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Spectra of Elliptic Operators in a Domain
with Unbounded Boundary

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0. Introduction

We shall study a spectral property of a 2nd order partial
differential operator

\[ L = -\Delta + \sum_{j,k=1}^{n} \alpha_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \beta_{j} \frac{\partial}{\partial x_j} + \gamma \]

in a domain \( \Omega \subset \mathbb{R}^n \) (n \( \geq 2 \)).

In case that \( \Omega = \mathbb{R}^n \) or \( \Omega \) = the exterior of a bounded hypersurface, this subject have been discussed by many authors under various conditions on \( L \). As for the case of unbounded boundary, we can refer to [1] [3,4] [6,11].

In § 4 of this note, we shall make the following conditions 1 and 2. (A list of notations will be given later.)

Condition 1. \( \Omega \) is characterized by \( x_n > \varphi(\vec{x}) \) (\( \vec{x} = (x_1, \ldots, x_{n-1}) \)), where \( \varphi(\vec{x}) \) is a single valued \( C^2 \)-function on \( \mathbb{R}^{n-1} \), and there are positive numbers \( M_1 \) and \( M_2 \) such that

\[ \varphi(\vec{x}) > M_1 (x_1^2 + \ldots + x_{n-1}^2)^{1/2}, \ |3\varphi/3x_j| < M_2, \ |3^2\varphi/3x_j 3x_k| < M_2 \]

(\( j, k = 1, \ldots, n-1 \)).

Condition 2. Put

\[ B = \sum_{j,k=1}^{n} \alpha_{jk} \frac{\partial^2}{\partial x_j \partial x_k} + \sum_{j=1}^{n} \beta_{j} \frac{\partial}{\partial x_j} + \gamma \]

* 1-5-1, Chofugaoka, Chofu-shi, Tokyo 182.
then $B$ is real and short range in the following sense.

1. If we put

$$D(H) = \{ u \in L^2(\Omega) \mid u \in H^1(\Omega), \ u|_{\Gamma} = 0, \ \Delta u \in L^2(\Omega) \}$$

and $Hu = Lu$ for $u \in D(H)$, then $H$ is a self-adjoint operator in $L^2(\Omega)$.

2. The inequality

$$\|u\|_2(Q, R+1) \leq C_1 \left\{ \|(-A+B)u\|_0(Q, R-1, R+2) + \|u\|_1(Q, R-1, R+2) \right\}$$

holds for $u \in H^2_{2, \text{loc}}(\Omega)$ with $u|_{\Gamma} = 0$, where $C_1$ is independent of $R$.

3. There are positive numbers $h$ and $C_2$ such that

$$\|Bu\|_0(Q, R+1) \leq C_2 R^{-h} \|u\|_2(Q, R-1, R+2)$$

for $\forall R > 0$ and $u \in H^2_{2, \text{loc}}(\Omega)$.

Under these conditions the discreteness of non-zero eigenvalues of $H$ and the absolute continuity of the continuous spectrum of $H$ will be shown. It should be noted that, under Condition 2, the coefficient $\gamma$ may have a high singularity of some type.

In §3, the conditions will be slightly weakened.

In §4, a short remark on the absence of positive eigenvalues will be given.

There are some possibilities that we can extend the arguments in §§1-2 to include the perturbations which are not necessarily short range. For example, we can put

$$B = B_1 + B_2,$$

$B_1$ is real and short range in the above sense,

$B_2$ is multiplication by a real $C^1$-function $q_2$, and

$$|q_2| + |\nabla q_2| \leq C|\mathbf{x}|^{-h}, \quad \frac{\partial q_2}{\partial |\mathbf{x}|} < C|\mathbf{x}|^{-1-h}$$

for some $C, h > 0$.

But, in order to avoid some tedious calculations, we shall omit
discussions about these possibilities. A full treatment will be given in [12].

List of notations

\( \Omega \): an unbounded domain in n-dimensional Euclidean space

\( \mathbb{R}^n = \{ x \mid x = (x_1, \ldots, x_n) \} \quad (n \geq 2) \).

