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APPROXIMATIONS FOR THE DISTRIBUTIONS OF THE
EXTREME ROOTS OF FOUR DETERMINANTAL
EQUATIONS IN MULTIVARIATE ANALYSIS

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3.1. INTRODUCTION AND SUMMARY

Simple approximations are presented for the
distribution of the extreme roots of four determinantal
equations in multivariate analysis. We consider the
extreme latent roots of three matrices, (i) $S_1 S_2^{-1}$ where $n_1 S_1$ and
$n_2 S_2$ are independently distributed as Wishart $W_m(n_1, \Sigma_1)$ and
$W_m(n_2, \Sigma_2)$ respectively, (ii) $S_1 S_2$ where $n_1 S_1$ and $n_2 S_2$ are
independently distributed as noncentral Wishart
$W_m(n_1, \Sigma, \Omega)$, $\Omega$ noncentrality matrix, and $W_m(n_2, \Sigma)$
respectively and (iii) $\Sigma S_1$ where $n S$ is distributed
as noncentral Wishart $W_m(n, \Sigma, \Omega)$, and (iv) the
extreme canonical correlation coefficients. The
approximations for cases (i), (ii) and (iii) take the form
of upper and lower bounds for the distribution.
functions of the largest and smallest latent roots respectively, and the approximations for case (ii) are valid for large sample size. The approximations are expressed in terms of products of (i) $F$, (ii) noncentral $F$, (iii) noncentral $\chi^2$ and (iv) noncentral $F$ and also normal probabilities.

§2. $S, S_i$ CASE (i)

Let $l_1 > l_2 > \cdots > l_m > 0$ be the latent roots of $S_i S_i'$. Let $A$ be an $m \times m$ non-singular matrix such that

$$A S_i A' = I_m$$

and

$$A S_i A' = \Lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_m)$$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_m > 0$ are the latent roots of $S_i S_i'$. Putting $S_i^* = A S_i A'$, it follows that $\eta_1 S_i^*$ and $\eta_2 S_i^*$ are independently distributed as $W_m (\eta_1, \Lambda)$ and $W_m (\eta_2, \Lambda)$ respectively, and $l_1, l_2, \ldots, l_m$ are the latent roots of $S_i^* S_i^{*-1}$. It is well-known (Roy [8]) that

$$l_1 = \frac{X' S_i^* X}{X' S_i^* X} = l_m, \quad X' S_i^{*1} X > 0.$$ 

Hence, if we let $S_i^* = (A_k^{(i)}) (i = 1, 2)$ it follows that

$$l_i = \max \left( \frac{A_{ii}^{(1)}}{A_{ii}^{(2)}}, \ldots, \frac{A_{mm}^{(1)}}{A_{mm}^{(2)}} \right)$$

and

$$-2$$
(2) \( l_m \leq \text{min} \left( \frac{A_{ii}^{(1)}}{\lambda_{ii}}, \ldots, \frac{A_{mm}^{(1)}}{\lambda_{mm}} \right) \).

Now, \( m, S_{ii}^{(1)} / \lambda_{ii} \) and \( m, S_{ii}^{(2)} / \lambda_{ii} \) \((i = 1, 2, \ldots, m)\) are all independently distributed as \( X_{m_i}^2 \) and \( X_{m_2}^2 \) respectively; hence the \( S_{ii}^{(1)} / \lambda_{ii}, S_{ii}^{(2)} / \lambda_{ii} \) \((i = 1, 2, \ldots, m)\) have independent \( F_{m_i, m_2} \) distributions. Thus using (1) and (2) we obtain the following.

**Theorem 1.** Upper and lower bounds for the distribution functions of \( l \) and \( l_m \) are respectively given by

\[ P \left( l \leq x \right) \leq \prod_{i=1}^{m} P \left( F_{m_i, m_2} \leq \frac{x}{\lambda_i} \right) \]

and

\[ P \left( l_m \leq x \right) \geq 1 - \prod_{i=1}^{m} P \left( F_{m_i, m_2} \geq \frac{x}{\lambda_i} \right). \]

The bounds are clearly exact when \( m = 1 \), and when \( \Lambda = I_m \), i.e., \( \Sigma_1 = \Sigma_2 \), they agree with bounds given by Mickey [4].

