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On the growing up problem for semilinear heat equations

by

Tunekiti Sirao

(Tokyo Metropolitan University)

§1. Introduction. This report is an extract of the joint paper by K. Kobayashi, T. Sirao and H. Tanaka [5], and the proofs of our theorems will be given in [5].

Let us consider the following Cauchy problem:

$$(1)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = \Delta u + f(u), \quad t > 0, \quad x \in R^d, \\ u(0,x) = a(x), \end{cases}$$

where Δ denotes Laplacian differential operator, f is a non-negative locally Lipschitz continuous function and a is a bounded non-negative continuous function. In this report, we limit the class of solutions of (1) as follows:

Definition 1.1. $u(t,x)$ is said to be a positive solution of (1) if there exists positive T_∞ ($\leqq \infty$) with the following properties (i), (ii) and (iii).

(i) For any positive $T < T_\infty$, $u(t,x)$ is bounded and continuous on $[0,T] \times R^d$.

(ii) $\dfrac{\partial u}{\partial t}$, $\dfrac{\partial^2 u}{\partial x_i \partial x_j}$ $(i,j = 1,2,\ldots,d)$ exists in $(0,T_\infty) \times R^d$ and $u(t,x)$ satisfies (1) in the classical sense.

(iii) $u(t,x) > 0$ in $(0,T_\infty) \times R^d$.

Though small $T_\infty > 0$ always satisfies the above conditions, we will take $T_\infty = T_\infty(a,f)$ as the supremum of T_∞ satisfying (i)-

(iii). Then T_∞ may or may not be infinity. If $T_\infty = \infty$, then u is said to be a global solution. Otherwise u is a local solution.

The purpose of this report is to consider "How does the behavior of f near the origin effect to the growth of positive solution as $t \longrightarrow \infty$?" The answer will be given in §2.

When $f(u) = u^{1+\alpha}$, $\alpha > 0$, this problem was first considered by H. Fujita [1]. The main result in [1] is stated as follows: If $\alpha d < 2$, then all the positive solutions of (1) blow up in finite times, that is, there is no global solution of (1) for any non-trivial $a(x) \geqq 0$. On the contrary, if $\alpha d > 2$ then there exist global solutions for small $a(x) \geqq 0$. About the critical case where $\alpha d = 2$, H. Hayakawa [3] proved the non-existence of global solution for non-trivial $a(x) \geqq 0$. Then S. Sugitani [6] considered Cauchy problem for the equation

$$\frac{\partial u}{\partial t} = -(-\Delta)^\beta u + u^{1+\alpha}, \quad t > 0, \quad x \in R^d,$$

where $0 < \beta < 1$, and obtained the same conclusion for $\alpha d \leqq 2\beta$.

On the other hand, Ya. I. Kaneli [4] discussed related problems about (1) in 1-dimensional case. Among others, he says that if

(2) $\quad f(0) = f(1) = 0, \quad f(u) > 0, \quad 0 < u < 1, \quad f'(0) > 0,$

(3) $\quad a(x) > 0$ on a certain interval and $0 \leqq a(x) \leqq 1$ everywhere,

then the solution $u(t,x)$ of (1)—(3) converges to 1 uniformly on every finite interval as $t \to \infty$. (Though another interesting results are stated in [4], they are slightly different from our present interest.)

§2. Results. Before stating our results, we give notations and terminologies.

$\mathcal{F}$ denotes the class of all functions satisfying the following conditions (A) and (B).

(A) f is a locally Lipschitz continuous function on $[0,\infty)$ and $f(0) = 0$, $f(u) > 0$ for $u > 0$.

(B) There exists a positive constant c_0 such that

$$f(uv) \geqq c_0 v^{1+\frac{2}{d}} f(u) \quad \text{for} \quad 0 \leqq u \leqq v, \quad u < c_0 \quad \text{and} \quad uv < c_0 .$$

$\tilde{\mathcal{F}}$ is the class of all non-decreasing functions f satisfying (A) and (C) stated below.

(C) There exists a positive constant c such that

(a) $f(uv) \geqq cv^{1+\frac{2}{d}} f(u)$ for $0 < u \leqq v$, $u < c$,

(b) $f(uv) \geqq cv^{2+\frac{2}{d}} f(u)$ for $0 < v \leqq u < c$.

Obviously $\tilde{\mathcal{F}} \subset \mathcal{F}$. (cf. Remark 3.)

Definition 2.1. $T_\infty = T_\infty(a,f)$ in §1 is said to be the blowing up time of the solution of $u(t,x)$ of (1). (i) If T_∞ is finite, then we say that $u(t,x)$ blows up (in finite time). (ii) If $T_\infty = \infty$ and $u(t,x) \to \infty$ uniformly in x on every compact set as $t \to \infty$, then we say that $u(t,x)$ grows up to infinity.

The solution of (1) corresponding to f and a is denoted by $u(t,x;\, a,f)$.

Now we can state the following

Theorem 1. Let $f \in \tilde{\mathcal{F}}$. If, for any $\varepsilon > 0$,

$$\int_0^{\varepsilon} f(u)/u^{2+\frac{2}{d}}\, du = \infty, \tag{4}$$

then the positive solution $u(t,x; a,f)$ of (1) blows up for any non-trivial $a(x) \geqq 0$.

Theorem 2. Let $f \in \tilde{\mathcal{F}}$. (i) If (4) holds for any $\varepsilon > 0$, then any positive solution $u(t,x; a,f)$ of (1) blows up or grows up to infinity. (ii) If the left hand side of (4) is finite for a certain $\varepsilon > 0$, then, for small initial data $a(x) = \alpha e^{-\beta|x|^2} > 0$, the solution $u(t,x; a,f)$ of (1) converges to 0 uniformly in x as $t \to \infty$.

