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On some evolution equations of subdifferential operators

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1. Introduction

In this paper we are concerned with nonlinear evolution equations of a form

\[ \frac{du}{dt} + \partial \psi^t u(t) + A(t)u(t) \equiv f(t), \quad 0 \leq t \leq T \quad (1.1) \]

in a real Hilbert space $H$. Here for each fixed $t$, $\partial \psi^t$ is subdifferential of a lower semicontinuous convex function $\psi^t$ from $H$ into $(-\infty, \infty]$, $\psi^t \not\equiv \infty$ and $A(t)$ is a monotone, single valued and hemicontinuous operator which is perturbation in a sense. The effective domain of $\psi^t$ defined by

\[ \{u \in H : \psi^t(u) < \infty\} = D \]

is independent of $t$. We denote the inner product and the norm in $H$ by $(\cdot, \cdot)$ and $\| \cdot \|$ respectively. Let $T$ be a positive constant.

We assume the following conditions for $\psi^t$ and $A(t)$.

$A \ - \ (1)$ For every $r > 0$ there exist a positive constant $L_1(r)$ such that

\[ |\psi^t(u) - \psi^s(u)| \leq L_1(r) |h(t) - h(s)| (\psi^t(u) + 1) \]

hold if $0 \leq s, t \leq T$, $u \in D$ and $\|u\| \leq r$, where $h(t)$ is continuous function with bounded total variation.

$A \ - \ (2)$ If $u(t) \in D$ is absolutely continuous on $[a, b]$ $(0 \leq a < b \leq T)$ then $A(t)u(t)$ is strongly measurable on $[a, b]$. 

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and for any fixed $t_0 \in [a, b]$ $A(t_0)u(t)$ is also strongly measurable on $[a, b]$. For any fixed $u \in D$, $A(t)u$ is continuous on $[0, T]$.

$A - (3)$ There are Riemann integrable functions $\omega^2_r(t)$ on $[0, T]$ and a constant $0 < k_r < 1/2$ such that

$$\|A(t)u\| \leq k_r \|\psi^t u\| + \omega^2_r(t) \quad \text{for any} \quad |u| \leq r.$$  

$A - (4)$ If $u(t)$ is absolutely continuous and $|\psi^t u| + \|u(t)\| \leq r$, then $A(t)u(t) \leq \omega^2_r(t)$.

Under the above assumptions we consider the uniqueness and existence of the solution of (1-1) where the solution is defined as follows:

Definition 1-1: We say that $u(t)$ is a solution of (1-1) if and only if $u(t)$ is continuous on $[0, T]$ and absolutely continuous on $(0, T]$ and if (1-1) holds almost everywhere on $[0, T]$.

Theorem 1-1. Suppose that the assumptions stated above are satisfied. Then we hold the unique solution of (1-1) where $f \in L_2[0, T; H]$ and the initial date $u_0 \in \mathcal{D}$.

Remark 1-1. The continuity assumption $A-(1)$ is weaker than those of J. Watanabe [3] and H. Attouch and A. Damlamian [1].

2. The outline of the proof.

Using $\psi^0(a) \geq C\|a\| + D'$ and $A-(1)$, we get the following lemma.
Lemma 2.1 There exist constants $C_1$ and $C_2$ which are independent of $t$ and $\alpha$ such that

$$\psi_t(\alpha) \geq C_1 \|\alpha\| + C_2$$

for any $\alpha \in H$.

We take a sequence $\{t_i\}_{i=1}^n$ such that $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$ and $t_i \in I$ for any $i = 0, 2, \cdots, n$ and $|t_i - t_{i-1}| \to 0$ as $n \to \infty$ for any $i = 1, 2, \cdots, n$.

