<table>
<thead>
<tr>
<th>Title</th>
<th>On Certain Nonlinear Parabolic Variational Inequalities in Hilbert Spaces (非線形問題の解析　非線形問題の解析　非線形問題の解析)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>KENMOCHI, NOBUYUKI</td>
</tr>
<tr>
<td>Citation</td>
<td>数理解析研究所講究録 (1975), 258: 88-96</td>
</tr>
<tr>
<td>Issue Date</td>
<td>1975-12</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/105784">http://hdl.handle.net/2433/105784</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>
On certain nonlinear parabolic variational inequalities
in Hilbert spaces

By
Nobuyuki KENMOCHI
Department of Mathematics, Faculty of Education,
Chiba University

1. Introduction. Let H be a (real) Hilbert space and T be
a fixed positive number. Let \( \{ \phi_t; 0 \leq t \leq T \} \) be a family of proper
l.s.c. (lower semicontinuous) convex functions on H. Assume that
for each \( v \in L^2(0, T; H) \) the function \( t \mapsto \phi_t(v(t)) \) is measurable
on \( (0, T) \). Then for any given \( u_0 \in H \) and \( f \in L^2(0, T; H) \) we
consider the Cauchy problem:

\[
\text{(E)} \quad \frac{d}{dt} u(t) + \partial \phi_t(u(t)) \ni f(t) \quad \text{on } [0, T],
\]

\[
\text{(I)} \quad u(0) = u_0,
\]

where for each \( t \), \( \partial \phi_t \) is the subdifferential of \( \phi_t \). This kind of
Cauchy problem has been studied by many mathematicians; for
instance, we can recall results of Brézis [4], Watanabe [10],
Moreau [8], Péraiba [9], Attouch-Damlamian [2], Attouch-Bénilan-
Damlamian-Picard [1] and the author [5].

In [4] Brézis treated the case of

\[ \phi_t = \phi + I_{K(t)}, \]

where \( \phi \) is a time-independent proper l.s.c. convex function on H,
\( K(t) \) is a closed convex subset of H with parameter \( t \) and \( I_{K(t)} \)
is the indicator function of \( K(t) \). Also, Watanabe [10] and
Attouch-Damlamian [2] dealt with this Cauchy problem. But they required that the effective domain $D(\phi_t)$ of $\phi_t$ is invariant with respect to the time $t$. By the effective domain of $\phi_t$ we mean the set of all $x \in H$ such that $\phi_t(x) < \infty$. In this paper we are going to treat the case where the effective domain of $\phi_t$ may change with the time $t$.

As is easily seen, the evolution equation (E) is translated into the following parabolic variational inequality:

$$
\begin{cases}
\left\{ \begin{array}{l}
\int_0^T (u'(t) - f(t), u(t) - v(t)) dt \leq \Phi(v) - \Phi(u) \\
\text{whenever } v \in D(\Phi) = \{ v \in L^2(0, T; H) ; \phi_t(v(t)) \in L^1(0, T) \},
\end{array} \right.
\end{cases}
$$

where $\Phi$ is a function on $L^2(0, T; H)$ given by

$$
\Phi(v) = \begin{cases} 
\int_0^T \phi_t(v(t)) dt & \text{if } v \in D(\Phi), \\
\infty & \text{otherwise}.
\end{cases}
$$

Therefore we consider the Cauchy problem for this parabolic variational inequality (V) instead of (E).

2. Formulation of a problem $P[\phi_t, f, u_0]$. Let us formulate a problem precisely. Denote by $D_0$ the effective domain of $\phi_0$, and by $D$ the closure of $D_0$ in $H$. Then, given $u_0 \in D$ and $f \in L^2(0, T; H)$ we formulate the problem $P[\phi_t, f, u_0]$ to find a function $u \in C([0, T]; H)$ such that

(a) $u(0) = u_0$;

(b) $u \in D(\phi)$ (and hence $\phi_t(u(t)) < \infty$ for a.e. $t \in [0, T]$);
(c) \( u' = (d/dt)u \in L^2(0, T; H); \)

(d) \((V)\) holds.

Such a function \(u\) is called a strong solution of \(P[\phi_t, f, u_0]\), while a function \(u \in C([0, T]; H)\) is often called a weak solution of \(P[\phi_t, f, u_0]\), if conditions (a), (b) and the following (e) are satisfied:

\[
(e) \quad \left\{ \begin{array}{l}
\int_0^T \left( (v' - f, u - v) dt - \frac{1}{2} ||u_0 - v(0)||^2 \right. \\
\left. \leq \phi(v) - \phi(u) \quad \text{whenever } v \in D(\phi) \text{ and } v' \in L^2(0, T; H). \right.
\end{array} \right.
\]

Before stating a sufficient condition for a strong or weak solution of \(P[\phi_t, f, u_0]\) to exist, we consider a simple example.

