\in H_{0,1+h}$,

$$\lim_{R\to\infty}R^{1+h}\int_{\Omega_{R,\infty}}\left|\frac{\partial u}{\partial x_n}\overline{Bu}\right|dx=0. \tag{1.7}$$

From (1.6) and (1.7),

$$\limsup_{R\to\infty}R^{1+h}\int_{S_R}\left(2\left|\frac{\partial u}{\partial x_n}\right|^2-|\nabla u|^2+\lambda|u|^2\right)dS\leqq 0. \tag{1.8}$$

Return to (1.1), and put $\phi=x_n^{1+2\varepsilon}$, $0<2\varepsilon\leqq\min\{h,1\}$, $R_1=0$. (1.8) allows us to take the limit $R_2\to\infty$, and we have

$$\begin{aligned}2(1+2\varepsilon)\int_{\Omega}x_n^{2\varepsilon}\left|\frac{\partial u}{\partial x_n}\right|^2dx&\leqq -2\,\mathrm{Re}\int_{\Omega}x_n^{1+2\varepsilon}\frac{\partial u}{\partial x_n}\overline{Bu}\,dx+\mathrm{Re}\,(1+2\varepsilon)\int_{\Omega}x^{2\varepsilon}(Bu)\bar{u}\,dx\\&\quad+(1+2\varepsilon)\varepsilon(2\varepsilon-1)\int_{\Omega}x_n^{-2+2\varepsilon}|u|^2dx\end{aligned} \tag{1.9}$$

From this we can show

$$\int_{\Omega}x_n^{2\varepsilon}\left|\frac{\partial u}{\partial x_n}\right|^2dx\leqq \text{const.}\,\|Bu\|^2_{0,1+\varepsilon} \tag{1.10}$$

Because of Condition 1, we have some latitude in choice of the direction of x_n-axis. So (1.10) means

(1.11) $$\int_{\Omega} x_n^{2\varepsilon}|\nabla u|^2 dx \leqq \text{const.}\|Bu\|^2_{0,1+\varepsilon}$$

Multiply the equation $(-\Delta-\lambda)u=-Bu$ by $x_n^{2\varepsilon}\bar{u}$ and integrate over Ω, then we have by partial integration and (1.11)

(1.12) $$\int_{\Omega} x_n^{2\varepsilon}|u|^2 dx \leqq \text{const.}\|Bu\|^2_{0,1+\varepsilon}$$

From (1.11) and (1.12),

(1.13) $$\|u\|_{1,\varepsilon} \leqq \text{const.}\|Bu\|_{0,1+\varepsilon} \ .$$

If we combine this with the inequality*

(1.14) $$\|w\|_{2,s} \leqq \text{const.}(\|Lw\|_{0,s} + \|w\|_{1,s}) \quad (s;\text{real},\ w\in H_{2,s},\ w_{|\Gamma}=0),$$

we have

(1.15) $$\|u\|_{2,\varepsilon} \leqq \text{const.}\|Bu\|_{0,1+\varepsilon}$$

(1.3),(1.4) and (1.15) give

$$\|u\|_{2,\varepsilon} \leqq \text{const.}\|u\|_{2,\varepsilon-h} \leqq \text{const.}\|u\|_{0,0} \ .$$

Thus we can conclude (1.2). (Q.E.D.)

Theorem 1.2. Let λ be a non-zero real number, $Lu=\lambda u$, $u_{|\Gamma}=0$, and $u\in H_{2,s}$ for some $s>-\frac{1+h}{2}$. Then u is in $H_{2,\infty} = \bigcap_{-\infty<t<+\infty} H_{2,t}$.

The proof of this theorem is based on the facts that (1.5) is valid also for $u\in H_{2,s}$ $(s>-\frac{1+h}{2})$, and that

$$\lim_{R\to\infty} R^{2s+1+h}\int_{\Omega_{R,\infty}} \left|\frac{\partial u}{\partial x_n}\overline{Bu}\right| dx = 0.$$

* This inequality follows from Condition 2-(ii).

