

Title	On Some Evolution Equations of Subdifferential Operators (非線形問題の解析 : Analysis of Nonlinear Problems, RIMS, 1974)
Author(s)	MARUO, KENJI
Citation	数理解析研究所講究録 (1975), 258: 97-103
Issue Date	1975-12
URL	<a href="http://hdl.handle.net/2433/105783">http://hdl.handle.net/2433/105783</a>
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

On some evolution equations of  
subdifferential operators

By Kenji Maruo

1. Introduction

In this paper we are concerned with nonlinear evolution equations of a form

$$\frac{du}{dt} + \partial\psi^t u(t) + A(t)u(t) \ni f(t), \quad 0 \leq t \leq T \quad (1.1)$$

in a real Hilbert space  $H$ . Here for each fixed  $t$ ,  $\partial\psi^t$  is subdifferential of a lower semicontinuous convex function  $\psi^t$  from  $H$  into  $(-\infty, \infty]$ ,  $\psi^t \not\equiv \infty$  and  $A(t)$  is a monotone, single valued and hemicontinuous operator which is perturbation in a sense. The effective domain of  $\psi^t$  defined by  $\{u \in H : \psi^t(u) < +\infty\} = D$  is independent of  $t$ . We denote the inner product and the norm in  $H$  by  $(\cdot, \cdot)$  and  $\|\cdot\|$  respectively. Let  $T$  be a positive constant.

We assume the following conditions for  $\psi^t$  and  $A(t)$ .

A - (1) For every  $r > 0$  there exist a positive constant  $L_1(r)$  such that

$$|\psi^t(u) - \psi^s(u)| \leq L_1(r) |h(t) - h(s)| \{\psi^t(u) + 1\}$$

hold if  $0 \leq s, t \leq T$ ,  $u \in D$  and  $\|u\| \leq r$ , where  $h(t)$  is continuous function with bounded total variation.

A - (2) If  $u(t) \in D$  is absolutely continuous on  $[a, b]$  ( $0 \leq a < b \leq T$ ) then  $A(t)u(t)$  is strongly measurable on  $[a, b]$

and for any fixed  $t_0 \in [a, b]$   $A(t_0)u(t)$  is also strongly measurable on  $[a, b]$ . For any fixed  $u \in D$ ,  $A(t)u$  is continuous on  $[0, T]$ .

A - (3) There are Riemann integrable functions  $w_r^2(t)$  on  $[0, T]$  and a constant  $0 < k_r < 1/2$  such that

$$\|A(t)u\| \leq k_r \|\partial \psi^t u\| + w_r(t) \quad \text{for any } \|u\| \leq r.$$

A - (4) If  $u(t)$  is absolutely continuous and  $|\psi^t(u)| + \|u(t)\| \leq r$ , then  $A(t)u(t) \leq W_r(t)$ .

Under the above assumptions we consider the uniqueness and existence of the solution of (1-1) where the solution is defined as follows:

Definition 1 - 1: We say that  $u(t)$  is a solution of (1-1) if and only if  $u(t)$  is continuous on  $[0, T]$  and absolutely continuous on  $(0, T]$  and if (1-1) holds almost everywhere on  $[0, T]$ .

Theorem 1 - 1. Suppose that the assumptions stated above are satisfied. Then we hold the unique solution of (1-1) where  $f \in L_2[0, T; H]$  and the initial date  $u_0 \in \bar{D}$ .

Remark 1 - 1. The continuity assumption A-(1) is weaker than those of J. Watanabe [3] and H. Attouch and A. Damlamian [1].

2. The outline of the proof.

Using  $\psi^0(a) \geq C\|a\| + D'$  and A-(1), we get the following lemma.

Lemma 2 - 1 There exist constants  $C_1$  and  $C_2$  which are independent of  $t$  and  $\alpha$  such that

$$\psi^t(\alpha) \geq C_1 \|\alpha\| + C_2 \quad \text{for any } \alpha \in H.$$

We take a sequence  $\{t_i\}_{i=1}^n$  such that  $0 = t_0 < t_1 < \dots < t_{n-1} < t_n = T$  and  $t_i \in I$  for any  $i = 0, 2, \dots, n$  and  $|t_i - t_{i-1}| \rightarrow 0$  as  $n \rightarrow \infty$  for any  $i = 1, 2, \dots, n$ . We denote by

$$\psi_n^t(u) = \psi^{t_i}(u), \quad A_n(t) = A(t_i), \quad \text{for } t_i \leq t < t_{i+1}.$$

