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Kyoto University
Spectra of Elliptic Operators in a Domain with Unbounded Boundary

Takao TAYOSHI
University of Electro-Communications*

0. Introduction
We shall study a spectral property of a 2nd order partial differential operator

\[ L = -\Delta + \sum_{j,k=1}^{n} \alpha_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \beta_j \frac{\partial}{\partial x_j} + \gamma \]

in a domain \( \Omega \subset \mathbb{R}^n \) (n \( \geq 2 \)).

In case that \( \Omega = \mathbb{R}^n \) or \( \Omega \) is the exterior of a bounded hypersurface, this subject have been discussed by many authors under various conditions on \( L \). As for the case of unbounded boundary, we can refer to [1] [3,4] [6-11].

In \( \S \S 1 \) and 2 of this note, we shall make the following conditions 1 and 2. (A list of notations will be given later.)

Condition 1. \( \Omega \) is characterized by \( x_n > \varphi(\vec{x}) \) (\( \vec{x} = (x_1, \ldots, x_{n-1}) \)), where \( \varphi(\vec{x}) \) is a single valued \( C^2 \)-function on \( \mathbb{R}^{n-1} \), and there are positive numbers \( M_1 \) and \( M_2 \) such that

\[ \varphi(\vec{x}) > M_1 (x_1^2 + \cdots + x_{n-1}^2)^{1/2}, \quad |\partial \varphi / \partial x_j| < M_2, \quad |\partial^2 \varphi / \partial x_j \partial x_k| < M_2 \]

\( (j,k=1, \ldots, n-1) \).

Condition 2. Put

\[ B = \sum_{j,k=1}^{n} \alpha_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \beta_j \frac{\partial}{\partial x_j} + \gamma \]

\* 1-5-1, Chofugaoka, Chofu-shi, Tokyo 182.
then $B$ is real and short range in the following sense.

(i) If we put $D(H) = \{ u \in L^2(\mathcal{R}) \mid u \in H^1(\mathcal{R}), u|_{\Gamma} = 0, \Delta u \in L^2(\mathcal{R}) \}$, and $H u = Lu$ for $u \in D(H)$, then $H$ is a self-adjoint operator in $L^2(\mathcal{R})$.

(ii) The inequality

$$
\| u \|_2(\mathcal{O}_{R,R+1}) \leq C_1 \| (-A + B)u \|_0(\mathcal{O}_{R-1,R+2}) + \| u \|_1(\mathcal{O}_{R-1,R+2})
$$

holds for $u \in H^2_{2,loc}(\bar{\mathcal{R}})$ with $u|_{\Gamma} = 0$, where $C_1$ is independent of $R$.

(iii) There are positive numbers $h$ and $C_2$ such that

$$
\| B u \|_0(\mathcal{O}_{R,R+1}) \leq C_2 R^{1-h} \| u \|_2(\mathcal{O}_{R-1,R+2})
$$

for $\forall R > 0$ and $u \in H^2_{2,loc}(\mathcal{R})$.

Under these conditions the discreteness of non-zero eigenvalues of $H$ and the absolute continuity of the continuous spectrum of $H$ will be shown. It should be noted that, under Condition 2, the coefficient $\gamma$ may have a high singularity of some type.

In §3, the conditions will be slightly weakened.

In §4, a short remark on the absence of positive eigenvalues will be given.

There are some possibilities that we can extend the arguments in §§1-2 to include the perturbations which are not necessarily short range. For example, we can put

$$
B = B_1 + B_2,
$$

$B_1$ is real and short range in the above sense,

$B_2$ is multiplication by a real $C^1$-function $q_2$, and

$$
|q_2| + |\nabla q_2| \leq C|x|^{-h}, \quad \frac{\partial q_2}{\partial x} \leq C|x|^{-1-h}
$$

for some $C, h > 0$.

But, in order to avoid some tedious calculations, we shall omit...
discussions about these possibilities. A full treatment will be given in [12].

List of notations

$\Omega$: an unbounded domain in n-dimensional Euclidean space

$\mathbb{R}^n = \{ x \mid x=(x_1, \ldots, x_n) \} \ (n \geq 2)$. 

