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Positively curved Kaehler submanifolds of a complex projective space

Koichi Ogiue
(Tokyo Metropolitan Univ.)

For a Kaehler manifold M we use the following notation:

n: the dimension of M

g: the Kaehler metric of M

K: the sectional curvature of M

H: the holomorphic sectional curvature of M

S: the Ricci tensor of M

ρ: the scalar curvature of M.

Moreover we define n scalars $\rho_1, \cdots, \rho_n$ by

$$\frac{\det(g_{i\bar{j}} + tR_{i\bar{j}})}{\det(g_{i\bar{j}})} = 1 + \sum_{k=1}^{n} \rho_k t^k,$$

where $g_{i\bar{j}}$ and $R_{i\bar{j}}$ denote the local components of g and S, respectively. It is easily seen that $\rho = 2\rho_1$.

Let $P_m(C)$ denote an m-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature 1. By a <u>Kaehler submanifold</u> we mean a complex submanifold with the induced Kaehler structure.

1. Problems

There are a lot of interesting problems in the theory of Kaehler submanifolds, among which we confine our attention to the following: Let M be an n-dimensional compact Kaehler submanifold immersed in $P_{n+p}(C)$.

(I) If $H > \frac{1}{2}$, is M totally geodesic?

(II) If $K > 0$ and $p < \frac{n(n+1)}{2}$ or if $K > \frac{1}{8}$ and $n \geq 2$, is M totally geodesic?

(III) If every Ricci curvature is $> \frac{n}{2}$, is M totally geodesic?

(IV) If $\rho_k > \binom{n}{k}(\frac{n}{2})^k$ for some k, is M totally geodesic?

2. Results in this direction

Using the vanishing theorem of Kodaira, Kobayashi and Ochai proved an outstanding result, as a corollary to which we first quote the following.

Theorem 1 ([5]). *Let M be an n-dimensional compact Kaehler submanifold imbedded in $P_{n+p}(C)$ as a complete intersection. If $H > \frac{1}{2}$ or if $K > 0$ and $n \geq 2$, then M is totally geodesic.*

Theorem 1 can be considered as a partial solution to Problems (I) and (II).

For a general Kaehler submanifold, we have the following result as a partial solution to Problem (I).

Theorem 2 ([7]). *Let* M *be an* n-*dimensional complete Kaehler submanifold immersed in* $P_{n+p}(C)$. *If* $H > 1 - \frac{n+2}{2(n+2p)}$, *then* M *is totally geodesic*.

Theorem 2 is best possible in the case $p = 1$.

The lower bound for H in Theorem 2 can be improved by imposing some additional assumptions, for example, the following can also be considered as a partial solution to (I).

Theorem 3 ([4, 7]). *Let* M *be an* n-*dimensional complete Kaehler submanifold immersed in* $P_{n+p}(C)$. *If* $H > \frac{1}{2}$, *then* M *is totally geodesic provided that* ρ_k *is constant for some* $k < n$.

The following is a result of the same type as Theorem 3 and is a partial solution to (II).

Theorem 4 ([4, 7]). *Let* M *be an* n-*dimensional complete Kaehler submanifold immersed in* $P_{n+p}(C)$. *If* $K > 0$ *and* $p < \frac{n(n+1)}{2}$ *or if* $K > \frac{1}{8}$ *and* $n \geq 2$, *then* M *is totally geodesic provided that* ρ_k *is constant for some* $k < n$.

Theorem 3 and Theorem 4 can be proved by combining the following three results.

Proposition 1 ([2]). *Let* M *be a compact Kaehler manifold. If* $K > 0$ *or* $1 \geq H > \frac{1}{2}$, *then the second Betti number of* M *is* 1.

Proposition 2 ([3, 4]). Let M be an n-dimensional compact Kaehler manifold ($n \geq 2$). If ρ_k is constant ($\neq 0$ when $k \neq 1$) for some $k < n$ and if the second Betti number of M is 1, then M is Einstein.

Proposition 3 ([1]). Let M be an n-dimensional compact Einstein Kaehler manifold. If $K > 0$ or $1 \geq H > \frac{1}{2}$, then $M = P_n(C)$.

As a partial solution to Problem (II), we have the following.

Theorem 5 ([9]). Let M be an n-dimensional complete Kaehler submanifold immersed in $P_{n+p}(C)$. If $K > \frac{(n+4)p+1}{4n(2p+1)}$, then M is totally geodesic.

For a hypersurface we have a simpler result.

Theorem 6 ([7]). Let M be a complete Kaehler hypersurface immersed (resp. imbedded) in $P_{n+1}(C)$. If $K > 0$ and $n \geq 4$ (resp. $n \geq 2$), then M is totally geodesic.

The following may also be considered as a partial solution to Problems (I) and (II).

Theorem 7 ([9]). Let M be an n-dimensional complete Kaehler submanifold immersed in $P_{n+p}(C)$. If $K > \frac{1}{8}$, $H > \frac{1}{2}$ and $p \neq \frac{n+2}{2}, \frac{n+3}{2}$, then M is totally geodesic.