We consider the following evolution equations

$$\begin{cases} \frac{d}{dt} u_n^i + (\partial \psi_n^t + A_n(t)) u_n^i(t) \ni f(t) & t_i \leq t < t_{i+1} \\ u_n^i(t_i) = u_n^{i-1}(t_i) \quad \text{and} \quad u_n^0(0) = u_0 \in D & \text{for } i = 0, 1, \dots \\ \dots n-1 \quad \text{and} \quad f(t) \in L^2[0, T; H]. & \end{cases} \quad (2-1)$$

The solution of (2-1) is defined inductively by the solution of a operator with constant coefficients. For the sake of simplicity we write  $u_n(t) = u_n^i(t)$ .

Using that  $\{u_n(t)\}$  are the solutions of (2-1) and lemma 1 we get the following lemma.

Lemma 2 - 2 There is a constant  $\gamma$  independent of  $n$  and  $t$  such that

$$\|u_n(t)\| \leq \gamma.$$

On the other hand since we get

$$\frac{d}{dt} \psi_n^t(u_n) + \left\| \frac{d}{dt} u_n \right\|^2 = (f(t) - A_n(t)u_n, \frac{d}{dt} u_n) \quad \text{a.e. } t$$

from H. Brezis [2],  $u_n(t)$  is a strong solution of (2-1) and A-(3) we see

$$\begin{aligned} \psi_n^t(u_n(t)) + \delta \int_{t_i}^t \left\| \frac{d}{dt} u_n \right\|^2 dt &\leq \psi_n^{t_i}(u_n(t_i)) \\ &+ \int_{t_i}^t C_\delta (\|f\| + w_r)^2 ds \end{aligned} \quad (2-2)$$

from our assumption A-(3) where  $\delta$  and  $C_\delta$  are positive constants independent of  $n$ ,  $t$  and  $t_i$ . Combining (2-2) and A-(1) we see

$$\begin{aligned} \psi_n^{t_i}(u_n(t_{i+1})) &\leq \psi_n^{t_i}(u_n(t_i)) \{1 + L_1(\gamma) |h(t_{i-1}) - h(t_i)|\} \\ &+ \int_{t_i}^{t_{i+1}} C_\delta (f(s) + W(t_i))^2 ds \\ &+ L_1(\gamma) |h(t_{i-1}) - h(t_i)|. \end{aligned} \quad (2-3)$$

We put

$$K = \left\{ \int_0^T 2C_\delta \|f\|^2 ds + 2 \int_0^T w_Y^2(t) dt + L_1(\gamma)V(h) + |\psi^0(u_0)| + 1 \right\}$$

then from (2-3) we see

$$|\psi_n^t(u_n(t))| < 3Ke^{KL_1(\gamma)V(h)} \quad (2-4)$$

where  $V(h) = \text{total variation of } h \text{ on } [0, T]$ .

Combining (2-3) and (2-4) we get the following lemma.

Lemma 2 - 3 We know

$$|\psi_n^t(u_n(t))| + \int_0^t \left\| \frac{du_n}{dt} \right\|^2 dt \leq C_3$$

where  $C_3$  is a constant independent of  $n$  and  $t$ .

From the above lemma we know that there exists subsequence  $\{\frac{d}{dt}u_{n_j}\}$  which is  $L_2$ -weakly convergent. For the sake of simplicity we put  $u_n = u_{n_j}$ . Thus we see that  $u_n(t)$  is weak convergence to  $u(t)$  and  $u(t)$  is absolutely continuous on  $[0, T]$ . On the other hand since  $u_n(t)$  is the solution of (2-1) we find

$$\begin{aligned} \int_0^T \psi_n^s(v(s)) ds - \int_0^T \psi_n^s(u_n(s)) ds &\geq \int_0^T (f(s) - A_n(s)u_n(s) \\ &\quad - \frac{d}{ds}u_n(s), v(s) - u_n(s)) ds \geq \\ \int_0^T (f(s) - A_n(s)v(s) - \frac{d}{ds}v(s), v(s) - u_n(s)) ds &+ \\ + 1/2 \|u_0 - v(0)\|^2. \end{aligned}$$

Then

$$\int_0^T (\psi^s(v(s)) - \psi^s(u(s))) ds \geq \int_0^T (f(s) - A(s)v(s) - \frac{d}{dt}v(s), v(s) - u(s)) ds + 1/2 \|u_0 - v(0)\|^2.$$

Next we put  $v(t) = pu(t) + (1-p)w(t)$  where  $w(t) \in D$  and is absolutely continuous.