2. Absolute continuity of the continuous spectrum

Let H_0 be the self-adjoint realization of $-\Delta$ defined as follows; $D(H_0) = D(H) = \{u \in L^2(\Omega) \mid u \in H_1(\Omega),\ u|_\Gamma = 0,\ \Delta u \in L^2(\Omega)\}$, $H_0 u = -\Delta u$ for $u \in D(H_0)$. And put $R_0(z) = (H_0 - z)^{-1}$ (Im $z \neq 0$).

Theorem 2.1. Let $\varepsilon > 0$ and K be a compact set in the complex plane. If the origin is not in K, there exists a positive constant $C = C(K, \varepsilon)$ such that

$$(2.1) \qquad \|R_0(z)f\|_{2,-\frac{1+\varepsilon}{2}} \leqq C\|f\|_{0,\frac{1+\varepsilon}{2}}$$

for any $f \in H_{0,\frac{1+\varepsilon}{2}}$ and $z \in K$ with Im $z \neq 0$.

The proof also can be carried out by use of (1.1). But we omit it because it is somewhat lengty and partly overlaps with the proof of Theorem 1.1.

Lemma 2.2. Let $0 < \varepsilon < h$ and K be a compact set in the complex plane. If the origin is not in K, there are $r_0 = r_0(K, \varepsilon) > 0$ and $C = C(K, \varepsilon) > 0$ such that

$$(2.2) \qquad \|BR(z)f\|_{0,\frac{1+\varepsilon}{2}} \leqq C\{\|f\|_{0,\frac{1+\varepsilon}{2}} + \|R(z)f\|_1(\Omega_{0,r_0})\}$$

for any $f \in H_{0,\frac{1+\varepsilon}{2}}$ and $z \in K$ with Im $z \neq 0$. Here $R(z) = (H - z)^{-1}$.

This lemma is obtained from Theorem 2.1 and the inequality

$$(2.3) \qquad \|Bw\|_{0,1+h+s} \leqq \text{const.}(\|(L-z)w\|_{0,s} + \|w\|_{1,s}) \quad (w \in H_{2,s},\ w|_\Gamma = 0)$$

by estimating

$$\int_{\Omega_{R,\infty}} (1+|x|^2)^{\frac{-1+\varepsilon}{2} - h} \sum_{|\alpha| \leq 1} |D^\alpha u|^2 \, dx$$

$(u = R(z)f = R_0(z)f - R_0(z)Bu)$.

In consequence of the ~~aveo~~ above lemma and Theorem 2.1 we have

Lemma 2.3. Let ε, K as above, then there are $r_0=r_0(K,\varepsilon)$ and $C=C(K,\varepsilon)$ such that

$$(2.4) \qquad \|R(z)f\|_{2,-\frac{1+\varepsilon}{2}} \leqq C\{\|f\|_{0,\frac{1+\varepsilon}{2}} + \|R(z)f\|_1(\Omega_{0,r_0})\}$$

for any f $H_{0,\frac{1+\varepsilon}{2}}$ and $z\in K$ with Im $z \neq 0$.

Theorem 2.4. Let K be a compact set in the complex plane. If K contains neither the origin nor eigenvalues of H, then, for any $\varepsilon>0$, one can take a constant $C=C(K,\varepsilon)>0$ such that the inequality

$$(2.5) \qquad \|R(z)f\|_{2,-\frac{1+\varepsilon}{2}} \leqq C\|f\|_{0,\frac{1+\varepsilon}{2}}$$

holds for any $f\in H_{0,\frac{1+\varepsilon}{2}}$ and $z\in K$ with Im $z \neq 0$.

Proof. Without loss of generality, we may assume $0<\varepsilon<h$. If $|\mathrm{Im}\ z|>\delta>0$, there is a constant C_δ which may depend on δ, and

$$\|R(z)f\|_1(\Omega_{0,r_0}) \leqq C_\delta\|f\|_{0,\frac{1+\varepsilon}{2}}$$

According to Lemma 2.3, to prove the theorem, it suffices to show the assertion: if $z\in K$ and Im $z \neq 0$, C_δ actually can be taken independently of $\delta>0$.