**§ 3.** \( S_1S_2^{-1} \); CASE (ii)

We can write \( S_1 \) as \( n_1S_1 = YY' \), where \( Y \) is an \( m \times n_1 \) matrix whose columns are independently
distributed as normal with covariance matrix $\Sigma$ and $E(Y) = M$, and $Q = \Sigma^{-1}MM'$. Let $l_1 > l_2 > \cdots > l_m > 0$ be the latent roots of $\Sigma\Sigma^{-1}$. Let $A$ be an $m \times m$ nonsingular matrix such that

$$A \Sigma A' = I_m$$

and $A M M A' = \Omega_D = \text{diag}(\omega_1, \omega_2, \ldots, \omega_m)$ where $\omega_1 > \omega_2 > \cdots > \omega_m > 0$ are the latent roots of $\Sigma M M' = \Omega_D$.

Putting $S_i^* = AS_iA'$ ($i = 1, 2$) we have that $\eta_1 S_1^*$ and $\eta_2 S_2^*$ are independently distributed as $W_m(n_1, I_m, \Omega_D)$ and $W_m(n_2, I_m)$ respectively, and $l_1, l_2, \ldots, l_m$ are the latent roots of $\Sigma_1^* \Sigma_2^{-1}$. Put $S_i^* = (S_{ki})$ ($i = 1, 2$); it follows that $\eta_1 S_{ii}^{(1)}$ and $\eta_2 S_{ii}^{(2)}$ are independently distributed as noncentral $\chi^2(n_i, \omega_i)$ with noncentrality parameter $\omega_i$ and $\chi^2(n_2)$ respectively. Hence the $S_{ii}^{(1)} / S_{ii}^{(2)}$ have independent noncentral $F_{n_1, n_2}(\omega_i)$ distributions. Thus, using (1) and (2) we obtain the following.

**Theorem 2.** Upper and lower bounds for the distribution functions of $l_i$ and $l_m$ are respectively given by
(5) \[ P \left( l_i \leq x \right) \leq \prod_{i=1}^{m} P \left( F_{n_i, \eta_i} \left( w_i \right) \leq x \right) \]

and

(6) \[ P \left( l_m \leq x \right) \geq 1 - \prod_{i=1}^{m} P \left( F_{n_i, \eta_i} \left( w_i \right) \geq x \right) \]

Numerical examinations showed that the bounds (3) and (5) appear quite reasonable as quick approximations to the exact probabilities.

§ 4. \( \mathcal{Z}^{-1} \mathbf{S} \); CASE (iii)

Let \( l_1 > l_2 > \cdots > l_m > 0 \) be the latent roots of \( \mathcal{Z}^{-1} \mathbf{S} \). As in Section 3, let \( \mathbf{A} \) be an \( m \times m \) nonsingular matrix such that

\[ \mathbf{A} \mathcal{Z} \mathbf{A}' = \mathbf{I}_m \quad \text{and} \quad \mathbf{A} \mathbf{M} \mathbf{M}' \mathbf{A}' = \mathbf{\Omega}_D = \text{diag} \left( w_1, w_2, \ldots, w_m \right) \]

where \( w_1 > w_2 > \cdots > w_m > 0 \) are the latent roots of \( \mathbf{\Omega} \left( = \mathcal{Z}^{-1} \mathbf{M} \mathbf{M}' \right) \). Then \( \mathbf{NS}^* = \mathbf{n} \mathbf{A} \mathbf{S}^* \mathbf{A}' \) has the \( \mathbf{W}_m \left( \mathbf{n}, \mathbf{I}_m, \mathbf{\Omega}_D \right) \) distribution and \( l_1, l_2, \ldots, l_m \) are the latent roots of \( \mathbf{S}^* \). Now it is well-known (Bellman [1], P111) that

\[ l_1 \left( \frac{\mathbf{z}' \mathbf{S}^* \mathbf{z}}{\mathbf{z}' \mathbf{z}} \right) > l_m \]

and hence that

\[ \frac{\mathbf{z}' \mathbf{S}^* \mathbf{z}}{\mathbf{z}' \mathbf{z}} > l_m \]

and hence that
\[ l \geq \max (a_{11}, \ldots, a_{nn}) \]

and

\[ l_m \leq \min (a_{11}, \ldots, a_{nm}) \]

where \( S^* = (s_{ij}) \). These inequalities, together with the fact that the \( n, S_i, (i = 1, 2, \ldots, m) \) have independent \( \chi^2_n (w_i) \) distributions, yield the following:

**Theorem 3.** Upper and lower bounds for the distribution functions of \( l \), and \( l_m \) are respectively given by

\[ P(l \leq x) \leq \prod_{i=1}^{m} P(\chi^2_n (w_i) \leq nx) \]

and

\[ P(l_m \leq x) \geq 1 - \prod_{i=1}^{m} P(\chi^2_n (w_i) \geq nx). \]

The bounds are exact when \( m = 1 \) and, when \( D = 0 \), i.e. \( nS^* \) is \( W_m (n, I_m) \), they agree with bounds given by Muirhead [5]. An approximation, valid for large \( n \), to \( P(l \leq x) \) somewhat similar to (7) has been given by Sugiyama [9] in terms of central \( \chi^2 \) probabilities.
§ 5. CANONICAL CORRELATION COEFFICIENTS; CASE (\(\hat{\eta}^2\))

Let \(\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_m \geq 0\) and \(r_1 \geq r_2 \geq \cdots \geq r_p \geq 0\) denote respectively the population canonical correlation coefficients and the sample canonical correlation coefficients, formed from a sample of size \(n + 1\).

We derive simple approximations for the distribution functions of \(\gamma_1^2\) and \(\gamma_p^2\) in the case when \(1 > r_1 > \cdots > r_p > 0\).

Using the results in Section 3, a representation of canonical correlation coefficient, conditional on the samples on one vector (Constantine [23]) and the asymptotic expansions, in terms of normal distributions, for the distributions of the latent roots of a Wishart matrix (Huiehead and Chikun [63]), we can obtain our results. Due to the limitation of space, we only summarize the results in

Theorem 4. Approximations for the distribution functions of \(\gamma_1^2\) and \(\gamma_p^2\), when \(1 > r_1 > \cdots > r_p > 0\), are given for large \(n\) by

\[
P(\gamma_1^2 \leq x) \approx \prod_{i=1}^{p} P(\tilde{F}_i, n-2 \left( n_1 \tilde{R}_1, (1 - \tilde{R}_1)^{-1}) \leq (n-2)^{\frac{1}{2}} x (1-x)^{-1})
\]
and
\[ P \left( r_i^2 \leq x \right) = 1 - \prod_{i=1}^{n-p} P \left( F_{i, m-n} \left( n\bar{f}_i^2 (1-\bar{f}_i^2)^{-1} \right) > (n-2)\bar{f}_i^2 x (1-x)^{-1} \right). \]

Alternative approximations are given for large \( n \) by
\[ P \left( r_i^2 \leq x \right) \approx \prod_{i=1}^{p} P \left( H_i \leq x \right) \]
and
\[ P \left( r_p^2 \leq x \right) \approx 1 - \prod_{i=1}^{p} P \left( H_i \geq x \right), \]
where \( H_i \) denotes a random variable distributed as normal \( N(\bar{f}_i^2, 4\bar{f}_i^2 (1-\bar{f}_i^2)^2 n^{-1}) \), and furthermore by
\[ P \left( r_i^2 \leq x \right) \approx P \left( H_i \leq x \right) \]
and
\[ P \left( r_p^2 \leq x \right) \approx P \left( H_p \leq x \right). \]

We note that the approximate distributions \( N(\bar{f}_i^2, 4\bar{f}_i^2 (1-\bar{f}_i^2)^2 n^{-1}) \)
of \( r_i^2, \ i = 1, p \), for large \( n \), given in (13) and (14),
are in fact the limiting distributions of \( r_i^2, \ i = 1, p \),
derived as a special case from results due to Hsu [3].

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REFERENCES


