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Hyperbolic automorphisms of a Lie algebra

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Let $\mathfrak{g}$ be a Lie algebra (of finite dimension) over $\mathbb{R}$. An automorphism σ of $\mathfrak{g}$ is said to be hyperbolic if no eigenvalue of σ is of absolute value one. Here we shall give a simple proof of the following theorem, which was stated in [2] in a slightly stronger form:

THEOREM. If $\mathfrak{g}$ has a hyperbolic automorphism, then $\mathfrak{g}$ is nilpotent.

Let σ be an automorphism of $\mathfrak{g}$, and let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ be the complexification of $\mathfrak{g}$. Then σ can be extended naturally to an automorphism of $\tilde{\mathfrak{g}}$, and if $\tilde{\mathfrak{g}}$ is nilpotent then so is $\mathfrak{g}$. Therefore in order to prove the theorem it suffices to consider a complex Lie algebra $\mathfrak{g}$. After this, by a Lie algebra we mean one over $\mathbb{C}$.

Known facts

(1) (S. Lie) Any representation of a solvable Lie algebra can be triangularized simultaneously.

(2) Let $\mathfrak{g}$ be a Lie algebra, and σ an automorphism of $\mathfrak{g}$. For $\alpha \in \mathbb{C}$, if α is an eigenvalue of σ we let $\mathfrak{g}(\alpha)$ denote the eigenspace of σ with eigenvalue α, and we put $\mathfrak{g}(\alpha) = \{0\}$ otherwise. Then

$$[\mathfrak{g}(\alpha), \mathfrak{g}(\beta)] \subset \mathfrak{g}(\alpha\beta).$$

(3) (M. Goto [1]) If a Lie algebra $\mathfrak{g}$ contains two nilpotent subalgebras $\mathfrak{n}_1$ and $\mathfrak{n}_2$ with $\mathfrak{n}_1 + \mathfrak{n}_2 = \mathfrak{g}$, then $\mathfrak{g}$ is solvable.

Proof of THEOREM

Let σ be a hyperbolic automorphism of $\mathfrak{g}$. Let $\{\alpha_1, \cdots, \alpha_k, \beta_h, \cdots, \beta_1\}$ be the totality of eigenvalues of σ, where

$$|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_k| > 1 > |\beta_h| \geq \cdots \geq |\beta_1|.$$

We put

$$\mathfrak{n}_1 = \mathfrak{g}(\alpha_1) + \mathfrak{g}(\alpha_2) + \cdots + \mathfrak{g}(\alpha_k),$$
$$\mathfrak{n}_2 = \mathfrak{g}(\beta_1) + \mathfrak{g}(\beta_2) + \cdots + \mathfrak{g}(\beta_h),$$
$$\mathfrak{g} = \mathfrak{n}_1 + \mathfrak{n}_2.$$

By (2), $\quad [\mathfrak{g}(\alpha_i), \mathfrak{g}(\alpha_j)] \subset \mathfrak{g}(\alpha_i \alpha_j) = \{0\}$ or $\mathfrak{g}(\alpha_l)$ with $l < \mathrm{Min}(i, j)$. Hence $\mathfrak{n}_1$ is a nilpotent subalgebra, and so is $\mathfrak{n}_2$. Therefore $\mathfrak{g}$ is solvable by (3).

We adopt the notation $(\mathrm{ad}X)Y = [X, Y]$ for $X, Y \in \mathfrak{g}$. For $X \in \mathfrak{g}(\gamma)$ and $Y \in \mathfrak{g}(\delta)$ where γ and δ are either α_i or β_j, we have

$$(\mathrm{ad}X)^s Y \in \mathfrak{g}(\gamma^s \delta) \qquad s = 1, 2, \cdots.$$

Since γ is not a root of unity,

$$\gamma\delta, \gamma^2\delta, \cdots, \gamma^s\delta, \cdots$$

are all distinct to each other, and we have that $\mathfrak{g}(\gamma^s\delta) = \{0\}$ for a sufficiently large s. This implies that $\mathrm{ad}X$ is nilpotent.

By a suitable choice of basis of $\mathfrak{g}$ $\mathrm{ad}X$ $(X \in \mathfrak{g})$ is represented by a matrix of the form

$$\begin{pmatrix} \lambda_1(X) & & * \\ & \lambda_2(X) & \\ & & \ddots \\ 0 & & & \lambda_n(X) \end{pmatrix}$$

simultaneously, by (1), where $n = \dim \mathfrak{g}$. But for $X \in \mathfrak{g}(\gamma)$, ad X is nilpotent and we have $\lambda_1(X) = \cdots = \lambda_n(X) = 0$. Hence $\lambda_1 = \cdots = \lambda_n = 0$ identically. Therefore $(\mathrm{ad}\,\mathfrak{g})^n = \{0\}$, that is $\mathfrak{g}$ is nilpotent. Q. E. D.

References

[1] M. Goto, Note on a characterization of solvable Lie algebras, J. Sci. Hiroshima Univ. 26 (1962), pp.1-2.

[2] S. Smale, Differentiable dynamical systems, Bull. A. M. S. 73 (1967), pp. 747-817.

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