\( \Gamma \): the boundary of \( \Omega \); \( \Gamma = \partial \Omega \)

\( \tilde{\Omega} = \Omega \cup \Gamma \), \( \Omega_{R_1, R_2} = \Omega \cap \{ x \mid R_1 < x_n < R_2 \} \), \( \Gamma_{R_1, R_2} = \Gamma \cap \{ x \mid R_1 < x_n < R_2 \} \)

\( S_{R_2} = \Omega_{R_1} \{ x \mid x_n = R_1 \} \).

\( \Delta \): the Laplace operator; \( \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \).

\( \phi \): a smooth function which depends only on \( x_n \);

\( \phi' = d\phi/dx_n \), \( \phi'' = d^2\phi/dx_n^2 \), \( \ldots \).

\( \gamma \): the angle between \( x_n \)-axis and outer normal.

\( |x| = (x_1^2 + x_n^2)^{1/2} \)

\( |\nabla u| = (\sum_{j=1}^{n} |\partial u/\partial x_j|^2)^{1/2} \)

\( H_{m,s} \): the space of functions with the norm

\[ \|u\|_{m,s} = \left( \int_{\Omega} (1+|x|^2)^s (\sum_{|\alpha| \leq m} |D^\alpha u|^2 dx) \right)^{1/2} \]

\( D^\alpha = \partial^{\alpha_1}_1 \ldots \partial^{\alpha_n}_n \), \( |\alpha| = \alpha_1 + \cdots + \alpha_n \).

\( H_m(G) \): the space of functions with the norm

\[ \|u\|_{m,G} = \left( \sum_{G_{\text{h}} \subseteq \Omega} (\sum_{|\alpha| \leq m} |D^\alpha u|^2 dx) \right)^{1/2} \]

\( L^2(\Omega) = H_0(\Omega) = H_{0,0} \).

1. Discreteness of non-zero eigenvalues

The next equality is important for our purpose (c.f. [3] chap.4).

Let \( u \in H^2_{\text{loc}}(\Omega) \) be a solution of the equation \((-\Delta - \varepsilon)u = f\) with the boundary condition \( u_{|\Gamma} = 0 \) (\( \varepsilon = \sigma + i\tau \), a complex number), then
\[ 2 \int_{\Omega_{R_0,R_2}} \phi \left( \frac{\partial u}{\partial x_n} \right)^2 dx = 2 \int_{\Omega_{R_0,R_2}} \phi \frac{\partial u}{\partial x_n} \bar{u} dx + \frac{1}{2} \int_{\Gamma_{R_0,R_2}} \phi \left| \frac{\partial u}{\partial x_n} \right|^2 \sigma d\sigma \]

\[ + \left( \phi | \nu u|^2 \cos \gamma_n d\sigma \right)_{\Gamma_{R_0,R_2}} + \int_{\Gamma_{R_0,R_2}} \phi \left( \frac{\partial^2 u}{\partial x_n^2} \right)^2 \cos \gamma_n d\sigma \]

\[ + \int_{\Gamma_{R_0,R_2}} \phi \left( \frac{\partial u}{\partial x_n} \right)^2 \sigma d\sigma + \frac{1}{2} \int_{\Gamma_{R_0,R_2}} \phi \left( \frac{\partial u}{\partial x_n} \right)^2 \sigma d\sigma \]

\[ + \int_{\Omega_{R_0,R_2}} \phi \left( \frac{\partial u}{\partial x_n} \right)^2 \sigma d\sigma \]

(1.1)

**Theorem 1.1.** Each non-zero eigenvalue of \( \mathcal{H} \) has a finite multiplicity. All of the non-zero eigenvalues of \( \mathcal{H} \) make a discrete set on the real axis.