$\Gamma$: the boundary of $\Omega$; $\Gamma = \partial \Omega$

$\bar{\Omega} = \Omega \cup \Gamma$, $\Omega_{R_1, R_2} = \Omega \cap \{ x \mid R_1 < x_n < R_2 \}$, $\Gamma_{R_1, R_2} = \Gamma \cap \{ x \mid R_1 < x_n < R_2 \}$, $\mathcal{S}_R = \Omega \cap \{ x \mid x_n = R \}$.

$\Delta$: the Laplace operator; $\Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}$.

$\phi$: a smooth function which depends only on $x_n$;

$\phi' = \frac{d\phi}{dx_n}$, $\phi'' = \frac{d^2\phi}{dx_n^2}$, $\ldots$.

$\gamma_n$: the angle between the $x_n$-axis and outer normal.

$|x| = (x_1^2 + x_n^2)^{1/2}$

$|\nabla u| = (\sum_{j=1}^{n} |\partial u / \partial x_j|^2)^{1/2}$

$H_{m,s}$: the space of functions with the norm

$$\| u \|_{m,s} = \left( \int_{\Omega} (1+|x|^2)^s \sum_{|a| \leq m} |D^a u|^2 dx \right)^{1/2}$$

$D^a = \partial^{a_1} / \partial x_1^{a_1} \ldots \partial x_n^{a_n}$, $|a| = a_1 + \ldots + a_n$.

$H_m(\Omega)$: the space of functions with the norm

$$\| u \|_{m}(\Omega) = \left( \int_{\Omega} \sum_{|a| \leq m} |D^a u|^2 dx \right)^{1/2}$$

$L^2(\Omega) = H^0(\Omega) = H^0_{0,0}$.

1. Discreteness of non-zero eigenvalues

The next equality is important for our purpose (c.f. [3] chap.4).

Let $u \in H_{2, \text{loc}}(\Omega)$ be a solution of the equation $(-\Delta - z)u = f$ with the boundary condition $u |_{\Gamma} = 0$ ($z = \sigma + i\tau$, a complex number), then

-3-
\[ 2 \int_{\Omega_{R_1 R_2}} \phi' \left| \frac{\partial u}{\partial x} \right|^2 \, dx = 2 \Re \int_{\Omega_{R_1 R_2}} \phi \frac{\partial u}{\partial x} \overline{f} \, dx + \frac{1}{2} \int_{\partial R_1 R_2} \phi u' |u|^2 \, d\tau \]

\[ + \Re \int_{\partial R_1 R_2} \phi' f \overline{u} \, d\tau + 2 \Im \int_{\partial R_1 R_2} \phi \frac{\partial u}{\partial x} \overline{u} \, d\tau \]

\[ + \left\{ \phi \left| \nabla u \right|^2 \cos \gamma_n \, dS + \left\{ \phi \left| \frac{\partial u}{\partial x} \right|^2 \right. - \left. \left| \nabla u \right|^2 \right| \right\} \frac{\partial u}{\partial x} \overline{u} \left. \right| dS \]

\[ + \Re \left\{ \left\{ - \left\{ \phi' \frac{\partial u}{\partial x} \overline{u} \, dS - \frac{1}{2} \left\{ \phi' \left| \nabla u \right|^2 \, dS \right. \right. \right. \right\} \left. \right. \right\} \]

(1.1)

**Theorem 1.1.** Each non-zero eigenvalue of \( \mathcal{H} \) has a finite multiplicity. All of the non-zero eigenvalues of \( \mathcal{H} \) make a discrete set on the real axis.

**Proof.** Let \( u \in D(\mathcal{H}) \), \( \mathcal{H}u = \lambda u \), \( \lambda \in [a, b] \), and \( 0 \notin [a, b] \). It suffices to show that \( u \in H^2_{2, \varepsilon} \) for some \( \varepsilon > 0 \), and

\[ (1.2) \quad \| u \|_{2, \varepsilon} \leq C \| u \|_{0, 0} \]

where \( \varepsilon \) and \( C \) may depend on the interval \([a, b]\), but are independent of \( \lambda \) and \( u \). Indeed, if we can show (1.2), the assertion of the theorem follows from the Rellich's compactness theorem (see the proof of Th. 3.1 in [2]).

