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Spectral and scattering theory for the J-selfadjoint operators associated with the perturbed Klein-Gordon equations

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§1. Introduction

In this lecture I shall investigate the spectral properties of the generators of the Klein-Gordon type equations and then study the asymptotic behavior of the generated semigroups, i.e., scattering problems.

The perturbed Klein-Gordon equation, to which we shall apply the abstract theory to be developed in the following, is

$$(1.1)\qquad \{(\frac{\partial}{\partial t}-ib_{\circ}(x))^2+(-\sum_{j=1}^{3}(\frac{\partial}{\partial x_j}-ib_j(x))^2+m^2+q(x))\}\psi(x,t)=0,$$

where $b_{\circ}(x)$, $b_j(x)$, $\frac{\partial}{\partial x_j}b_j(x)$ and $q(x)$ are real functions which decrease to zero with the order $O(|x|^{-2-\varepsilon})$ when $|x|$ tends to infinity. The first order equation in t, which is associated with (1.1), is in an abstract form

$$(1.2)\qquad \frac{d}{dt}\begin{pmatrix} f^1(t) \\ f^2(t)\end{pmatrix}= i\begin{pmatrix} 0 & -i \\ iH & K\end{pmatrix}\begin{pmatrix} f^1(t) \\ f^2(t)\end{pmatrix}.$$

For the Klein-Gordon equation, the operators H and K in the above

equation are specialized as $H=-\sum_{j=1}^{3}(\frac{\partial}{\partial x_j}-ib_j(x))^2+m^2+q(x)-b_o(x)^2$ and $K=2b_o(x)$, and $f^1(t)$ is $\psi(x,t)$ and $f^2(t)$ is $\frac{\partial}{\partial t}\psi(x,t)$. If H is positive or K is zero, then we can construct a nice Hilbert space in which the generator of the equation (1.2) is selfadjoint (see [2],[3],[5],[9],[12],[17],[19],[23],[26] and [27]). In general we can not find out such a space a priori and the generator may have non-real spectrum (see [9],[17] and [24]). We therefore handle the equation (1.2) in the space for the unperturbed equation

$$(1.3)\qquad \frac{d}{dt}\begin{pmatrix} f_o^1 \\ f_o^2 \end{pmatrix} = i\begin{pmatrix} 0 & -i \\ iH_o & 0 \end{pmatrix}\begin{pmatrix} f_o^1 \\ f_o^2 \end{pmatrix},\ H_o \geq c>0.$$

In the case of the Klein-Gordon equation, we take $H_o= -\Delta+m^2$. We must treat a non-selfadjoint problem, while we have a nice sesqui-linear form (not necessarily positive definite) for which the generator of (1.2) is symmetric, i.e., J-selfadjoint[1]. Using the limiting absorption method, we then construct a perturbed spectral measure and invariant subspaces which reduce the equation (1.2). The above mentioned form is positive definite in the subspaces and we develope the scattering theory with two Hilbert spaces.

§2. The unperturbed equation

Let X be a Hilbert space with an inner product $(\cdot,\cdot)$ and the corresponding norm $\|\cdot\|$. Let H_o be a positive definite selfadjoint operator in X: $H_o \geq c>0$. We denote the square root

of H_o by h_o. The Hilbert space $\mathfrak{Y}_o$ is then defined as the direct sum of spaces D and X, where D is the domain of h_o, $D(h_o)$, which is a Hilbert space with an inner product $(h_o\cdot, h_o\cdot)$. We denote here and hereafter the domain and the range of an operator A by D(A) and R(A) respectively and in the case of sesqui-linear form $h[\cdot,\cdot]$, the domain by D[h]. We consider the equation (1.3) in this space $\mathfrak{Y}_o$.