**Proof.** Let \( u \in D(\mathcal{H}), \mathcal{H}u = \lambda u, \lambda \in [a,b], \) and \( 0 \in [a,b] \). It suffices to show that \( u \in H_{2,\varepsilon} \) for some \( \varepsilon > 0 \), and

\[ \|u\|_{2,\varepsilon}^2 \leq C \|u\|_{0,0}^2, \]

where \( \varepsilon \) and \( C \) may depend on the interval \([a,b]\), but are independent of \( \lambda \) and \( u \). Indeed, if we can show (1.2), the assertion of the theorem follows from the Rellich's compactness theorem (see the proof of Th. 3.1 in [2]).

Note at first that

\[ \|u\|_{2,0}^2 \leq \text{const.} \|u\|_{0,0}^2. \]

This follows from Condition 2 not directly, but easily.

For any \( w \in H_{2,s} \) (\( s \) a real number),

\[ \|Bw\|_{0,1+h+s} \leq \text{const.} \|w\|_{2,s}. \]

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This follows from Condition 2-(iii).

Note that \((-\Delta - \lambda)u = Bu\), and \(u\mid_{\Omega} = 0\). Put \(\phi \equiv 1\) in (1.1) and let \(R_\perp \to \infty\) (along suitable sequence if necessary), then we have

\[
0 = -2 \mathcal{R} \left\{ \frac{\partial u}{\partial x_n} \quad Bu \right\} d\chi + \int_{\partial R_\perp, \infty} (2|u^2| - |\nabla u| + |\lambda w|^2) d\Sigma
\]

(1.5)

for any \(R_\perp > 0\). In view of \(\cos \gamma_n < 0\) (Condition 1), we see

\[
\int_{\partial R_\perp} \left( \frac{2}{\partial x_n} - |\nabla u|^2 + \lambda |w|^2 \right) d\Sigma \leq -2 \mathcal{R} \left\{ \frac{\partial u}{\partial x_n} \quad Bu \right\} d\chi,
\]

(1.6)

In consequence of \(Bu \in H_{0,1+h}\),

\[
\lim_{R \to \infty} \mathcal{R} \left\{ \frac{\partial u}{\partial x_n} \quad Bu \right\} d\chi = 0.
\]

(1.7)

From (1.6) and (1.7),

\[
\lim_{R \to \infty} \mathcal{R} \left\{ \frac{2}{\partial x_n} - |\nabla u|^2 + \lambda |w|^2 \right\} d\Sigma \leq 0.
\]

(1.8)

Return to (1.1), and put \(\phi = x_n^{1+2\varepsilon}\), \(0 < 2\varepsilon \leq \min\{h, 1\}\), \(R_\perp = 0\).

(1.8) allows us to take the limit \(R_\perp \to \infty\), and we have

\[
2(1+2\varepsilon) \int_{\partial R_\perp} x_n^{2\varepsilon} \left( \frac{\partial u}{\partial x_n} \right)^2 d\chi \leq -2 \mathcal{R} \left\{ \frac{\partial u}{\partial x_n} \quad Bu \right\} d\chi + \mathcal{R} \left\{ (1+2\varepsilon) \left( x_n^{1+2\varepsilon} \frac{\partial u}{\partial x_n} \right) d\chi
\]

(1.9)

\[
+ (1+2\varepsilon) \varepsilon (2\varepsilon - 1) \int_{\partial R_\perp} x_n^{2\varepsilon - 22\varepsilon} |u|^2 d\chi
\]

From this we can show

\[
\int_{\partial R_\perp} x_n^{2\varepsilon} \left( \frac{\partial u}{\partial x_n} \right)^2 d\chi \leq \text{const.} B u \|_{L^2_{0,1+\varepsilon}}^2
\]

(1.10)

Because of Condition 1, we have some latitude in choice of the direction of \(x_n\)-axis. So (1.10) means
Multiply the equation \((-\Delta - \lambda)u = Bu\) by \(x_n^2 u\) and integrate over \(\Omega\), then we have by partial integration and (1.11)

\[
\int_{\Omega} x_n^2 |u|^2 \, dx \leq \text{const.} \| Bu \|_{0, 1+\varepsilon}^2
\]

