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<th>On the Growing up Problem for Semilinear Heat Equations (非線形問題の解析)</th>
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<td>Author(s)</td>
<td>SHIRAO, TSUNEKICHI</td>
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<tr>
<td>Citation</td>
<td>RIMS, 1974</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1975-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/105781">http://hdl.handle.net/2433/105781</a></td>
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<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
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On the growing up problem for semilinear heat equations
by
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§1. Introduction. This report is an extract of the joint paper by K. Kobayashi, T. Sirao and H. Tanaka [5], and the proofs of our theorems will be given in [5].

Let us consider the following Cauchy problem:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + f(u), \quad t > 0, \quad x \in \mathbb{R}^d, \\
u(0,x) &= a(x),
\end{align*}
\]

where \( \Delta \) denotes Laplacian differential operator, \( f \) is a non-negative locally Lipschitz continuous function and \( a \) is a bounded non-negative continuous function. In this report, we limit the class of solutions of (1) as follows:

Definition 1.1. \( u(t,x) \) is said to be a positive solution of (1) if there exists positive \( T_\infty (\geq \infty) \) with the following properties (i), (ii) and (iii).

(i) For any positive \( T < T_\infty \), \( u(t,x) \) is bounded and continuous on \([0,T] \times \mathbb{R}^d\).
(ii) \( \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x_i \partial x_j} (i,j = 1,2,\ldots,d) \) exists in \((0,T_\infty) \times \mathbb{R}^d\) and \( u(t,x) \) satisfies (1) in the classical sense.
(iii) \( u(t,x) > 0 \) in \((0,T_\infty) \times \mathbb{R}^d\).

Though small \( T_\infty > 0 \) always satisfies the above conditions, we will take \( T_\infty = T_\infty(a,f) \) as the supremum of \( T_\infty \) satisfying (i)-
(iii). Then $T_{\infty}$ may or may not be infinity. If $T_{\infty} = \infty$, then $u$ is said to be a global solution. Otherwise $u$ is a local solution.

The purpose of this report is to consider "How does the behavior of $f$ near the origin effect to the growth of positive solution as $t \to \infty"? The answer will be given in §2.

When $f(u) = u^{1+\alpha}$, $\alpha > 0$, this problem was first considered by H. Fujita [1]. The main result in [1] is stated as follows: If $ad < 2$, then all the positive solutions of (1) blow up in finite times, that is, there is no global solution of (1) for any non-trivial $a(x) \geq 0$. On the contrary, if $ad > 2$ then there exist global solutions for small $a(x) \geq 0$. About the critical case where $ad = 2$, H. Hayakawa [3] proved the non-existence of global solution for non-trivial $a(x) \geq 0$. Then S. Sugitani [6] considered Cauchy problem for the equation

$$\frac{\partial u}{\partial t} = -(-\Delta)^{\beta} u + u^{1+\alpha}, \ t > 0, \ x \in \mathbb{R}^d,$$

where $0 < \beta < 1$, and obtained the same conclusion for $ad \leq 2\beta$.

On the other hand, Ya. I. Kaneli [4] discussed related problems about (1) in $1$-dimensional case. Among others, he says that if

1. $f(0) = f(1) = 0$, $f(u) > 0$, $0 < u < 1$, $f'(0) > 0$,
2. $a(x) > 0$ on a certain interval and $0 \leq a(x) \leq 1$ everywhere,
3. then the solution $u(t,x)$ of (1)---(3) converges to 1 uniformly on every finite interval as $t \to \infty$. (Though another interesting
results are stated in [4], they are slightly different from our present interest.)

§ 2. Results. Before stating our results, we give notations and terminologies.

$\tilde{F}$ denotes the class of all functions satisfying the following conditions (A) and (B).

(A) $f$ is a locally Lipschitz continuous function on $[0, \infty)$ and $f(0) = 0$, $f(u) > 0$ for $u > 0$.

(B) There exists a positive constant $c_0$ such that

$$f(uv) \geq c_0^{v^{1+2d}} f(u) \quad \text{for} \quad 0 \leq u \leq v, \quad u < c_0 \quad \text{and} \quad uv < c_0.$$

$\hat{F}$ is the class of all non-decreasing functions $f$ satisfying (A) and (C) stated below.

(C) There exists a positive constant $c$ such that

(a) $f(uv) \geq cv^{1+2d} f(u) \quad \text{for} \quad 0 < u \leq v, \quad u < c,$

(b) $f(uv) \geq cv^{2+2d} f(u) \quad \text{for} \quad 0 < v \leq u < c.$

Obviously $\hat{F} \subset \tilde{F}$. (cf. Remark 3.)

Definition 2.1. $T_\infty = T_\infty(a, f)$ in §1 is said to be the blowing up time of the solution of $u(t, x)$ of (1). (i) If $T_\infty$ is finite, then we say that $u(t, x)$ blows up (in finite time). (ii) If $T_\infty = \infty$ and $u(t, x) \to \infty$ uniformly in $x$ on every compact set as $t \to \infty$, then we say that $u(t, x)$ grows up to infinity.

The solution of (1) corresponding to $f$ and $a$ is denoted by $u(t, x; a, f)$.

Now we can state the following
Theorem 1. Let \( f \in \mathcal{F} \). If, for any \( \epsilon > 0 \),

\[
(4) \quad \int_0^\epsilon f(u)/u^{2+\frac{2}{d}} \, du = \infty,
\]

then the positive solution \( u(t,x; a,f) \) of (1) blows up for any non-trivial \( a(x) \geq 0 \).