We denote by

$$\psi^n_t(u) = \psi^{t_i}(u), \quad A_n(t) = A(t_i), \text{ for } t_i \leq t < t_{i+1}.$$

We consider the following evolution equations

$$\begin{cases}
\frac{d}{dt} u_i^i + (\psi^n_t + A_n(t))u_i^i(t) \geq f(t), & t_i \leq t < t_{i+1} \\
u_i^i(t_i) = u_{i-1}^i(t_i) \text{ and } u_0^i(0) = u_0 \in D \text{ for } i = 0, 1, \cdots, n-1 \text{ and } f(t) \in L^2[0, T : H].
\end{cases} \tag{2-1}
$$

The solution of (2-1) is defined inductively by the solution of a operator with constant coefficients. For the sake of simplicity we write $u_n^i(t) = u_i^i(t)$.

Using that $\{u_n^i(t)\}$ are the solutions of (2-1) and lemma 1 we get the following lemma.

Lemma 2.2 There is a constant $\gamma$ independent of $n$ and $t$ such that

$$\|u_n^i(t)\| \leq \gamma.$$

On the other hand since we get
\[
\frac{d}{dt} \psi_n(u_n) + \|\frac{d}{dt} u_n\|^2 = (f(t) - A_n(t)u_n, \frac{d}{dt} u_n) \quad \text{a.e.}\ t
\]

from H. Brezis [2], \( u_n(t) \) is a strong solution of (2-1) and \( \lambda \)- (3) we see

\[
\psi_n(u_n(t)) + \delta \int_{t_{i-1}}^{t_i} \|\frac{d}{dt} u_n\|^2 dt \leq \psi_n(u_n(t_i))
\]

\[
+ \int_{t_{i-1}}^{t_i} C_\delta (\|f\| + w_t)^2 ds \quad (2-2)
\]

from our assumption \( \lambda \)- (3) where \( \delta \) and \( C_\delta \) are positive constants independent of \( n, t \) and \( t_i \). Combining (2-2) and \( \lambda \)- (1) we see

\[
\psi_n(u_n(t_{i+1})) \leq \psi_n(u_n(t_i)) \{1 + L_1(\gamma)|h(t_{i-1}) - h(t_i)|\} + \int_{t_i}^{t_{i+1}} C_\delta (f(s) + W(t_i))^2 ds
\]

\[
+ L_1(\gamma)|h(t_{i-1}) - h(t_i)|. \quad (2-3)
\]

We put

\[
K = \{ \int_0^T 2C_\delta \|f\|^2 ds + 2 \int_0^T w_\gamma^2(t)dt + L_1(\gamma)V(h) + |\psi^0(u_0)| + 1 \}
\]

then from (2-3) we see

\[
|\psi_n(u_n(t))| \leq 3Ke^{KL_1(\gamma)V(h)} \quad (2-4)
\]

where \( V(h) = \text{total variation of } h \text{ on } [0, T] \). Combining (2-3) and (2-4) we get the following lemma.

Lemma 2 - 3 We know

\[
|\psi_n(u_n(t))| + \int_0^t \|\frac{d}{dt} u_n\|^2 dt \leq C_3
\]
where \( C_3 \) is a constant independent of \( n \) and \( t \).

From the above lemma we know that there exists subsequence \( \{ \frac{d}{dt}u_n \} \) which is \( L_2 \)-weakly convergent. For the sake of simplicity we put \( u_n = u_{n_j} \). Thus we see that \( u_n(t) \) is weak convergence to \( u(t) \) and \( u(t) \) is absolutely continuous on \([0, T]\). On the other hand since \( u_n(t) \) is the solution of (2-1) we find

\[
\int_0^T \psi^S_n(v(s)) ds - \int_0^T \psi^S_n(u_n(s)) ds \geq \int_0^T (f(s) - A_n(s)u_n(s) - \frac{d}{ds}u_n(s), v(s) - u_n(s)) ds \geq
\]

\[
\int_0^T (f(s) - A_n(s)v(s) - \frac{d}{ds}v(s), v(s) - u_n(s)) ds +
\]

\[+ \frac{1}{2} \| u_0 - v(0) \|^2.
\]

Then

\[
\int_0^T (\psi^S(v(s)) - \psi^S(u(s))) ds \geq \int_0^T (f(s) - \Lambda(s)v(s) - \frac{d}{dt}v(s), v(s) - u(s)) ds + \frac{1}{2} \| u_0 - v(0) \|^2.
\]

Next we put \( v(t) = pu(t) + (1-p)w(t) \) where \( w(t) \in D \) and is absolutely continuous.

