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Spontaneously broken symmetry and the cusp catastrophe

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Spontaneously broken symmetry is discussed from the viewpoint of Thom's catastrophe theory. It is just the Maxwell convention in cusp catastrophe and is a special case of the usual symmetry breaking. The relation between usual symmetry breaking and 'bag theory' is suggested briefly.

1. Introduction

It is well known that a kind of symmetry can be obtained from an exactly symmetric Lagrangian, provided that the physical vacuum is not invariant under the symmetry group $U(1)$. Such a symmetry is popularly called 'spontaneously broken symmetry' (hereafter we call it s.b.s.) and is considered as a basic concept of the Higgs mechanism (Higgs 1964; Weinberg 1967; Salam 1968), phase transitions and so on.

In this note, s.b.s. as well as the usual symmetry breaking is discussed from the viewpoint of Thom's catastrophe theory (Thom 1972). As a simple case, we consider the Lagrangian density with only a single real scalar field,

$$\mathscr{L} = \tfrac{1}{2}(\partial^\mu\phi\,\partial_\mu\phi) - \tfrac{1}{2}\mu^2\phi^2 - \tfrac{1}{4}\lambda\phi^4.$$

The mechanism of s.b.s. is caused by the following symmetric potential (Abers & Lee 1973):

$$V(\phi) = \tfrac{1}{4}\lambda\phi^4 + \tfrac{1}{2}\mu^2\phi^2, \tag{1}$$

where λ (>0) and μ^2 are the coupling constant and the square of 'mass' respectively. The variable ϕ may be considered as 'c-number' because quantization gives only a small correction effect near the stable equilibrium (Lee 1974).

2. Catastrophe theory

We can find that the potential (1) is strongly connected with the universal unfolding of catastrophe theory. The universal unfolding, $V(x, c_i)$, are written for single variable x as (Thom 1972; Woodcock & Poston 1973)

$$V(x, c_i) = \frac{x^{n+2}}{n+2} + c_1\frac{x^n}{n} + c_2\frac{x^{n-1}}{n-1} + \ldots + c_n x,$$

where c_i is called the control parameter (we call it 'parameter'). The number of parameters is regarded as the co-dimension of $V_0(x) = V(x, 0)$, codim V_0, by Thom's catastrophe theory and it is enough to consider the case codim $V_0 \leqslant 4$. The universal unfoldings $V(x, c_i)$ and the name of their bifurcation sets are shown in table 1 with codim $V_0 \leqslant 4$. The geometrical considerations of these catastrophe are demonstrated in the lecture notes (Woodcock & Poston 1973).

We are then concerned with the cusp (Riemann–Hugoniot) catastrophe for which

$$V_0(x) = \tfrac{1}{4}x^4, \tag{2}$$

TABLE 1. THE UNIVERSAL UNFOLDINGS AND THE NAME OF THEIR BIFURCATION SETS

codim	$V_0(x)$	$V(x, c_i)$	name
0	$\frac{x^2}{2}$	$\frac{x_i^2}{2}$	object
1	$\frac{x^3}{3}$	$\frac{x^3}{3}+ux$	fold
2	$\frac{x^4}{4}$	$\frac{x^4}{4}+u\frac{x^2}{2}+vx$	cusp
3	$\frac{x^5}{5}$	$\frac{x^5}{5}+u\frac{x^3}{3}+v\frac{x^2}{2}+wx$	swallowtail
4	$\frac{x^6}{6}$	$\frac{x^6}{6}+u\frac{x^3}{4}+v\frac{x^3}{3}+w\frac{x^2}{2}+tx$	butterfly

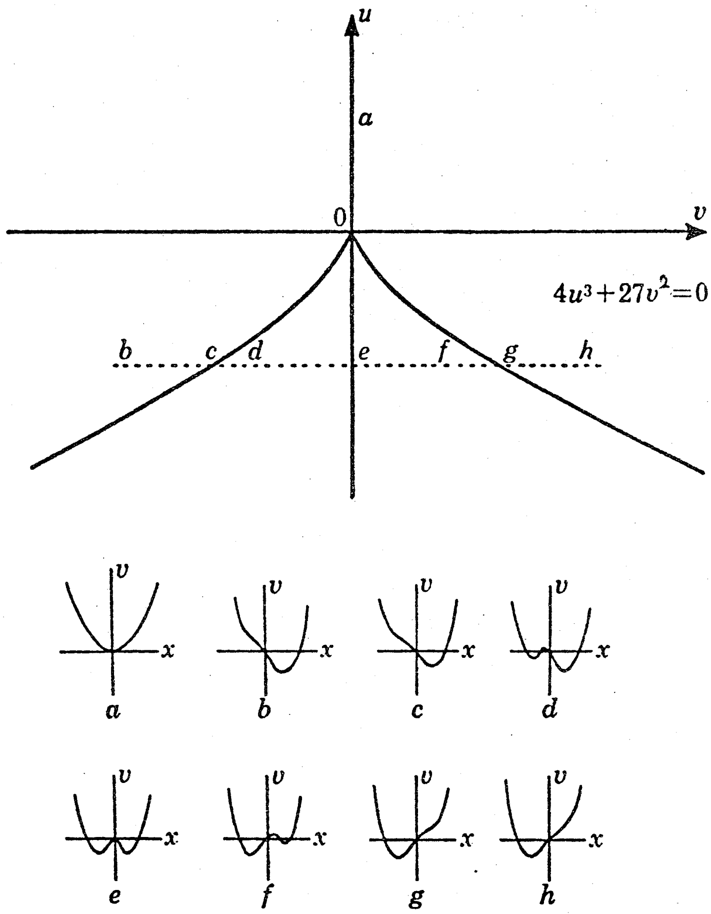

FIGURE 1. The bifurcation set and the potential $V(x)$.