As for Problem (III) we have a complete solution.

Theorem 8 ([7]). <u>Let</u> M <u>be an</u> n-<u>dimensional complete Kaehler submanifold immersed in</u> $P_{n+p}(C)$. <u>If</u> $S > \frac{n}{2} g$, <u>then</u> M <u>is totally geodesic</u>.

As an immediate consequence of Theorem 8 we have the following result which gives a best possible solution to (I) in the case of complex curves.

Corollary 9 ([7]). <u>Let</u> M <u>be a complete complex curve immersed in</u> $P_{1+p}(C)$. <u>If</u> $K > \frac{1}{2}$, <u>then</u> M <u>is totally geodesic</u>.

The special case of this result where $p = 1$ and M is non-singular (or imbedded) was proved by Nomizu and Smyth ([6]).

The theorem of Gauss-Bonnet gives a relation between curvature (differential geometric invariant) and the Euler number (topological invariant). The following result is of Gauss-Bonnet type in the sense that it provides a relation between differential geometric invariant and more primitive invariant.

Theorem 10 ([7, 8]). <u>Let</u> M <u>be an</u> n-<u>dimensional compact Kaehler submanifold imbedded in</u> $P_{n+p}(C)$. <u>If</u> M <u>is a complete intersection of</u> p <u>non-singular hypersurfaces of degrees</u> $a_1, \cdots,$ <u>in</u> $P_{n+p}(C)$, <u>then</u>

$$\int_M \rho_k * 1 = \binom{n}{k}\left(\frac{n+p+1-\sum a_\alpha}{2}\right)^k \int_M * 1$$

$$= \binom{n}{k}\left(\frac{n+p+1-\sum a_\alpha}{2}\right)^k \left(\prod a_\alpha\right)\frac{(4\pi)^n}{n!} .$$

This result can be proved by using the properties of the first Chern class and some elementary facts in harmonic integral theory.

The following result is an immediate consequence of Theorem 10, which provides a partial solution to (IV).

Corollary 11 ([7, 8]). *Let M be an n-dimensional compact Kaehler submanifold imbedded in $P_{n+p}(C)$ as a complete intersection. If $\rho_k > \binom{n}{k}\left(\frac{n}{2}\right)^k$ for some k, then M is totally geodesic.*

Theorem 10 implies that the integral of the scalar curvature depends only on (the sum and the product of) the degrees. But the scalar curvature itself depends strongly on the equations defining M. In fact, we have the following.

Theorem 12 ([7]). *Let M be a compact Kaehler hypersurface of $P_{n+1}(C)$ defined by a homogeneous equation $F(z_0, \cdots, z_{n+1}) = 0$. Then*

$$\rho = n(n+1) - \left(\sum z_i \bar{z}_i\right)\left\{ \frac{\mathrm{tr} A\bar{A}}{{}^t\bar{\mathfrak{a}}\mathfrak{a}} - 2\,\frac{{}^t(\bar{A}\mathfrak{a})(A\bar{\mathfrak{a}})}{({}^t\bar{\mathfrak{a}}\mathfrak{a})^2} + \frac{({}^t\bar{\mathfrak{a}}A\bar{\mathfrak{a}})({}^t\mathfrak{a}\bar{A}\mathfrak{a})}{({}^t\bar{\mathfrak{a}}\mathfrak{a})^3}\right\},$$

where $\mathfrak{a} = \left(\frac{\partial F}{\partial z_i}\right)$ *and* $A = \left(\frac{\partial^2 F}{\partial z_i \partial z_j}\right)$.

<u>Example</u> Let $M = \left\{(z_0, \cdots, z_{n+1}) \in P_{n+1}(C) \,\middle|\, z_0^2 + \cdots + z_n^2 + az_{n+1}^2 = 0\right\}$. Then Theorem 10 and Theorem 12 imply

$$n^2 + 1 - a^2 \leqq \rho \leqq n(n+1) - \frac{n}{a} \qquad (a \geqq 1)$$

$$n(n+1) - \frac{n}{a} \leqq \rho \leqq n^2 + 1 - a^2 \qquad (0 < a \leqq 1)$$

$$\int_M \rho * 1 = n^2 \int_M * 1 = n^2 \frac{2(4\pi)^n}{n!} \quad (\underline{\text{independent}}\ \underline{\text{of}}\ a).$$

Corollary 11 states that Problem (IV) is affirmative for some special class of Kaehler submanifolds. The following result due to Tanno gives a partial solution to (IV) in general case.

<u>Theorem</u> 13 ([10]). <u>Let</u> M <u>be</u> <u>an</u> n-<u>dimensional</u> <u>compact</u> <u>Kaehler</u> <u>submanifold</u> <u>immersed</u> <u>in</u> $P_{n+p}(C)$. <u>If</u> $\rho > n(n+1) - \frac{n+2}{3}$, <u>then</u> M <u>is</u> <u>totally</u> <u>geodesic</u>.

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