Thus we obtain the following inequality

\[
\int_0^T (\psi^S(w(s)) - \psi^S(u(s))) ds \geq \int_0^T (f(s) - \Lambda(s)u(s) - \frac{d}{dt}u(s), w(s) - u(s)) ds.
\]

Next for any fixed \( \xi \in D \) and \( 0 \leq t_1 < t_2 \leq T \) we put
\[ w(t) = \begin{cases} 
\xi : t_1 + \epsilon \leq t \leq t_2 - \epsilon \\
p\psi(t_1) + q\xi : t = pt_1 + q(t_1 + \epsilon) \\
u(t) : 0 \leq t \leq t_1, \ t_2 \leq t \leq T \\
p\psi(t_2) + q\xi : t = pt_2 + (t_2 - \epsilon)q 
\end{cases} \]

where \( p + q = 1 \), \( p > 0 \), \( q > 0 \) and \( \epsilon > 0 \).

If \( \epsilon \to 0 \) we get

\[
\int_{t_1}^{t_2} \psi^t(\xi)dt - \int_{t_1}^{t_2} \psi^t(u(t))dt \geq \int_{t_1}^{t_2} (f(t) - A(t)u(t) - \frac{d}{dt}u(t), \xi - u(t)) dt.
\]

For any Lebesgue points of \( \psi^t u(t), f(t), A(t)u(t), \frac{d}{dt}u(t) \), and \( u(t) \) we know

\[
\psi^t(\xi) - \psi^t u(t) \geq (f(t) - A(t)u(t) - \frac{d}{dt}u(t), \xi - u(t)).
\]

Considering that \( \partial \psi^t + A(t) \) is monotone operator we can show the uniqueness of (1-1). If \( u_0 \in D \) we can proved the theorem.

Next if \( u_0 \in \mathcal{D} \). We put \( u_{m,0} = (1 + 1/m\psi^0)^{-1}u_0 \). We denote by \( u_m(t) \) the solution of (1-1) of initial date \( u_{m,0} \).

Since \( \partial \psi^t + A(t) \) is monotone operator we see that \( u_m(t) \) is uniformly convergent on \( [0, T] \) then \( \lim_{m \to \infty} u_m(t) = u(t) \).

Using that \( u_m(t) \) are strong solutions of (1-1) and A-(3) we know for any \( 0 < \delta < T \),

\[
\int_{0}^{\delta} \psi^t(u_m(t))dt \leq C_4
\]
where $C_4$ is a constant independent of $\delta$ and $m$.

There exist $0 < \delta_m < \delta$ such that

$$\psi^m_r(u_m(\delta_m)) \leq \frac{1}{\delta} \int_0^\delta \psi^t_r(u_m(t)) dt \leq \frac{C_4}{\delta} = C_5.$$ 

We denote by $V_m(t)$ the solution of (1-1) for the initial date $V(\delta_m) = u_m(\delta_m) \in D$ on $[\delta_m, T]$. Then we find $V_m(t) = u_m(t)$ on $[\delta_m, T]$ from the uniqueness of the solution of (1-1).

On the other hand noting the method of Lemma 2-3 we get

$$|\psi_{m,n}^t(V_m^n(t))| \leq C_6 \text{ for } t \in [\delta_m, T]$$

where $C_6$ is independent of $n$ and $m$.

Thus we get

$$\int_0^T \| \frac{dV_m}{dt}(t) \|^2 dt \leq \int_0^T \| \frac{dV_m}{dt}(t) \|^2 dt \leq C_7$$

Using the above same method on $[\delta, T]$ we can prove the Theorem.

Bibliography


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