Theorem 3. Let f be a Lipschitz continuous function on $[0,1]$ such that $f(u) > 0$ for $0 < u < 1$ and $f(0) = f(1) = 0$. Moreover we assume that f satisfies the conditions (B) and (4). Then, for each continuous initial data $a(x)$ with $0 \leqq a(x) \leqq 1$, $a(x) \not\equiv 0$, the solution $u(t,x; a,f)$ of (1) converges to 1 uniformly on every compact set ($\subset R^d$) as $t \to \infty$.

Remark 1. The assumptions of (ii) in Theorem 2 can be weakend as follows: f is a locally Lipschitz continuous function satisfying (iia) $f(u) \geqq 0$ and $f(0) = 0$, (iib) $f(uv) \geqq vf(u)$ for $u \geqq 0$, $v \geqq 1$, and (iic) the left hand side of (4) is finite.

Remark 2. For each $f \in \mathcal{F}$ satisfying (4), there exists $\tilde{f} \in \tilde{\mathcal{F}}$ such that (4) holds for $\tilde{f}$ and

$$\liminf_{u \downarrow 0} \frac{f(u)}{\tilde{f}(u)} > 0.$$

Remark 3. As an application of Theorem 2, let us consider the case when f is given by

$$f(u) = u^{1+\frac{2}{d}}\{\log \frac{1}{u} \log_{(2)} \frac{1}{u} \cdots \log_{(n-1)} \frac{1}{u} (\log_{(n)} \frac{1}{u})^{\delta}\}^{-1}$$

near the origin and smooth and positive in the whole of $(0,\infty)$, where $\delta > 0$ and $\log_{(k)}u = \log\log \cdots \log u$ (k-times). If $0 < \delta \leqq 1$, then $f \in \mathcal{F}$ and hence any positive solution of (1) blows up or grows up to infinity by Theorem 2, (i). On the other hand, if $\delta > 1$, then some positive solution $u(t,x)$ of (1) converges to 0 uniformly in x as $t \to \infty$ by Theorem 2, (ii).

Remark 4. The conditions (B) and (4) of Theorem 1 are concerned with the local behavior of f near the origin only apart from $f(u) > 0$ $(u > 0)$, while the condition (a) of (C) is concerned with the behavior of f for large u, that is, it implies that

(5) $\quad f(u) > \text{const.}u^{1+\frac{2}{d}}$ for all sufficiently large u.

Some condition on the behavior of $f(u)$ for large u such as (5) is required for the blowing up conclusion. This aspect will be made much clear by the following theorem which is a slight extension of one of results due to Fujita [2].

Theorem 4. Assume that

(i) $\quad \int^{\infty} \frac{d\lambda}{f(\lambda)} < \infty,$

(ii) there exist constants $c > 0$ and $u_0 > 0$ such that

$$f(u) \geqq cf(v) \quad \text{for} \quad u_0 < v < u.$$

Let $u(t,x)$ be a positive solution of (1). If for any $M > 0$ there exists $t_M > 0$ such that $u(t_M,x) > M$ for $|x| < 1$, then

$u(t,x)$ blows up.

Remark 5. The following two theorems were used to prove Theorem 1-3.

Theorem 5. Let f, $\tilde{f}$ be locally Lipschitz continuous functions on $[0,\infty)$, and assume that (i) $f(u) > 0$ for $u > 0$, (ii) $\tilde{f}$ is non-decreasing with $\tilde{f}(0) = 0$, and (iii)

$$\liminf_{u \downarrow 0} \frac{f(u)}{\tilde{f}(u)} > 0.$$

Suppose that, for each bounded continuous initial data $a(x) \geqq 0$, the solution $u(t,x;\, a,\tilde{f})$ of (1) either blows up or satisfies

$$\limsup_{t \to \infty} \| u(t,x;\, a,\tilde{f})\|_{\infty} = \infty,$$

where $\| \;\; \|_{\infty}$ denotes the supremum norm. Then any positive solution $u(t,x;\, a,f)$ of (1) blows up or grows up to infinity.

Theorem 6. Let f be a Lipschitz continuous function on $[0,1]$ such that $f(u) > 0$ for $0 < u < 1$ and $f(1) = 0$, and let f satisfy the same assumptions as in Theorem 5. Moreover we assume that, for any non-negative bounded continuous $a(x) \geqq 0$ ($\not\equiv 0$), the solution $u(t,x;\, a,\tilde{f})$ of (1) has the same property as in Theorem 5. Then, for any continuous function $a(X)$ with $0 \leqq a(x) \leqq 1$, $a(x) \not\equiv 0$, the solution $u(t,x;\, a,f)$ of (1) converges to 1 uniformly on each compact set of R^d as $t \to \infty$.

References

[1] H. Fujita: On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci. Univ. of Tokyo, Sec.I, 13(1966), 109-124.

[2] _____ : On some nonexistence and nonuniqueness theorems for nonlinear parabolic equation. Proc. Symp. Pure Math., AMS, 18(1969), 105-113.

[3] K. Hayakawa: On nonexistence of global solutions of some semilinear parabolic differential equations. Proc. Japan Acad., 49(1973), 503-505.

[4] Ya. I. Kaneli: On the stability of solutions of the equation of combustion theory with finite initial functions. Mat. Sbor. 65(1964), 398-413. (Russian)

[5] K. Kobayashi, T. Sirao and H. Tanaka: On the growing up problem for semilinear heat equations. (to appear)

[6] S. Sugitani: On nonexistense of global solutions for some nonlinear integral equations. Osaka Jour. of Math., 12, No.1(1975), 45-51.