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On the spectral properties of tensor products of linear operators

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1. Introduction

Let X and Y be Hilbert spaces and $X\hat{\otimes}Y$ the completion of the tensor product $X\otimes Y$ with respect to the prehilbertian norm, which is a Hilbert space. Given densely defined closed linear operators A in X with domain D[A] and B in Y with domain D[B], consider two kinds of linear operators defined in $X\hat{\otimes}Y$

(1.1) $$A\otimes I + I\otimes B$$

with domain $D[A]\otimes D[B]$ and

(1.2) $$A\hat{\otimes}I + I\hat{\otimes}B$$

with domain $D[A\hat{\otimes}I]\cap D[I\hat{\otimes}B]$. Here the symbol I stands for the identity operators in both X and Y. $A\hat{\otimes}I$ and $I\hat{\otimes}B$ are the closures of $A\otimes I$ and $I\otimes B$, since they are closable in $X\hat{\otimes}Y$. Both (1.1) and (1.2) are also closable in $X\hat{\otimes}Y$.

It is an interesting problem which spectral contributions A and B make to the closures of (1.1) and (1.2).

The spectra of these operators in $X\hat{\otimes}Y$ have been determined in [3]. The result has been applied, by E.Balslev and J.M.Combes [1], to the study of spectral properties of many-body Schrödinger operators.

The present note announces some results on fine structures of the spectra of these operators. It is attempted to represent their essential spectra (and hence the complementary sets) in terms of the parts of the spectra of A and B. By the essential spectra are meant those in the sense of F.E.Browder[2], F.Wolf [6], M.Schechter[5] and T.Kato[4]. Further, those formulae expressing their nullity, deficiency and index in terms of the quantities concerning A and B are obtained.

The results may be useful as basic principles in the spectral theory of many-body Schrödinger operators.

Finally, it is remarked that the results will be extended to the case where X and Y are complex Banach spaces as well as to the operators $\sum_{jk}c_{jk}A^j\otimes B^k$ and $\sum_{jk}c_{jk}A^j\hat{\otimes}B^k$ associated with $P(\xi,\eta)=\sum_{jk}c_{jk}\xi^j\eta^k$ in a certain class of polynomials.

2. The main results.

Let T be a densely defined closed linear operator in a Hilbert space X. The spectrum and resolvent set of T are denoted by $\sigma(T)$ and $\rho(T)$, respectively.

The <u>Browder essential spectrum</u> of T, $\sigma_{eb}(T)$, is the subset of the spectrum $\sigma(T)$ of T excluding all isolated, finite-dimensional eigenvalues. The <u>Kato</u> (resp. <u>Wolf</u>) <u>essential spectrum</u> of T, $\sigma_{ek}(T)$ (resp. $\sigma_{ew}(T)$), is the complementary set of the semi-Fredholm (resp. Fredholm) domain of T. The <u>Schechter essential spectrum</u> of T, $\sigma_{em}(T)$, is the union of $\sigma_{ew}(T)$ and the set of all λ in the Fredholm domain of T for which the index of $T-\lambda I$ does not vanish. For T selfadjoint all these four essential spectra coincide.

T said to be _of type_ $(\theta_T, M_T(\theta))$, $0 \leq \theta_T < \pi$, if the resolvent set $\rho(T)$ includes the complementary set of the sector $S(\theta_T) = \{\zeta; |\arg\zeta| \leq \theta_T\}$ and $\|\zeta(\zeta I-T)^{-1}\| \leq M_T(\theta)$, $\theta=\arg\zeta$, outside $S(\theta_T)$, where $M_T(\theta)$ is a constant depending only on $\theta=\arg\zeta$.

For the statements of the main results, _it is assumed throughout that_ X _and_ Y _are Hilbert spaces and that_ A _and_ B _are of type_ $(\theta_A, M_A(\theta))$ _and_ $(\theta_B, M_B(\theta))$, _respectively, with_ $0 \leq \theta_A + \theta_B < \pi$. In this case, both the operators (1.1) and (1.2) have the same closure, which we denote by $\mathcal{A}$, and we have

$$\sigma(\mathcal{A}) = \sigma(A) + \sigma(B)$$

(see [3]). The identity operator in $X \hat{\otimes} Y$ is denoted by $\mathcal{J}(=I \hat{\otimes} I)$.