Note at first that

\[ (1.3) \quad \| u \|_{2, 0} \leq \text{const.} \| u \|_{0, 0} \]

This follows from Condition 2 not directly, but easily.

For any \( w \in H^2_{2, s} \) \((s = a \text{ a real number})\),

\[ (1.4) \quad \| Bw \|_{0, 1+h+s} \leq \text{const.} \| w \|_{2, s} \]
This follows from Condition 2-(iii).

Note that \((-\Delta - \lambda)u = -Bu\), and \(u|_{R_1} = 0\). Put \(\phi = 1\) in (1.1) and let \(R_\infty \to \infty\) (along suitable sequence if necessary), then we have

\[
0 = -2 \Re \int_{\Omega_{R_\infty}} \frac{\partial u}{\partial x_n} \overline{Bu} \, dx + \int_{\Omega_{R_\infty}} |\nabla u|^2 \cos \gamma_n \, dS - \int_{\Sigma_{R_1}} \left( 2 \left| \frac{\partial u}{\partial x_n} \right|^2 - |\nabla u|^2 + \lambda |u|^2 \right) dS
\]

for any \(R_1 > 0\). In view of \(\cos \gamma_n < 0\) (Condition 1), we see

\[
\int_{\Sigma_{R_1}} \left( 2 \left| \frac{\partial u}{\partial x_n} \right|^2 - |\nabla u|^2 + \lambda |u|^2 \right) dS \leq -2 \Re \int_{\partial \Omega_{R_\infty}} \frac{\partial u}{\partial x_n} \overline{Bu} \, dx.
\]

In consequence of \(Bu \in H_0, l + h\),

\[
\lim_{R \to \infty} \left( \int_{\Omega_{R_\infty}} \left| \frac{\partial u}{\partial x_n} \right|^2 \overline{Bu} \right) \, dx = 0.
\]

From (1.6) and (1.7),

\[
\limsup_{R \to \infty} \int_{\Sigma_{R_1}} \left( 2 \left| \frac{\partial u}{\partial x_n} \right|^2 - |\nabla u|^2 + \lambda |u|^2 \right) dS \leq 0.
\]

Return to (1.1), and put \(\phi = x_n^{1+2\varepsilon} \), \(0 < 2\varepsilon \leq \min\{h, 1\}\), \(R_1 = 0\). (1.8) allows us to take the limit \(R_\infty \to \infty\), and we have

\[
2 (1 + 2\varepsilon) \int_{\Omega} x_n^{2\varepsilon} \left| \frac{\partial u}{\partial x_n} \right|^2 \, dx \leq -2 \Re \int_{\Omega} x_n^{1+2\varepsilon} \frac{\partial u}{\partial x_n} \overline{Bu} \, dx + \Re \int_{\Omega} x_n^{2\varepsilon} \left( 2 \frac{\partial u}{\partial x_n} \overline{Bu} \right) \, dx
\]

\[
+ (1 + 2\varepsilon) \varepsilon (2\varepsilon - 1) \int_{\Omega} x_n^{-2+2\varepsilon} |u|^2 \, dx.
\]

From this we can show

\[
\int_{\Omega} x_n^{2\varepsilon} \left| \frac{\partial u}{\partial x_n} \right|^2 \, dx \leq \text{const.} \, Bu \, \|u\|_{H_0, l + 4\varepsilon}^2
\]

Because of Condition 1, we have some latitude in choice of the direction of \(x_n\)-axis. So (1.10) means
(1.11) \[ \int_{\Omega} x_n^2 |v_n|^2 \, dx \leq \text{const.} \|Bu\|_{0,1+\varepsilon}^2 \]

Multiply the equation \((-\Delta - \lambda)u = -Bu\) by \(x_n^2 u\) and integrate over \(\Omega\), then we have by partial integration and (1.11)

(1.12) \[ \int_{\Omega} x_n^2 |u|^2 \, dx \leq \text{const.} \|Bu\|_{0,1+\varepsilon}^2 \]