<u>Proposition 2.1.</u> The operator B_o defined as

$$D(B_o)=\begin{pmatrix} D(H_o) \\ D \end{pmatrix},$$

$$B_o f_o=\begin{pmatrix} 0 & -i \\ iH_o & 0 \end{pmatrix} f_o,\quad f_o=\begin{pmatrix} f_o^1 \\ f_o^2 \end{pmatrix} \varepsilon D(B_o),$$

is selfadjoint in $\mathfrak{Y}_o$.

Using this proposition, we can integrate the equation (1.3) and obtain a unitary group $\mathfrak{V}_o(t)$ with a generator B_o. A simple proof of this proposition is given as follows. We transform the equation (1.3) into a "diagonal" form. Let $\mathfrak{X}_o$ be another Hilbert space which is a direct sum of two copies of X, and let T be a unitary operator from $\mathfrak{Y}_o$ to $\mathfrak{X}_o$ given as

$$Tf_o=\frac{1}{\sqrt{2}}\begin{pmatrix} h_o & -i \\ h_o & i \end{pmatrix} f_o.$$

The unitarity of T is an easy culculation. Then the equation (1.3) is transformed into

$$(2.1)\qquad \frac{d}{dt}g_o = i\begin{pmatrix} h_o & 0 \\ 0 & -h_o \end{pmatrix} g_o,\quad g_o = Tf_o.$$

We denote the operator $\begin{pmatrix} h_o & 0 \\ 0 & -h_o \end{pmatrix} = TB_oT^{-1}$ by A_o. Then $D(A_o)=TD(B_o)$ $=\begin{pmatrix} D(h_o) \\ D(h_o) \end{pmatrix}$ and A_o is selfadjoint in $\mathfrak{X}_o$, and consequently B_o is selfadjoint in $\mathfrak{Y}_o$. We denote a unitary operator $T\mathfrak{V}_o(t)T^{-1}$ by $\mathfrak{U}_o(t)$.

§3. The perturbed equation and the indefinite inner product

We shall now investigate the perturbed equation (1.2) on the following

Assumption 3.1. (1) H is a selfadjoint operator in X with domain $D(H)=D(H_o)$ and bounded from below; (2) K is a closed symmetric operator in X with domain $D(K)\supset D=D(h_o)$.

We define an operator V as $D(V)=D(H)=D(H_o)$ and $Vf=Hf-H_of$ for $f\varepsilon D(V)$. This assumption implies the following

Theorem 3.2. The operator B which is defined as

$$D(B)=D(B_o),$$

$$Bf=\begin{pmatrix} 0 & -i \\ iH & K \end{pmatrix} f,\quad f=\begin{pmatrix} f^1 \\ f^2 \end{pmatrix} \varepsilon D(B),$$

is a closed operator in $\mathfrak{Y}_o$ and generates a C^o-semigroup $\mathfrak{V}(t)$.

In order to prove this theorem, we need the next lemma.

Lemma 3.3. In the case that H is positive definite in Theorem 3.2, B is selfadjoint in $\mathfrak{Y}_o$ which is equipped with the inner product $(f,g)_{\mathfrak{Y}}=(\sqrt{H}f^1,\ \sqrt{H}g^1)+(f^2,g^2)$.

The operator A in $\mathfrak{X}_o$, which corresponds to B in $\mathfrak{Y}_o$, is defined as

$$D(A)=D(A_o),$$

$$Af= TBT^{-1}f= A_of+Gf,\ f\varepsilon D(A),$$

where $G=\frac{1}{2}Vh_o^{-1}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}+\frac{1}{2}K\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Then we obtain the next theorem in $\mathfrak{X}_o$ which corresponds to Theorem 3.2 in $\mathfrak{Y}_o$.

<u>Theorem 3.4.</u> The operator A is a closed operator in $\mathfrak{X}_o$ and generates a C^o-semigroup $\mathfrak{U}(t)$.