Suppose that this is not true, then there would be sequences $\{z_N\}$ and $\{u_N\}$ which satisfy the following (2.6).

$$(2.6) \qquad \begin{cases} \mathrm{Im}\ z_N \neq 0,\ \{z_N\}\subset K,\ \{z_N\} \text{ converges to a real number } \sigma_0\in K. \\ \{u_N\}\subset D(H),\ \|u_N\|_1(\Omega_{0,r_0})=1,\ f_N=(H-z)u_N \text{ converges} \\ \text{strongly to 0 in } H_{0,\frac{1+\varepsilon}{2}} \quad . \end{cases}$$

We must show that (2.6) leads to a contradiction. Note that $(-\Delta-z_N)u_N=f_N-B_N$. $\|Bu_N\|_{0,(1+\varepsilon)/2} \leqq \mathrm{const.}(\|f_N\|_{0,(1+\varepsilon)/2}+1)$ by

Lemma 2.2, so $\{(-\Delta-z_N)u_N\}$ is bounded in $H_{0,(1+\varepsilon)/2}$. Consequently, $\{u_N\}$ is bounded in $H_{2,-(1+\varepsilon)/2}$ (Theorem 2.1). Let us take ε' such that $\varepsilon<\varepsilon'<h$. Then, taking a subsequences if necessary, we may assume that $\{u_N\}$ converges strongly to a function u_0 in $H_{1,-(1+\varepsilon')/2}$ (Rellich's compactness theorem). Moreover we can see

$$(2.7)\qquad \begin{cases} \|u_0\|_1(\Omega_{0,r_0})=1 \\ (L-\sigma_0)u_0=0 \\ \quad u_0|_\Gamma=0 \end{cases}$$

In view of (1.14), $u_0\in H_{2,-\frac{1+\varepsilon'}{2}}$, and so $u_0\in H_{2,\infty}$ (Theorem 1.2). This contradicts the assumption that K contains no eigenvalues of H.

(Q.E.D.)

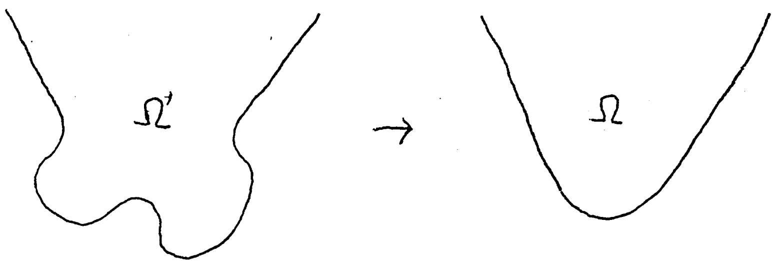
<u>Theorem 2.5.</u> The continuous spectrum of H is sbsolutely continuous.

This is a consequence of Theorem 2.4. See, for example, [3] Chap.1 or [8] Appendix C.

3. An extension

Let us suppose L' to be a differential operator in a domain $\Omega'\subset R^n$. We make the following conditions.

(i) There is a diffeomorphism $\Omega'\to\Omega$: $x_1=x_1(X_1,\dots,X_n),\dots, x_n=x_n(X_1,\dots,X_n)$, and Ω satisfies Condition 1.



(ii) If we define the transform of functions as follows; $(Uu)(X)=u(x(X))|J(X)|^{1/2}$, where $J(X)$ is the Jacobian $\partial(x_1,\cdots,x_n)/\partial(X_1,\cdots,X_n)$. Then $L=U^{-1}L'U$ satisfies Condition 2.

In this situation, $H'=UHU^{-1}$ is a self-adjoint realization of L' in $L^2(\Omega')$ under the Dirichlet boundary condition, and we can replace H by H' in Theorem 1.1 and 2.5. This follows from the fact that U defines a unitary transform $L^2(\Omega)\to L^2(\Omega')$.

4. A remark on the absence of positive eigenvalues

In this section, we write L in the form

$$L = \sum_{j,k=1}^{n}\left(\sqrt{-1}\frac{\partial}{\partial x_j}+b_j\right)a_{jk}\left(\sqrt{-1}\frac{\partial}{\partial x_k}+b_k\right) + q$$