From (1.11) and (1.12),

\[
\| u \|_{1, \varepsilon} \leq \text{const.} \| Bu \|_{0, 1+\varepsilon}.
\]

If we combine this with the inequality

\[
\| u \|_{2, \varepsilon} \leq \text{const.} (\| Lw \|_{0, s} + \| w \|_{1, s}) \quad (s; \text{real}, w \in H_{2, s}, w_\| = 0),
\]

we have

\[
\| u \|_{2, \varepsilon} \leq \text{const.} \| Bu \|_{0, 1+\varepsilon}
\]

(1.3), (1.4) and (1.5) give

\[
\| u \|_{2, \varepsilon} \leq \text{const.} \| u \|_{2, \varepsilon-h} \leq \text{const.} \| u \|_{0, 0}.
\]

Thus we can conclude (1.2). (Q.E.D.)

**Theorem 1.2.** Let \(\lambda\) be a non-zero real number, \(Lu = \lambda u, u_\| = 0,\) and \(u \in H_{2, s}\) for some \(s > -\frac{1+h}{2}\). Then \(u\) is in \(H_{2, \infty} = \bigcap_{t > 0} H_{2, t}\).

The proof of this theorem is based on the facts that (1.5) is valid also for \(u \in H_{2, s}, (s > -\frac{1+h}{2})\), and that

\[
\lim_{R \to \infty} \int_{R^{2+h}} \int_{R_{s, \infty}} \frac{2u}{2s} |\nabla (x_u d x = 0.
\]

* This inequality follows from Condition (ii).
2. Absolute continuity of the continuous spectrum

Let $H_0$ be the self-adjoint realization of $-\Delta$ defined as follows;
$D(H_0) = D(H) = \{ u \in L^2(\Omega) \mid u|_{\partial \Omega} = 0, \Delta u \in L^2(\Omega) \}$, $H_0 u = -\Delta u$ for $u \in D(H_0)$. And put $R_0(z) = (H_0 - z)^{-1}$ (Im $z \neq 0$).

**Theorem 2.1.** Let $\varepsilon > 0$ and $K$ be a compact set in the complex plane. If the origin is not in $K$, there exists a positive constant $C = C(K, \varepsilon)$ such that

$$
\|(R_0(z) f)_2, - \frac{l+\varepsilon}{2} \| \leq C\|f\|_0, \frac{l+\varepsilon}{2}
$$

for any $f \in H_0, \frac{l+\varepsilon}{2}$ and $z \in K$ with Im $z \neq 0$.

The proof also can be carried out by use of (1.1). But we omit it because it is somewhat lengthy and partly overlaps with the proof of Theorem 1.1.

**Lemma 2.2.** Let $0 < \varepsilon < h$ and $K$ be a compact set in the complex plane. If the origin is not in $K$, there are $r_0 = r_0(K, \varepsilon) > 0$ and $C = C(K, \varepsilon) > 0$ such that

$$
\|BR(z)f\|_0, \frac{l+\varepsilon}{2} \leq C\|f\|_0, \frac{l+\varepsilon}{2} + \|R(z)f\|_1(0, r_0)
$$

for any $f \in H_0, \frac{l+\varepsilon}{2}$ and $z \in K$ with Im $z \neq 0$. Here $R(z) = (H - z)^{-1}$.

This lemma is obtained from Theorem 2.1 and the inequality

$$
\|Bw\|_0, l+h+s \leq \text{const.} (\|(L-z)w\|_0, s + \|w\|_1, s) (w \in H_2, s, w_i = 0)
$$

by estimating

$$
\int_{\Omega} (1 + |x|^2)^{-l+\varepsilon} \sum_{|i| \leq l} |\nabla^i u|^2 \, dx
$$

($u = R(z)f = R_0(z)f - R_0(z)Bu$).
In consequence of the above lemma and Theorem 2.1 we have

**Lemma 2.3.** Let \( \varepsilon, K \) as above, then there are \( r_0 = r_0(K, \varepsilon) \) and \( C = C(K, \varepsilon) \) such that

\[
\| R(z)f \|_{2, -\frac{1+\varepsilon}{2}} \leq C \left\{ \| f \|_{0, \frac{1+\varepsilon}{2}} + \| R(z)f \|_{1, (0, r_0)} \right\}
\]

for any \( f \in H_{0, \frac{1+\varepsilon}{2}} \) and \( z \in K \) with \( \text{Im } z \neq 0 \).