We shall now introduce "inner product"s for which the operator A or B is "symmetric". The sesqui-linear forms $(\cdot,\cdot)_{\mathfrak{Y}}$ in $\mathfrak{Y}_o$ and $(\cdot,\cdot)_{\mathfrak{X}}$ in $\mathfrak{X}_o$, which we call (indefinite) inner products, are defined as follows. Let $h[f^1,g^1]$ be a sesqui-linear form on the product space $D\times D$ defined as

$$h[f^1,g^1]=(\sqrt{H+\gamma}f^1,\sqrt{H+\gamma}g^1)-\gamma(f^1,g^1),\ f^1,g^1\varepsilon D,$$

where $-\gamma$ is a lower bound of H ($H>-\gamma$). Then the desired form is defined as

$$(f,g)_{\mathfrak{Y}}=h[f^1,g^1]+(f^2,g^2),\ f,g\varepsilon\mathfrak{Y}_o.$$

We also define the form $(\cdot,\cdot)_{\mathfrak{X}}$ on the product space $\mathfrak{X}_o\times\mathfrak{X}_o$ as

$$(f,g)_{\mathfrak{X}}=(T^{-1}f,T^{-1}g)_{\mathfrak{Y}}=(f,g)_{\mathfrak{X}_o}+V[f,g],\ f,g\varepsilon\mathfrak{X}_o,$$

where $V[f,g]=\frac{1}{2}\{h-h_o\}[h_o^{-1}(f^1+f^2),h_o^{-1}(g^1+g^2)]$ with $h_o[f^1,g^1]=(h_of^1,h_og^1)$. Since $D(H)=D(H_o)$ by Assumption 3.1, $h_o^{-1}Vh_o^{-1}$ has a bounded extension $(h_o^{-1}Vh_o^{-1})^a$, and $V[f,g]$ is expressed as

$$V[f,g]=\frac{1}{2}((h_o^{-1}Vh_o^{-1})^a(f^1+f^2),g^1+g^2).$$

Then we have the following propositions.

Proposition 3.5. The forms $(\cdot,\cdot)_{\mathfrak{Y}}$ and $(\cdot,\cdot)_{\mathfrak{X}}$ are bounded in $\mathfrak{Y}_o$ and $\mathfrak{X}_o$ respectively.

Proposition 3.6. The operator B (A) is symmetric with respect to the form $(\cdot,\cdot)_{\mathfrak{Y}}$ $((\cdot,\cdot)_{\mathfrak{X}})$:

$$(Bf,g)_{\mathfrak{Y}}=(f,Bg)_{\mathfrak{Y}},\ f,g\varepsilon D(B),$$
$$\left((Af,g)_{\mathfrak{X}}=(f,Ag)_{\mathfrak{X}},\ f,g\varepsilon D(A)\right).$$

§4. Structure of the perturbed operator

4.1. Discrete spectrum

First we state the basic condition which we shall always assume from now on.

Assumption 4.1. The operators V and K are compact from $D(H_o)$ to X and from D to X respectively, where $D(H_o)$ is equipped with the norm $\|H_o\cdot\|$.

From this assumption, using the interpolation theorem (see Hayakawa[7]), we have the following

Proposition 4.2. The operator $(h_o^{-1}Vh_o^{-1})^a$ is compact.

Now we shall proceed to the investigation of a discrete spectrum of B. From Assumption 4.1, H has a finite dimensional negative subspace $e(0-)X$, where $e(\lambda)$ is the spectral measure of H: $H=\int_{-\infty}^{\infty}\lambda de(\lambda)$. This implies that the space $\mathfrak{Y}_o$ equipped with the indefinite inner product $(\cdot,\cdot)_{\mathfrak{Y}}$ is a Pontrjagin space[2] when null space of H, N(H), is empty. So, modifying slightly the proofs in the book of Bognár, we can prove the finiteness of the non-real spectrum of B. Furthermore, using the standard argument about the meromorphic family of compact operators $\{B-B_o\}R(z;B_o)$ (here we denote the resolvent of B_o, $(B_o-z)^{-1}$, by $R(z;B_o)$), we obtain the following

Theorem 4.3. The intersection of the spectrum of B and the resolvent set of B_o is discrete and $(B-z)$ is a Fredholm operator with an index zero at such a point. Further, the points which belong to the non-real spectrum of B are finite and lie symmetrically with respect to the real axis and are included in the circle $C=\{|z|\leq\sqrt{\gamma}\}$. And, if we replace B_o by A_o and B by A, all the preceding statements are still valid.