Theorem 2. Let \( f \in \mathcal{F} \). (i) If (4) holds for any \( \epsilon > 0 \), then any positive solution \( u(t,x; a,f) \) of (1) blows up or grows up to infinity. (ii) If the left hand side of (4) is finite for a certain \( \epsilon > 0 \), then, for small initial data \( a(x) = \alpha e^{-\beta |x|^2} > 0 \), the solution \( u(t,x; a,f) \) of (1) converges to 0 uniformly in \( x \) as \( t \to \infty \).

Theorem 3. Let \( f \) be a Lipschitz continuous function on \([0,1]\) such that \( f(u) > 0 \) for \( 0 < u < 1 \) and \( f(0) = f(1) = 0 \). Moreover we assume that \( f \) satisfies the conditions (B) and (4). Then, for each continuous initial data \( a(x) \) with \( 0 \leq a(x) \leq 1 \), \( a(x) \neq 0 \), the solution \( u(t,x; a,f) \) of (1) converges to 1 uniformly on every compact set \((\subset \mathbb{R}^d)\) as \( t \to \infty \).

Remark 1. The assumptions of (ii) in Theorem 2 can be weakened as follows: \( f \) is a locally Lipschitz continuous function satisfying (iiia) \( f(u) \geq 0 \) and \( f(0) = 0 \), (iiib) \( f(uv) \geq vf(u) \) for \( u \geq 0 \), \( v \geq 1 \), and (iiic) the left hand side of (4) is finite.

Remark 2. For each \( f \in \mathcal{F} \) satisfying (4), there exists \( \tilde{f} \in \mathcal{F} \) such that (4) holds for \( \tilde{f} \) and

\[
\liminf_{u \to 0} \frac{f(u)}{\tilde{f}(u)} > 0.
\]

Remark 3. As an application of Theorem 2, let us consider
the case when \( f \) is given by
\[
f(u) = u^{1+\delta \over 4} \prod_{i=1}^{n-1} \log \left( {1 \over u} \log \left( {1 \over u} \log \cdots \log \left( {1 \over u} \log u \right) \right) \right) \delta - 1
\]

near the origin and smooth and positive in the whole of \((0, \infty)\), where \( \delta > 0 \) and \( \log_k(u) = \log \log \cdots \log u \) \((k\text{-times})\). If \( 0 < \delta \leq 1 \), then \( f \in \mathcal{F} \) and hence any positive solution of (1) blows up or grows up to infinity by Theorem 2, (i). On the other hand, if \( \delta > 1 \), then some positive solution \( u(t, x) \) of (1) converges to 0 uniformly in \( x \) as \( t \to \infty \) by Theorem 2, (ii).

Remark 4. The conditions (B) and (4) of Theorem 1 are concerned with the local behavior of \( f \) near the origin only apart from \( f(u) > 0 \) \((u > 0)\), while the condition (a) of (C) is concerned with the behavior of \( f \) for large \( u \), that is, it implies that
\[
(5) \quad f(u) > \text{const.} u^{1+\delta \over 4} \quad \text{for all sufficiently large } u.
\]

Some condition on the behavior of \( f(u) \) for large \( u \) such as (5) is required for the blowing up conclusion. This aspect will be made much clear by the following theorem which is a slight extension of one of results due to Fujita [2].

Theorem 4. Assume that
\[
(1) \quad \int_{-\infty}^{\infty} {d\lambda \over f(\lambda)} < \infty,
\]
(\( i \)) there exist constants \( c > 0 \) and \( u_0 > 0 \) such that
\[
f(u) \gtrsim c f(v) \quad \text{for } u_0 < v < u.
\]
Let \( u(t, x) \) be a positive solution of (1). If for any \( M > 0 \) there exists \( t_M > 0 \) such that \( u(t_M, x) > M \) for \( |x| < 1 \), then

\[ -5 - \]
u(t,x) blows up.

Remark 5. The following two theorems were used to prove Theorem 1-3.

Theorem 5. Let \( f, \tilde{f} \) be locally Lipschitz continuous functions on \([0, \infty)\), and assume that (1) \( f(u) > 0 \) for \( u > 0 \), (ii) \( \tilde{f} \) is non-decreasing with \( \tilde{f}(0) = 0 \), and (iii)

\[
\lim \inf_{u \downarrow 0} \frac{f(u)}{\tilde{f}(u)} > 0.
\]

Suppose that, for each bounded continuous initial data \( a(x) \geq 0 \), the solution \( u(t,x; a, \tilde{f}) \) of (1) either blows up or satisfies

\[
\lim_{t \to \infty} \| u(t,x; a, \tilde{f}) \|_{\infty} = \infty,
\]

where \( \| \cdot \|_{\infty} \) denotes the supremum norm. Then any positive solution \( u(t,x; a, f) \) of (1) blows up or grows up to infinity.

Theorem 6. Let \( f \) be a Lipschitz continuous function on \([0,1]\) such that \( f(u) > 0 \) for \( 0 < u < 1 \) and \( f(1) = 0 \), and let \( f \) satisfy the same assumptions as in Theorem 5. Moreover we assume that, for any non-negative bounded continuous \( a(x) \equiv 0 \) (\( \not\equiv 0 \)), the solution \( u(t,x; a, \tilde{f}) \) of (1) has the same property as in Theorem 5. Then, for any continuous function \( a(X) \) with \( 0 \leq a(x) \leq 1 \), \( a(x) \not\equiv 0 \), the solution \( u(t,x; a, f) \) of (1) converges to 1 uniformly on each compact set of \( R^d \) as \( t \to \infty \).

References