From (1.11) and (1.12),

(1.13) \[ \|u\|_{1,\varepsilon} \leq \text{const.} \|Bu\|_{0,1+\varepsilon} \]

If we combine this with the inequality*

(1.14) \[ \|w\|_{2,s} \leq \text{const.} (\|Lw\|_{0,s} + \|w\|_{1,s}) \quad (s \text{ real}, w \in H^{2,s}_{\varepsilon}, w_{\|}=0), \]

we have

(1.15) \[ \|u\|_{2,\varepsilon} \leq \text{const.} \|Bu\|_{0,1+\varepsilon} \]

(1.3), (1.4) and (1.15) give

\[ \|u\|_{2,\varepsilon} \leq \text{const.} \|u\|_{2,\varepsilon-h} \leq \text{const.} \|u\|_{0,0} \]

Thus we can conclude (1.2). \(\square\)

Theorem 1.2. Let \(\lambda\) be a non-zero real number, \(L = \lambda u\), \(u_{\|}=0\), and

\(u \in H^{2,s}_{\varepsilon}\) for some \(s > -\frac{1+h}{2}\). Then \(u\) is in \(H^{2,\infty} = \bigcap_{-\infty < t < \infty} H^{2,t}_{\varepsilon}\).

The proof of this theorem is based on the facts that (1.5) is valid also for \(u \in H^{2,s}_{\varepsilon}\) \((s > -\frac{1+h}{2})\), and that

\[ \lim_{R \to \infty} \frac{1}{R} \int_{-R_{\varepsilon},0} |\frac{\partial}{\partial x_n} u| \, dx = 0. \]

* This inequality follows from Condition 2-(ii).
2. Absolute continuity of the continuous spectrum

Let \( H_0 \) be the self-adjoint realization of \(- \Delta\) defined as follows;
\[ D(H_0) = D(H) = \{ u \in L^2(\Omega) \mid u \in H_1(\Omega), u_{\mid \Gamma} = 0, \Delta u \in L^2(\Omega) \} , \quad H_0 u = -\Delta u \text{ for } u \in D(H_0). \]
And put \( R_0(z) = (H_0 - z)^{-1} (\text{Im } z \neq 0). \)

**Theorem 2.1.** Let \( \varepsilon > 0 \) and \( K \) be a compact set in the complex plane. If the origin is not in \( K \), there exists a positive constant \( C = C(K, \varepsilon) \) such that
\[ \| R_0(z) f \|_2 , \frac{1 + \varepsilon}{2} \leq C \| f \|_0 , \frac{1 + \varepsilon}{2} \]
for any \( f \in L^2_0, \frac{1 + \varepsilon}{2} \) and \( z \in K \) with \( \text{Im } z \neq 0 \).

The proof also can be carried out by use of (1.1). But we omit it because it is somewhat lengthy and partly overlaps with the proof of Theorem 1.1.

**Lemma 2.2.** Let \( 0 < \varepsilon < h \) and \( K \) be a compact set in the complex plane. If the origin is not in \( K \), there are \( r_0 = r_0(K, \varepsilon) > 0 \) and \( C = C(K, \varepsilon) > 0 \) such that
\[ \| BR(z) f \|_0 , \frac{1 + \varepsilon}{2} \leq C \| f \|_0 , \frac{1 + \varepsilon}{2} + \| R(z) f \|_1 (r_0, r_0) \}
for any \( f \in L^2_0, \frac{1 + \varepsilon}{2} \) and \( z \in K \) with \( \text{Im } z \neq 0 \). Here \( R(z) = (H - z)^{-1} \).