4.2. Construction of a perturbed spectral measure

In the following part of this lecture we use the $\mathfrak{X}_o$ representation exclusively unless otherwise stated, and use the notations: $R_o(z)=R(z;A_o)=(A_o-z)^{-1}$, $R(z)=R(z;A)$ and $r_o(z)=R(z;h_o)$. We now state here another basic assumption.

Assumption 4.4. There is a bounded selfadjoint operator d in X such that: (1) d is one to one and has dense range; (2) $R(V), R(K) \subset R(d) = D(d^{-1})$; (3) the operators $dr_\circ(\lambda \pm i\varepsilon)d$, $dr_\circ(0)r_\circ(\lambda \pm i\varepsilon)d$, $d^{-1}Kr_\circ(\lambda \pm i\varepsilon)d$ and $d^{-1}Vr_\circ(0)r_\circ(\lambda \pm i\varepsilon)d$ with real λ and ε are compact and have boundary values in the operator norm topology as $\varepsilon \downarrow 0$, where the convergence is uniform for λ belonging to any compact interval of the real axis R^1; (4) The operators d, $d^{-1}K$ and $d^{-1}Vr_\circ(0)$ are $h_\circ$-smooth in the sense that

$$\int_{R^1} \|\tilde{d}r_\circ(\lambda \pm i\varepsilon)f\|^2 d\lambda \leq c\|f\|^2,$$

where $\tilde{d}$ stands for one of d, $d^{-1}K$ and $d^{-1}Vr_\circ(0)$, and c does not depend on ε.

We define an operator M in $\mathfrak{X}_\circ$ as $Mf = \begin{pmatrix} d & 0 \\ 0 & d \end{pmatrix} f$. Then we have the following proposition from Assumption 4.4.

Proposition 4.5. Under Assumption 4.4, we have: (1) $R(VR_\circ(z)) \subset R(M)$; (2) the operator $Q(z)$, which is defined as $Q(z) = M^{-1}GR_\circ(z)M$, is bounded for non-real z and has a boundary value $Q(\lambda \pm i0) = \lim_{\varepsilon \downarrow 0} Q(\lambda \pm i\varepsilon)$ in the operator norm topology and $Q(\lambda \pm i0)$ is continuous in λ; (3) there exists a closed set $\Gamma \subset R^1$ with Lebesgue measure zero such that $(1+Q(\lambda \pm i\varepsilon))$ ($\lambda \varepsilon \Delta$ and $\varepsilon \varepsilon (0,\delta)$) is invertible and continuous on $\overline{\Pi_\pm(\delta;\Delta)}$= the closure of $\Pi_\pm(\delta,\Delta)$ for Δ with $\Delta \cap \Gamma = \phi$, where $\Pi_\pm(\delta,\Delta) = \{z = \lambda + i\varepsilon;\ \lambda \varepsilon \Delta,\ \varepsilon \varepsilon (0, \pm\delta)\}$, and for sufficiently small $\delta > 0$.