**Theorem 2.4.** Let \( K \) be a compact set in the complex plane. If \( K \) contains neither the origin nor eigenvalues of \( H \), then, for any \( \varepsilon > 0 \), one can take a constant \( C = C(\varepsilon, \delta, K) > 0 \) such that the inequality

\[
\| R(z)f \|_{2, -\frac{1+\varepsilon}{2}} \leq C \| f \|_{0, \frac{1+\varepsilon}{2}}
\]

holds for any \( f \in H_{0, \frac{1+\varepsilon}{2}} \) and \( z \in K \) with \( \text{Im } z \neq 0 \).

**Proof.** Without loss of generality, we may assume \( 0 < \varepsilon < \delta \). If \( |\text{Im } z| > \delta > 0 \), there is a constant \( C_0 \) which may depend on \( \delta \), and

\[
\| R(z)f \|_{1, (0, r_0)} \leq C_0 \| f \|_{0, \frac{1+\varepsilon}{2}}
\]

According to Lemma 2.3, to prove the theorem, it suffices to show the assertion: if \( z \in K \) and \( z \neq 0 \), \( C_0 \) actually can be taken independently of \( \delta > 0 \).

Suppose that this is not true, then there would be sequences \( \{ z_N \} \) and \( \{ u_N \} \) which satisfy the following (2.6).

\[
\begin{align*}
\text{Im } z_N &\neq 0, \{ z_N \} \subset K, \{ z_N \} \text{ converges to a real number } \alpha \in K, \\
\{ u_N \} &\subset D(H), \| u_N \|_{1, (0, r_0)} = 1, f_N = (H-z)u_N \text{ converges strongly to 0 in } H_{0, \frac{1+\varepsilon}{2}}.
\end{align*}
\]

We must show that (2.6) leads to a contradiction. Note that

\[
(-\Delta - z_N)u_N = f_N - B_N. \quad \| B_N \|_{0, (1+\varepsilon)/\varepsilon} \leq \text{const.} \left( \| f_N \|_{0, (1+\varepsilon)/\varepsilon} + 1 \right)
\]

by

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Lemma 2.2, so \((-\Delta - z_N)u_N\) is bounded in $H_0, (1+\varepsilon)/2$ . Consequently, 
\(\{u_N\}\) is bounded in $H, (1+\varepsilon)/2$ (Theorem 2.1). Let us take $\varepsilon'$ such 
that $\varepsilon < \varepsilon' < h$. Then, taking a subsequences if necessary, we may assume 
that $\{u_N\}$ converges strongly to a function $u_0$ in $H, -(1+\varepsilon)/2$ (Rellich's 
compactness theorem). Moreover we can see 
\[
\begin{cases}
\|u_0\|_1(\Omega, r_0) = 1 \\
(L - \mathcal{Q})u_0 = 0 \\
u_0|_{\partial \Omega} = 0
\end{cases}
\] 
(2.7) 
In view of (1.14), $u_0 \notin H, -1+\varepsilon'$, and so $u_0 \notin H, \infty$ (Theorem 1.2). This 
contradicts the assumption that $K$ contains no eigenvalues of $H$. 
(Q.E.D.) 

Theorem 2.5. The continuous spectrum of $H$ is absolutely 
continuous. 

This is a consequence of Theorem 2.4. See, for example, [3] 
Chap.1 or [8] Appendix C. 