This lemma is obtained from Theorem 2.1 and the inequality
\[ \| B w \|_0 , l + h + s \leq \text{const.} \left( \| (L-z) w \|_0 , s + \| w \|_1 , s \right) (w \in L^2_0, s , w_{\mid \Gamma} = 0) \]
by estimating
\[ \int_{\Omega_{k, \infty}} \left( 1 + |x|^2 \right)^{-\frac{l + \varepsilon}{2}} \sum_{i=1}^{n} |\nabla^2 u|^2 \, dx \]
\( (u = R(z) f = R_0(z) f - R_0(z) B u) \).
In consequence of the above lemma and Theorem 2.1 we have

**Lemma 2.3.** Let ε, K as above, then there are \( r_0 = r_0(K, \varepsilon) \) and 
\[ C = C(K, \varepsilon) \] such that

\[ (2.4) \quad \| R(z)f \|_{L^2, -\frac{1+\varepsilon}{2}} \leq C \left\{ \| f \|_{0, \frac{1+\varepsilon}{2}} + \| R(z)f \|_1(Q_0, r_0) \right\} \]

for any \( f \in H_{0, \frac{1+\varepsilon}{2}} \) and \( z \in K \) with \( \text{Im} \ z \neq 0 \).

**Theorem 2.4.** Let K be a compact set in the complex plane. If K contains neither the origin nor eigenvalues of \( H \), then, for any \( \varepsilon > 0 \), one can take a constant \( C = C(K, \varepsilon) > 0 \) such that the inequality

\[ (2.5) \quad \| R(z)f \|_{L^2, -\frac{1+\varepsilon}{2}} \leq C \| f \|_{0, \frac{1+\varepsilon}{2}} \]

holds for any \( f \in H_{0, \frac{1+\varepsilon}{2}} \) and \( z \in K \) with \( \text{Im} \ z \neq 0 \).

**Proof.** Without loss of generality, we may assume \( 0 < \varepsilon < \chi \). If \( |\text{Im} \ z| > \delta > 0 \), there is a constant \( C_\delta \) which may depend on \( \delta \), and

\[ \| R(z)f \|_1(Q_0, r_0) \leq C_\delta \| f \|_{0, \frac{1+\varepsilon}{2}} \]

According to Lemma 2.3, to prove the theorem, it suffices to show the assertion: if \( z \in K \) and \( \text{Im} \ z \neq 0 \), \( C_\delta \) actually can be taken independently of \( \delta > 0 \).

Suppose that this is not true, then there would be sequences \( \{ z_N \} \) and \( \{ u_N \} \) which satisfy the following \( (2.6) \).

\[ (2.6) \quad \begin{cases} 
\text{Im} \ z_N \neq 0, \quad \{ z_N \}, \{ z_N \} \text{ converges to a real number } \alpha \in K. \\
\{ u_N \} \subset D(K), \quad \| u_N \|_1(Q_0, r_0) = 1, \quad f_N = (H-z)u_N \text{ converges strongly to } 0 \text{ in } H_{0, \frac{1+\varepsilon}{2}}.
\end{cases} \]

We must show that \( (2.6) \) leads to a contradiction. Note that

\[ (-\Delta - z_N)u_N = f_N - B_N, \quad \| B_N \|_0, (1+\varepsilon)/2 < \text{const.} (\| f_N \|_0, (1+\varepsilon)/2 + 1) \] by
Lemma 2.2, so \((-A_{\lambda})u_{\lambda}\) is bounded in \(H_{0},(1+\varepsilon)/2\). Consequently, 
\(\{u_{\lambda}\}\) is bounded in \(H_{2},-(1+\varepsilon)/2\) (Theorem 2.1). Let us take \(\varepsilon'\) such 
that \(\varepsilon<\varepsilon'<2\). Then, taking a subsequences if necessary, we may assume 
that \(\{u_{\lambda}\}\) converges strongly to a function \(u_{0}\) in \(H_{1},-(1+\varepsilon)/2\) (Rellich's 
compactness theorem). Moreover we can see

\[
\begin{align*}
\|u_{0}\|_{L} & = 1 \\
\left(L-\varphi\right)u_{0} & = 0 \\
u_{0}|_{\partial} & = 0
\end{align*}
\]

(2.7)

In view of (1.14), \(u_{0}\in H_{2},-\varepsilon'\), and so \(u_{0}\in H_{2,\infty}\) (Theorem 1.2). This 
contradicts the assumption that \(K\) contains no eigenvalues of \(L\).

(Q.E.D.)