After these preparatory works we can define the following sesqui-linear form $E(\Delta;\varepsilon)[f,g]$ on $\mathfrak{X}_o\times\mathfrak{X}_o$:

$$E(\Delta;\varepsilon)[f,g]=\frac{1}{2\pi i}\int_\Delta(\{R(\lambda+i\varepsilon)-R(\lambda-i\varepsilon)\}f,g)_{\mathfrak{X}_o}d\lambda$$

$$=\frac{1}{2\pi i}\int_\Delta(\{R_o(\lambda+i\varepsilon)-R_o(\lambda-i\varepsilon)\}f,g)_{\mathfrak{X}_o}d\lambda$$

$$-\frac{1}{2\pi i}\int_\Delta(R(\lambda+i\varepsilon)GR_o(\lambda+i\varepsilon)f,g)_{\mathfrak{X}_o}d\lambda$$

$$+\frac{1}{2\pi i}\int_\Delta(R(\lambda-i\varepsilon)GR_o(\lambda-i\varepsilon)f,g)_{\mathfrak{X}_o}d\lambda.$$

The first term of these integrals is bounded and has a limit as $\varepsilon\downarrow 0$ which equals to $(E_o(\Delta)f,g)_{\mathfrak{X}_o}$, where $E_o(\Delta)$ is a spectral measure of a selfadjoint operator A_o (here we used the absolute continuity of A_o which is a consequence of the smoothness condition in Assumption 4.4). The second and the third integrals are estimated as follows. By the resolvent equation, the integrand of the second integral is rewritten as

$$(R(\lambda+i\varepsilon)GR_o(\lambda+i\varepsilon)f,g)_{\mathfrak{X}_o}$$

$$=((1+Q(\lambda+i\varepsilon))^{-1}M^{-1}GR_o(\lambda+i\varepsilon)f,MR_o(\lambda-i\varepsilon)g)_{\mathfrak{X}_o}$$

So that, using the concrete expression of $M^{-1}GR_o(z)$;

$$M^{-1}GR_o(z)=\begin{pmatrix} d^{-1} & 0 \\ 0 & d^{-1} \end{pmatrix}\{\tfrac{1}{2}Vr_o(z)\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}+\tfrac{1}{2}K\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\}\begin{pmatrix} r_o(z) & 0 \\ 0 & -r_o(-z) \end{pmatrix},$$

and the smoothness condition, we obtain, for a compact Δ, $\Delta\cap\Gamma=\phi$,

$$\left|\frac{1}{2\pi i}\int_\Delta (R(\lambda+i\varepsilon)GR_o(\lambda+i\varepsilon)f,g)_{\mathfrak{X}_o}d\lambda\right|<c\|f\|_{\mathfrak{X}_o}\|g\|_{\mathfrak{X}_o}.$$

Then we have the following

<u>Theorem 4.6.</u> A set of sesqui-linear forms $E(\Delta;\varepsilon)[f,g]$ with compact Δ which do not intersect with Γ are bounded in $\mathfrak{X}_o\times\mathfrak{X}_o$ uniformly in ε, and have limits $E(\Delta)[f,g]$ when $\varepsilon\downarrow 0$ which are also bounded.

This theorem shows that $E(\Delta)[f,g]$ defines a bounded operator $E(\Delta)$ which is given as

$$(E(\Delta)f,g)_{\mathfrak{X}_o}=E(\Delta)[f,g].$$

We call this family of operators $E(\Delta)$ ($\overline{\Delta}$; compact and $\overline{\Delta}\cap\Gamma=\phi$) the perturbed spectral measure and shall investigate its properties in the following.

4.3. Properties of the perturbed spectral measure

We shall show that the family of bounded operators $E(\Delta)$ has most of the properties of a usual spectral measure. Namely we can prove the next theorem.