Thus we obtain the following inequality

$$\int_0^T (\psi^s(w(s)) - \psi^s(u(s))) ds \geq \int_0^T (f(s) - A(s)u(s) - \frac{d}{dt}u(s), w(s) - u(s)) ds.$$

Next for any fixed  $\xi \in D$  and  $0 \leq t_1 < t_2 \leq T$  we put

$$w(t) = \begin{cases} \xi & : t_1 + \varepsilon \leq t \leq t_2 - \varepsilon \\ pu(t_1) + q\xi & : t = pt_1 + q(t_1 + \varepsilon) \\ u(t) & : 0 \leq t \leq t_1, t_2 \leq t \leq T \\ pu(t_2) + q\xi & : t = pt_2 + (t_2 - \varepsilon)q \end{cases}$$

where  $p + q = 1$ ,  $p > 0$ ,  $q > 0$  and  $\varepsilon > 0$ .

If  $\varepsilon \rightarrow 0$  we get

$$\int_{t_1}^{t_2} \psi^t(\xi) dt - \int_{t_1}^{t_2} \psi^t(u(t)) dt \geq \int_{t_1}^{t_2} (f(t) - A(t)u(t) - \frac{d}{dt}u(t) - u(t)) dt.$$

For any Lebesgue points of  $\psi^t u(t)$ ,  $f(t)$ ,  $A(t)u(t)$ ,  $\frac{d}{dt}u(t)$ , and  $u(t)$  we know

$$\psi^t(\xi) - \psi^t u(t) \geq (f(t) - A(t)u(t) - \frac{d}{dt}u(t), \xi - u(t)).$$

Considering that  $\partial \psi^t + A(t)$  is monotone operator we can show the uniqueness of (1-1). If  $u_0 \in D$  we can prove the theorem.

Next if  $u_0 \in \bar{D}$ . We put  $u_{m,0} = (1 + 1/m \partial \psi^0)^{-1} u_0$ . We denote by  $u_m(t)$  the solution of (1-1) of initial data  $u_{m,0}$ . Since  $\partial \psi^t + A(t)$  is monotone operator we see that  $u_m(t)$  is uniformly convergent on  $[0, T]$  then  $\lim_{m \rightarrow \infty} u_m(t) = u(t)$ .

Using that  $u_m(t)$  are strong solutions of (1-1) and A-(3) we know for any  $0 < \delta < T$ ,

$$\int_0^\delta \psi^t(u_m(t)) dt \leq C_4$$

where  $C_4$  is a constant independent of  $\delta$  and  $m$ .

There exist  $0 < \delta_m < \delta$   $m = 1, 2, \dots$  such that

$$\psi^{\delta_m}(u_m(\delta_m)) \leq \frac{1}{\delta} \int_0^{\delta} \psi^t(u_m(t)) dt \leq \frac{C_4}{\delta} = C_5.$$

We denote by  $V_m(t)$  the solution of (1-1) for the initial data

$V(\delta_m) = u_m(\delta_m) \in D$  on  $[\delta_m, T]$ . Then we find  $v_m(t) = u_m(t)$

on  $[\delta_m, T]$  from the uniqueness of the solution of (1-1).

On the other hand noting the method of Lemma 2-3 we get

$$|\psi_n^t(v_m^n(t))| \leq C_6 \quad \text{for } t \in [\delta_m, T]$$

where  $C_6$  is independent of  $n$  and  $m$ .

Thus we get

$$\int_{\delta}^T \left\| \frac{du_m}{dt}(t) \right\|^2 dt \leq \int_{\delta_m}^T \left\| \frac{dv_m}{dt}(t) \right\|^2 dt \leq C_7$$

Using the above same method on  $[\delta, T]$  we can prove the Theorem.

#### Bibliography

- [1] H. Attouch et Damlamian: Problèmes dévolution dans Les Hilbert et applications. to appear.
- [2] H. Brezis: Propriétés régularisantes de certains semi groupes non linéaires, Israel. J. Math. 9 (1971), 123-144.
- [3] J. Watanabe: On certain nonlinear evolution equations J. Math. Soc. Japan, 25 (1973), 446-463.