**Theorem 2.5.** The continuous spectrum of \(L\) is absolutely 
continuous.

This is a consequence of Theorem 2.4. See, for example, [3] 
Chap.1 or [8] Appendix C.

3. An extension

Let us suppose \(L'\) to be a differential operator in a domain 
\(\Omega'\subset R^{N}\). We make the following conditions.

(i) There is a diffeomorphism \(\Omega'\rightarrow\Omega\): \(x_{1}=x_{1}(X_{1},\ldots,X_{n}),\ldots,x_{n}= 
x_{n}(X_{1},\ldots,X_{n})\), and \(\Omega\) satisfies Condition 1.
(ii) If we define the transform of functions as follows;

\((\bar{u}, u, \tilde{u})(x) = u(x(X)) \cdot J(X) |^{1/2} \), where \(J(X)\) is the Jacobian

\(\Delta(x_1, \ldots, x_n) / \Delta(x_1, \ldots, x_n)\). Then \(L = U^{-1}L'U\) satisfies Condition \(\mathcal{C}\).

In this situation, \(H' = UHU^{-1}\) is a self-adjoint realization of \(L'\)
in \(L^2(\Omega')\) under the Dirichlet boundary condition, and we can replace
\(H\) by \(H'\) in Theorem 1.1 and 2.5. This follows from the fact that \(U\)
defines a unitary transform \(L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})\).

4. A remark on the absence of positive eigenvalues

In this section, we write \(L\) in the form

\[ L = \sum_{j,k=1}^{r} \left( \frac{1}{2} \frac{\partial^2}{\partial x_j^2} + b_j \right) a_{jk} \left( \frac{1}{2} \frac{\partial^2}{\partial x_k^2} + b_k \right) + q \]

Let us assume that there exists a positive constant \(r_0\) such
that the following conditions are satisfied for \(|x| \geq r_0\). (a) \(a_{jk}\)
and \(b_j\) are real valued \(C^1\)-functions and \(a_{jk} = a_{kj}\). (b) \(\exists \epsilon > 0\) such
that \(c|\xi|^2 \leq \sum a_{jk} \xi_j^2 \xi_k^2 \leq \frac{1}{\epsilon}|\xi|^2\) for any complex vector \((\xi_1, \ldots, \xi_n)\).
(c) \(a_{jk} / \partial x_k = 0(|x|^{-1})\). (d) \(a_{jk} \rightarrow b_{jk} (|x| \rightarrow \infty)\). (e) \(\frac{\partial^2}{\partial x_j^2} - \frac{\partial^2}{\partial x_k^2} = o(|x|)\)
(f) \(q(x) = o(|x|^{-1})\). (g) The unique continuation property holds.
(h) \(\sum a_{jk} x_j x_k \leq 0\) for each boundary point \(x (|x| > r_0)\), where
\((\nu_1, \ldots, \nu_n)\) is the outer normal to \(\Gamma\) at \(x\).

The conditions (a)-(g) are nothing but those which are proposed
by Ikebe and Uchiyama in [5].

**Theorem 4.1.** If \(u\) is a not identically vanishing solution of
\(Lu = \lambda u\ (\lambda > 0)\) in \(\Omega \cap \{x : |x| > r_0\}\) and \(u|_{\Gamma \cap \{x : |x| > r_0\}} = 0\), then we have for
any \(\epsilon > 0\)

\[ \lim_{r \to \infty} \int_{\Omega} \left( \sum a_{jk} \hat{x}_j \hat{x}_k \right) (\lambda |u|^2 + |\Sigma a_{jk} (D_{jk} u) \hat{x}_k|^2) dS = \infty, \]

-10-
where $S(r) = \Omega \cap \{x \mid |x| = r\}$, $\hat{X}_j = \frac{x_j}{|x|}$, and $\partial_x^2 = \frac{1}{2x_j} \partial_x x_j$. The proof can be carried out in the same way of [5]. Because of the condition (h), the unboundedness of $\Gamma$ brings about no essential difficulties.

If $L$ enjoys the unique continuation property on the whole of $\Omega$, the absence of positive eigenvalues of $H$ follows directly from this theorem.

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