Theorem 4.7. Let Δ and Δ_j $(j=1,2)$ be bounded sets of R^1 whose closures do not intersect with Γ. Then we have: (1) $E(\Delta_1)\cap E(\Delta_2)=E(\Delta_1\cap\Delta_2)$; (2) $(E(\Delta)f,g)_{\mathfrak{X}}=(f,E(\Delta)g)_{\mathfrak{X}}$; (3) $(E(\Delta)f,f)_{\mathfrak{X}}\geq 0$ and, if $0\notin\Delta$, $(E(\Delta)f,f)_{\mathfrak{X}}=0 \iff E(\Delta)f=0$; (4) if Δ contains a point of the spectrum of A_o, $R(E(\Delta))$ is non-trivial; (5) if $0\notin\Delta$,

$$c_1\|E(\Delta)f\|_{\mathfrak{X}}\leq\|E(\Delta)f\|_{\mathfrak{X}_o}\leq c_2\|E(\Delta)f\|_{\mathfrak{X}}$$

with some positive constants c_1 and c_2, and consequently $E(\Delta)\mathfrak{X}_o$ is closed with respect to each of the topologies defined through $\|\cdot\|_{\mathfrak{X}}$ and $\|\cdot\|_{\mathfrak{X}_o}$.

In proving this theorem we use the following facts. First, $V[R(\lambda+i\varepsilon)Mf, R(\lambda+i\varepsilon)Mf]$ is uniformly bounded in ε by Assumption 4.4 and Proposition 4.5. Second, $(E(\Delta)Mf,Mf)_{\mathfrak{X}}$ is expressed as

$$(4.1)\qquad (E(\Delta)Mf,Mf)_{\mathfrak{X}}=\lim_{\varepsilon\downarrow 0}\frac{\varepsilon}{\pi}\int_\Delta\|R(\lambda\pm i\varepsilon)Mf\|^2_{\mathfrak{X}_o}d\lambda$$

$$=\int_\Delta\|\mathfrak{M}_o(\lambda)(1+Q(\lambda\pm i0))^{-1}f\|^2_{\mathfrak{X}_o}d\lambda,$$

where $\mathfrak{M}_o(\lambda)$ is defined as

$$\mathfrak{M}_o(\lambda)=\sqrt{\frac{1}{2\pi i}M\{R_o(\lambda+i0)-R_o(\lambda-i0)\}M},$$

which is bounded by Assumption 4.4.

Now next, we shall examine the properties of A on $R(E(\Delta))$.

Theorem 4.8. Let $\Delta\cap\Gamma=\phi$, then we have: (1) $E(\Delta)A\subset AE(\Delta)$; (2) A is bounded on $R(E(\Delta))$.

These theorems show that $R(E(\Delta))$ is a Hilbert space with a definite inner product $(\cdot,\cdot)_{\mathfrak{X}}$. We denote this space by $\mathfrak{X}(\Delta)$ The restriction of A on $\mathfrak{X}(\Delta)$, $A_{|X(\Delta)}$, is a bounded selfadjoint operator. Namely we have the following

Theorem 4.9. The operator $A_{|\mathfrak{X}(\Delta)}$ is selfadjoint in $\mathfrak{X}(\Delta)$ and bounded. The family of operators $E(\Omega)$ (Ω; $\Omega\subset\Delta$) is a spectral measure of $A_{|\mathfrak{X}(\Delta)}$ in $\mathfrak{X}(\Delta)$.

Remark. Using the representation of $(E(\Delta)f,f)_{\mathfrak{X}}$ in (4,1), we have that A is also absolutely continuous in $\mathfrak{X}(\Delta)$.

§5. Unitary equivalence and scattering theory

Now we shall establish the unitary equivalence between $A_{\circ|R(E_\circ(\Delta))}$ and $A_{|\mathfrak{X}(\Delta)}$, using the abstract stationary method of scattering theory (see Kato-Kuroda[13] and [14]). Namely, first we introduce the Hilbert space which is a completion of

$$L^2_c(\mathfrak{X}_\circ;\Delta)=\{f(\lambda);\ \lambda\varepsilon\Delta,\ f(\lambda)=\mathfrak{M}_\circ(\lambda)g(\lambda),\ g(\lambda) \text{ is a strongly continuous function of } \lambda \text{ with its values in } \mathfrak{X}_\circ\},$$