3. An extension 
Let us suppose $L'$ to be a differential operator in a domain 
$\mathfrak{g}' \subset R^N$. We make the following conditions. 
(i) There is a diffeomorphism $\mathfrak{g}' \rightarrow \mathfrak{g}: x_1 = x_1(X_1, \ldots, X_n), \ldots, 
x_n = x_n(X_1, \ldots, X_n)$, and $\mathfrak{g}$ satisfies Condition 1.
(ii) If we define the transform of functions as follows;

\((\mathcal{U}u)(X) = u(x(X)) |J(X)|^{1/2}\)

where \(J(X)\) is the Jacobian

\(\partial(x_1, \ldots, x_n)/\partial(X_1, \ldots, X_n)\). Then \(L = \mathcal{U}^* L \mathcal{U}\) satisfies Condition \(\mathcal{C}\).

In this situation, \(H' = \mathcal{U}H \mathcal{U}^{-1}\) is a self-adjoint realization of \(L'\)

in \(L^2(\Omega')\) under the Dirichlet boundary condition, and we can replace

\(H\) by \(H'\) in Theorem 1.1 and 2.5. This follows from the fact that \(\mathcal{U}\)

defines a unitary transform \(L^2(\mathcal{M}) \to L^2(\Omega')\).

4. A remark on the absence of positive eigenvalues

In this section, we write \(L\) in the form

\[ L = \sum_{j, k=1}^n \left( \frac{1}{2} \frac{\partial}{\partial x_j} + b_j \right) a_{jk} \left( \frac{1}{2} \frac{\partial}{\partial x_k} + b_k \right) + q \]

Let us assume that there exists a positive constant \(r_0\) such that the following conditions are satisfied for \(|x| \geq r_0\):

(a) \(a_{jk}\)

and \(b_j\) are real valued \(C^1\)-functions and \(a_{jk} = a_{kj}\).

(b) \(\exists c > 0\) such that \(c |\xi| \leq \sum a_{jk} \xi_j \xi_k \leq \frac{1}{c} |\xi|^2\) for any complex vector \((\xi_1, \ldots, \xi_n)\).

(c) \(\exists a_{jk}/\partial x_k = o(|x|^{-1})\). (d) \(a_{jk} \to 5_{jk} (|x| \to \infty)\). (e) \(\frac{\partial^2 \xi}{\partial x_j \partial x_k} \to 0\) as \(|x| \to r_0\).

(f) \(q(x) = o(|x|^{-1})\). (g) The unique continuation property holds.

(h) \(\sum a_{jk} x_j v_k \leq 0\) for each boundary point \(x\) (\(|x| > r_0\), where \((v_1, \ldots, v_n)\) is the outer normal to \(\Gamma\) at \(x\).

The conditions (a)-(g) are nothing but those which are proposed by Ikebe and Uchiyama in [5].

**Theorem 4.1.** If \(u\) is a not identically vanishing solution of

\(Lu = \lambda u\ (\lambda > 0)\) in \(\Omega \cap \{|x| > r_0\}\) and \(u |_{\Gamma} \cap \{|x| > r_0\} = 0\), then we have for any \(\epsilon > 0\)

\[\lim_{r \to \infty} r^\epsilon \int_{\Omega} \left( \sum a_{jk} \hat{\xi}_j \hat{\xi}_k \right) (|u|^2 + |\Sigma a_{jk} (D_j u)^2|) dS = \infty,\]

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where \( S(r) = \Omega \cap \{ x \mid |x| = r \} \), \( \hat{\mathcal{H}}_j = \frac{\mathcal{H}_j}{|x|} \), and \( \mathcal{D}_j = \frac{\hat{\mathcal{D}}_j}{2x_j} + b_j \).

The proof can be carried out in the same way of [5]. Because of the condition (h), the unboundedness of \( \Gamma \) brings about no essential difficulties.

If \( L \) enjoys the unique continuation property on the whole of \( \Omega \), the absence of positive eigenvalues of \( \mathcal{H} \) follows directly from this theorem.

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