Then we have the following theorems on fine structures of the spectrum of $\mathcal{A}$. The proofs are omitted. We only note that in the proofs of the inclusions $\supset$ for the essential spectra, a frequent use is made of the Fredholm, semi-Fredholm theory and perturbation theory by I.C.Gohberg-M.G.Kreĭn and T.Kato, while the other inclusions are proved with $\mathcal{A}$ reduced by some invariant subspaces.

The following convention is used. For the subsets σ_A of $\sigma(A)$ and σ_B of $\sigma(B)$, the set $\sigma_A + \sigma_B$ is empty if at least one of σ_A and σ_B is empty.

Theorem 1. a) For the Browder essential spectrum:

(2.1) $$\sigma_{eb}(\mathcal{A}) = (\sigma_{eb}(A) + \sigma(B)) \cup (\sigma(A) + \sigma_{eb}(B)).$$

b) For the Wolf essential spectrum:

(2.2) $$\sigma_{ew}(\mathcal{A}) = (\sigma_{ew}(A) + \sigma(B)) \cup (\sigma(A) + \sigma_{ew}(B)).$$

Theorem 2. a) For the set of isolated, finite-dimensional eigenvalues:

$$\begin{aligned}(2.3)\quad &\sigma(\mathcal{A})\setminus\sigma_{eb}(\mathcal{A})\\ &=\{(\sigma(A)\setminus\sigma_{eb}(A))+(\sigma(B)\setminus\sigma_{eb}(B))\}\setminus\{(\sigma_{eb}(A)+\sigma(B))\cup(\sigma(A)+\sigma_{eb}(B))\}.\end{aligned}$$

Let λ be in the set (2.3). Then λ is an isolated, finite-dimensional eigenvalue of $\mathcal{A}$ with algebraic multiplicity $\sum_{(\mu,\nu)\in\Delta(\lambda)} s(\mu)t(\nu)$, and

$$(2.4)\quad \mathrm{ind}(\mathcal{A}-\lambda\mathcal{I})=0,$$

$$\begin{aligned}(2.5)\quad &\mathrm{nul}(\mathcal{A}-\lambda\mathcal{I})\\ &=\sum_{(\mu,\nu)\in\Delta(\lambda)}\sum_{p=1}^{\infty}(\mathrm{nul}(A-\mu I)^{p}-\mathrm{nul}(A-\mu I)^{p-1})(\mathrm{nul}(B-\nu I)^{p}-\mathrm{nul}(B-\nu I)^{p-1}),\end{aligned}$$

$$\begin{aligned}(2.6)\quad &\mathrm{def}(\mathcal{A}-\lambda\mathcal{I})\\ &=\sum_{(\mu,\nu)\in\Delta(\lambda)}\sum_{p=1}^{\infty}(\mathrm{def}(A-\mu I)^{p}-\mathrm{def}(A-\mu I)^{p-1})(\mathrm{def}(B-\nu I)^{p}-\mathrm{def}(B-\nu I)^{p-1}).\end{aligned}$$

Here

$$\begin{aligned}\Delta(\lambda) &= \{(\mu,\nu)\in(\sigma(A)\setminus\sigma_{eb}(A))\times(\sigma(B)\setminus\sigma_{eb}(B));\ \mu+\nu=\lambda\}\\ &= \{(\mu,\nu)\in\sigma(A)+\sigma(B);\ \mu+\nu=\lambda\}\ ;\end{aligned}$$

the sum is finite and the summation in p is in fact taken over $1\le p\le\min(s(\mu),t(\nu))$, where $s(\mu)$ and $t(\nu)$ are the algebraic multiplicities of the eigenvalues μ and ν, respectively.