and then define a unitary operator $J_\circ(\Delta)$ from $\mathfrak{X}_\circ(\Delta)$ $(=R(E_\circ(\Delta)))$ to the above defined Hilbert space $L^2(\mathfrak{X}_\circ;\Delta)$ as follows. $J_\circ(\Delta)$ is an extention of an operator: $J_\circ(\Delta)E_\circ(\Delta)Mf=\chi_\Delta(\lambda)\mathfrak{M}_\circ(\lambda)f$, where $\chi_\Delta(\Delta)$ is a

characteristic function of Δ. Next, we define two unitary operators $J_{\pm}(\Delta)$ from $\mathcal{H}(\Delta)$ to $L^2(\mathcal{H}_o;\Delta)$ as the extensions of operators: $J_{\pm}(\Delta)E(\Delta)Mf= \chi_{\Delta}(\lambda)\mathcal{M}_o(\lambda)(1+Q(\lambda\pm i0))^{-1}f$. Then we have the following

Theorem 5.1. The operators $J_{\pm}(\Delta)$ have the properties: $J_{\pm}(\Delta)AJ_{\pm}(\Delta)^{*}=\lambda$, where * denotes the adjoint. The wave operators $W_{\pm}(\Delta)$ defined as

$$W_{\pm}(\Delta)=J_{\pm}(\Delta)^{*}J_o(\Delta)$$

are unitary and have the intertwining properties:

$$W_{\pm}(\Delta)A_o|_{\mathcal{H}_o(\Delta)}=A|_{\mathcal{H}(\Delta)}W_{\pm}(\Delta).$$

Furthermore, using the smoothness condition, we have another representation of $W_{\pm}(\Delta)$, and it shows the asymptotic behavior of $\mathcal{U}(t)|_{\mathcal{H}(\Delta)}$. Namely, we have the following

Theorem 5.2. The wave operators $W_{\pm}(\Delta)$ have the representation:

$$W_{\pm}(\Delta)f=\lim_{t\to\pm\infty} U(-t)E(\Delta)U_o(t)f, \quad f\in\mathcal{H}_o(\Delta),$$

where the limits are taken with respect to the norm $\|\cdot\|_{\mathcal{H}}$.

We can prove this theorem by using the smoothness conditions.

Remark. We can omit $E(\Delta)$ in Theorem 5.2.

6. Applications

We can apply the preceding results to the Klein-Gordon equation (1.1), on the assumptions described in the introduction.

Theorem 6.1. Let the functions $b_j(x)$ $(j=0,1,2,3)$ and $q(x)$ be bounded real functions in R^3 and satisfy the conditions: (1) $|b_j(x)|\le c|x|^{-2-\varepsilon}$ $(j=0,1,2,3)$, $\varepsilon>0$; (2) $b_j(x)$ $(j=1,2,3)$ are differentiable and $|\frac{\partial}{\partial x_j}b_j(x)|\le c|x|^{-2-\varepsilon}$; (3) $|q(x)|\le c|x|^{-2-\varepsilon}$. Then Assumptions 3.1, 4.1 and 4.4 are valid, so that the results in the preceding sections are valid.

Further, we have in this situation the following

Proposition 6.2. If λ belongs to the set Γ in Proposition 4.5 and $\lambda\ne m$, there exists a non-zero vector $g\in X_0$ such that $Ag=\lambda g$.

Since the equation $Ag=\lambda g$ implies the existence of a non-zero vector g^1 in X such that $(H+\lambda K)g^1=\lambda^2 g^1$, using the result of Ikebe-Uchiyama[8] for the non-existence of a positive eigenvalue, we have the following

Proposition 6.3. Under the assumptions in Theorem 6.1 and the unique continuation property of $(H+\lambda K)$, A has no singular spectrum in $(-\infty,-m)\cup(m,\infty)$.

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1),2) see a book of Bognár.

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