Let us assume that there exists a positive constant r_0 such that the following conditions are satisfied for $|x|\geqq r_0$. (a) a_{jk} and b_j are real valued C^1-functions and $a_{jk}=a_{kj}$. (b) $\exists\, c>0$ such that $c|\xi|^2\leqq \sum a_{jk}\xi_j\bar{\xi}_k \leqq \frac{1}{c}|\xi|^2$ for any complex vector $(\xi_1,\cdots,\xi_n)$. (c) $\partial a_{jk}/\partial x_\ell = o(|x|^{-1})$. (d) $a_{jk}\to\delta_{jk}$ ($|x|\to\infty$). (e) $\frac{\partial b_k}{\partial x_j}-\frac{\partial b_j}{\partial x_k}=o(|x|^{-1})$ (f) $q(x)=o(|x|^{-1})$. (g) The unique continuation property holds. (h) $\sum a_{jk}x_j\nu_k \leqq 0$ for each boundary point x ($|x|>r_0$), where $(\nu_1,\cdots,\nu_n)$ is the outer normal to Γ at x.

The conditons (a)-(g) are nothing but those which are proposed by Ikebe and Uchiyama in [5].

Theorem 4.1. If u is a not identically vanishing solution of $Lu=\lambda u$ ($\lambda>0$) in $\Omega\cap\{x\mid |x|>r_0\}$ and $u|_{\Gamma\cap\{x\mid |x|>r_0\}}=0$, then we have for any $\varepsilon>0$

$$\lim_{r\to\infty} r^{\varepsilon}\int_{S(r)}\left(\sum a_{jk}\hat{x}_j\hat{x}_k\right)\left(\lambda|u|^2+\left|\sum a_{jk}(\mathcal{D}_j u)\hat{x}_k\right|^2\right)dS = \infty,$$

where $S(r)=\Omega\cap\{x \mid |x|=r\}$, $\hat{x}_j=\frac{x_j}{|x|}$, and $\mathcal{D}_j=\sqrt{-1}\frac{\partial}{\partial x_j}+b_j$.

The proof can be carried out in the same way of [5]. Because of the condition (h), the unboundedness of Γ brings about no essential difficulties.

If L enjoys the unique continuation property on the whole of Ω, the absence of positive eigenvalues of H follows directly from this theorem.

References

[1] Agmon, S., Lower bounds for solutions of Schrödinger type equations in unbounded domains, Proceedings of the International Conference on Functional Analysis and Related Topics, Tokyo, 1969, 216-224.

[2] ——, Spectral properties of Schrödinger operators and Scattering theory, Lectures given at the Scuola Normale Superiore of Pisa, (1973).

[3] Éidus, D. M., The principle of limit amplitude, Russ. Math. Surveys, 24 (1969), 97-169.

[4] Goldstein, C., Eigenfunction expansion associated with the Laplacian for certain domains with infinite boundaries. I, II, III, Trans. Amer. Math. Soc. 135 (1969), 1-31, 135, 33-50, 143 (1969), 283-301.

[5] Ikebe, T. and J. Uchiyama, On the asymptotic behavior of eigenfunctions of second-order elliptic operators, J. Math. Kyoto. Univ. 11 (1971), 425-448.

[6] Jones, D. C., The eigenvalues of $(\nabla^2+\lambda)u=0$ when the boundary conditions are given on semi-infinite domains, Proc. Cambridge Philos. Soc. 49 (1953), 668-684.

[7] Konno, R., Non-existence of positive eigenvalues of Schrödinger operators in infinite domains, J. Fac. Sci. Univ. Tokyo, Sec. IA, 19 (1972), 393-402.

[8] Kuroda, S. T. et al., Spectra of elliptic partial differential operators, Tōdai Sūgaku Seminari Nōto 27 (1972). (Japanese)

[9] Rellich, F., Uber das asymptotische Verhalten der Losungen von $(\Delta+\lambda)u=0$ in unendlichen Gebieten, Jber. Deutsch. Math. Verein.53 (1943), 57-65.

[10] ———, Das Eigenwerteproblem von $\Delta u+\lambda u=0$ in Halbrölen, Studies and esseys presented to R. Courant (New York 1948), 329-344.

[11] Tayoshi, T. The asymptotic behavior of the solutions of $(\Delta+\lambda)u=0$ in a domain with the unbounded boundary, Publ. RIMS. Kyoto Univ. 8 (1972) 375-391.

[12] ———, Limiting absorption method for second order elliptic equations in a domain with unbounded boundary (tentative title), to appear.