b) For the intersection of the Browder essential spectrum and the Fredholm domain:

$$\begin{aligned}(2.7)\quad &\sigma_{eb}(\mathcal{A})\setminus\sigma_{ew}(\mathcal{A})\\ &=[\{(\sigma(A)\setminus\sigma_{eb}(A))+(\sigma_{eb}(B)\setminus\sigma_{ew}(B))\}\cup\{(\sigma_{eb}(A)\setminus\sigma_{ew}(A))+(\sigma(B)\setminus\sigma_{eb}(B))\}]\\ &\qquad\setminus\{(\sigma_{ew}(A)+\sigma(B))\cup(\sigma(A)+\sigma_{ew}(B))\}\ .\end{aligned}$$

For every λ in the set (2.7), $\mathcal{A}-\lambda\mathcal{I}$ has the nullity (2.5) and the deficiency (2.6) with

$$\Delta(\lambda) = \Delta_{10}(\lambda)\cup\Delta_{01}(\lambda) = \{(\mu,\nu)\in\sigma(A)\times\sigma(B);\ \mu+\nu=\lambda\},$$

$$\Delta_{10}(\lambda) = \{(\mu,\nu)\in(\sigma(A)\setminus\sigma_{eb}(A))\times(\sigma(B)\setminus\sigma_{ew}(B));\ \mu+\nu=\lambda\},$$

$$\Delta_{01}(\lambda) = \{(\mu,\nu)\in(\sigma(A)\setminus\sigma_{ew}(A))\times(\sigma(B)\setminus\sigma_{eb}(B));\ \mu+\nu=\lambda\};$$

the sum is finite as well and the summation in p is in fact taken over $1\le p\le s(\mu)$ for $(\mu,\nu)\in\Delta_{10}(\lambda)$ and $1\le p\le t(\nu)$ for $(\mu,\nu)\in\Delta_{01}(\lambda)$.

The index of $\mathcal{A}-\lambda\mathcal{I}$ is given by

$$\begin{aligned}(2.8)\quad &\operatorname{ind}(\mathcal{A}-\lambda\mathcal{I})\\ &=\sum_{(\mu,\nu)\in\Delta_{10}(\lambda)}\operatorname{ind}(B-\nu I)\sum_{p=1}^{s(\mu)}(\operatorname{nul}(A-\mu I)^{p}-\operatorname{nul}(A-\mu I)^{p-1})\\ &\quad+\sum_{(\mu,\nu)\in\Delta_{01}(\lambda)}\operatorname{ind}(A-\mu I)\sum_{p=1}^{t(\nu)}(\operatorname{nul}(B-\nu I)^{p}-\operatorname{nul}(B-\nu I)^{p-1}).\end{aligned}$$

Theorem 3. a) For the Schechter essential spectrum:

$$\sigma_{em}(\mathcal{A}) = \sigma_1\cup\sigma_2\ ,$$

where

$$\sigma_1 = (\sigma_{ew}(A) + \sigma(B))\cup(\sigma(A) + \sigma_{ew}(B))$$

and σ_2 is the set of all λ in the set

$$\{(\sigma_{em}(A) + \sigma(B))\cup(\sigma(A) + \sigma_{em}(B))\}\setminus\sigma_1$$

for which the index (2.8) does not vanish.

b) For the Kato essential spectrum:

$$\sigma_{ek}(\mathcal{A})\supset(\sigma_{ek}(A) + (\sigma(B)\setminus\Psi_o(B)))\cup((\sigma(A)\setminus\Psi_o(A)) + \sigma_{ek}(B)),$$

where $\Psi_o(A)$ is the set of all μ in the semi-Fredholm domain of A for which either $\operatorname{nul}(A-\mu I)=0$, $\operatorname{def}(A-\mu I)>0$ or $\operatorname{nul}(A-\mu I)>0$, $\operatorname{nul}(A-\mu I)=0$, and $\Psi_o(B)$ is defined similarly.

Remark. $\sigma_{ek}(\mathcal{A})$ can be estimated by a set as small as possible in which it is included, expressed by the parts of $\sigma(A)$ and $\sigma(B)$ with the aid of the nullities and deficiencies for A and B. In general, the set

$$(\sigma_{ek}(A) + \sigma(B)) \cup (\sigma(A) + \sigma_{ek}(B))$$

neither includes nor is included in $\sigma_{ek}(\mathcal{A})$.

An exact representation of the Kato essential spectrum of $\mathcal{A}$ in terms of the parts of the spectra of A and B will be complicated.

References

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