with minimum $x = 0$. Then any nearby potential function, universal unfolding, can be expressed as

$$V(x) = \tfrac{1}{4}x^4 + \tfrac{1}{2}ux^2 + vx. \tag{3}$$

Varying the parameters (u, v) can give rise to two different types of potential function: if (u, v) lies outside the cusp, $4u^3 + 27v^2 = 0$, then there will be one minimum: if (u, v) lies inside, then two minima. These are shown in figure 1 with the potential shapes. Such a situation is clearly explained by the manifold, M,

$$\frac{\mathrm{d}V(x)}{\mathrm{d}x} = x^3 + ux + v = 0 \tag{4}$$

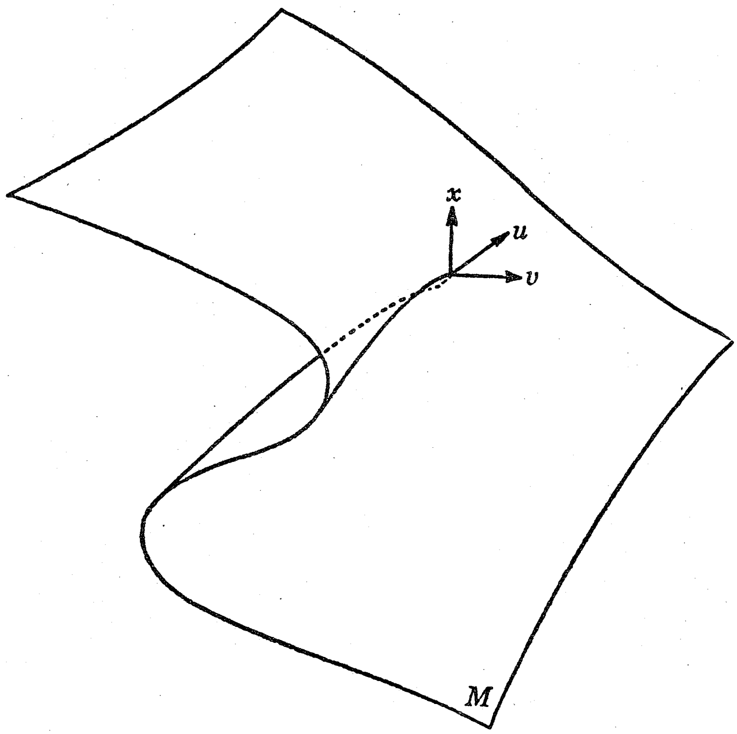

FIGURE 2. The manifold M; $x^3 + ux + v = 0$.

shown in figure 2. M is called the bifurcation set. The projection from the manifold M to the control plane (u–v plane) in figure 1 is defined as the catastrophe projection. The catastrophe with this type is called the cusp catastrophe. Two minima of the potential have the same value on the line $v = 0$ as shown in figure 1. Such a situation is called the Maxwell convention and is specified by

$$v = 0, \quad u = -x^2 \tag{5}$$

by equating the two minima of (3).

3. Spontaneously broken symmetry

Now we return the s.b.s. potential (1). To consider (1) as the universal unfolding, we should rewrite it as

$$V(\phi) = \tfrac{1}{4}\phi^4 + \frac{\mu^2}{\lambda}\tfrac{1}{2}\phi^2 + \frac{\gamma}{\lambda}\phi, \tag{6}$$

where the additional term, $(\gamma/\lambda)\phi$, breaks the symmetry under the transformation $\phi \to -\phi$. The potential (6) corresponds to the usual symmetry breaking potential as the σ-model of Gell-Mann & Levy (1960). If we put

$$u = \mu^2/\lambda, \quad v = \gamma/\lambda \quad \text{and} \quad x = \phi \tag{7}$$

in (6), we obtain the cusp catastrophe. Especially, the concept of the Maxwell convention for the potential (6) is just the s.b.s. and we obtain by (5)

$$\gamma = 0, \quad \phi^2 = -\mu^2/\lambda \quad (\text{then}\, \mu^2 < 0). \tag{8}$$

This is exactly the same as the result in the s.b.s. (Abers & Lee 1973).

It is found from the viewpoint of catastrophe theory that 'spontaneously broken symmetry' is explained as the special case (the Maxwell convention) of the usual symmetry breaking. Furthermore, the following are given: for the usual symmetry breaking, two minima of the potential appear in the case of

$$-(\gamma/\lambda)^2 > (4/27)(\mu^6/\lambda^3) \quad (\mu^2 < 0). \tag{9}$$

Two different vacuum expectation values $\langle 0|\phi|0\rangle$, are then expected for $\gamma \neq 0$ in (9). One is 'stable' and the other is 'metastable'. These two vacuum states correspond to the 'true vacuum state' and the 'vacuum excitation state' in a somewhat different approach (Lee & Wick 1974). It seems that the results of our simple and topological analysis are closely connected with the usual 'bag theory' in particle physics.

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