Example. Let us take \(H = L^2(0, 1)\) and consider a function \(\beta\) as follows:

\[
\beta(r) = \begin{cases} 
  r & \text{if } r < 0, \\
  \tan r & \text{if } 0 \leq r < \pi/2, \\
  \infty & \text{if } r \geq \pi/2.
\end{cases}
\]

Define proper l.s.c. convex functions \(\phi^1\) and \(\phi^2\) on \(L^2(0, 1)\) by the following:

\[
\phi^1(v) = \frac{1}{2} \|v\|^2,
\]

\[
\phi^2(v) = \int_0^1 \int_0^1 v(x) \beta(r) dx dr.
\]

Then we set

\[
\phi_t(v) = \begin{cases} 
  \phi^1(v) & \text{if } t \in [0, \pi/2), \\
  \phi^2(v) & \text{if } t \in [\pi/2, 2].
\end{cases}
\]
and consider the Cauchy problem:

\[
\begin{cases}
(a) & \int_0^2 (u', u - v)dt \leq \Phi(v) - \Phi(u) \quad \text{for all } v \in D(\Phi), \\
(b) & u(0) = u_0 \in L^2(0, 1).
\end{cases}
\]

Clearly, the inequality (a) is equivalent to the evolution equation

\[
u' + \Phi_t(u) = 0 \quad \text{on } [0, 2].
\]

If this Cauchy problem (*) has a strong solution \( u \), then we have

\[
u(t) = u_0 e^{-t} \quad \text{on } [0, \pi/2],
\]

because \( \Phi_t \) is the identity for any \( t \in [0, \pi/2] \). Moreover, the function \( u \) must satisfy

\[
\begin{cases}
\quad u' + \Phi^2(u) = 0 \quad \text{on } [\pi/2, 2], \\
\quad u(\pi/2) = u_0 e^{-\pi/2} (\in D(\Phi^2)),
\end{cases}
\]

that is, \( u \) is a strong solution of the Cauchy problem (**) on \([\pi/2, 2]\). Therefore \( u_0 e^{-\pi/2} \) must be contained in the effective domain \( D(\Phi^2) \) of \( \Phi^2 \). But this is impossible if \( u_0 \) is sufficiently large, because

\[
D(\Phi^2) \subset \{ v \in L^2(0, 1); v(x) < \pi/2 \text{ a.e. } x \in (0, 1) \}.
\]

Thus for a sufficiently large initial data, the Cauchy problem (*) cannot have a strong or even weak solution. Such a phenomenon arises from the fact that the effective domain of \( \Phi_t \) undergoes a change from a large set into a small set suddenly at the time \( \pi/2 \), so we can say about the problem \( P[\Phi_t, f, u_0] \) that in order for a strong solution to exist the effective domain of \( \Phi_t \) should move smoothly with the time in a sense, in particular when the
effective domain of $\phi_t$ is decreasing.

In this note, we require the following assumption on the time-dependence of the family $\{\phi_t\}$:

**Assumption.** For each $t \in [0, T]$, $x \in H$ with $\phi_t(x) < \infty$ and $s \in [t, T]$, there is an element $\hat{x} \in H$ such that

$$\|\hat{x} - x\| \leq \text{const.} |t - s|,$$

$$\phi_s(\hat{x}) \leq \phi_t(x) + \text{const.} |t - s|(1 + \|x\|^2 + |\phi_t(x)|),$$

where these constants are independent of $t, x, s$ and $\hat{x}$.

By the way, the family $\{\phi_t\}$ in the Example does not satisfy the Assumption at $t = \pi/2$. If we exchange $\phi^1$ for $\phi^2$ in the Example, the family $\{\phi_t\}$ given by this exchange satisfies the Assumption. More generally, if $\phi_t(x)$ is a decreasing function in $t$, then the Assumption is trivially satisfied.

3. **Main results.** Under the Assumption mentioned in the previous section, we establish the following existence theorem.

**Theorem 1.** 1) If $u_0 \in D_0$ and $f \in L^2(0, T; H)$, then $P[\phi_t, f, u_0]$ has a unique strong solution $u$ such that $t \to \phi_t(u(t))$ is bounded on $[0, T]$.

2) If $u_0 \in D$ and $f \in L^2(0, T; H)$, then $P[\phi_t, f, u_0]$ has a unique weak solution $u$ such that for any positive number $\delta$,

$$u' \in L^2(\delta, T; H),$$

$$t \to \phi_t(u(t))$$

is bounded on $[\delta, T]$.

So far as a weak solution is concerned, we see the following:
Let \( u_0 \) be any element of \( D \) and \( f \) be any function in \( L^2(0, T; H) \). Then a function \( u \in L^2(0, T; H) \) is a weak solution of \( P[\phi_t, f, u_0] \) if and only if there are sequences \( \{f_n\} \subseteq L^2(0, T; H) \), \( \{u_{0,n}\} \subseteq D \) and \( \{u_n\} \subseteq C([0, T]; H) \) such that each \( u_n \) is a strong solution of \( P[\phi_t, f_n, u_{0,n}] \) and

\[
\begin{align*}
  f_n & \to f \text{ in } L^2(0, T; H), \\
  u_{0,n} & \to u_0 \text{ in } H, \\
  u_n & \to u \text{ in } L^2(0, T; H)
\end{align*}
\]

as \( n \to \infty \).

Moreover, for any given \( u_0 \in D \), define a multivalued operator \( M_{u_0} \) from \( L^2(0, T; H) \) into itself by the following:

\[
  f \in M_{u_0}(u) \iff u \text{ is a weak solution of } P[\phi_t, f, u_0].
\]

Then we see that \( f \in M_{u_0}(u) \) if and only if \( u \in D(\phi) \) and \( (\text{e}) \) holds, and have an interesting result about the operator \( M_{u_0} \).

**Theorem 2.** For each \( u_0 \in D \), \( M_{u_0} \) is a maximal monotone operator in \( L^2(0, T; H) \).

Remark. In particular, when \( \phi_t \) is time-independent, Theorem 2 was proved by Brézis [3].

Remark. Detail proofs of Theorems 1 and 2 are found in [6] and [7], respectively.

4. **Construction of a strong solution.** Finally we state how to construct a strong solution of \( P[\phi_t, f, u_0] \). Here we
employ a finite difference method with respect to \( t \).

For each positive integer \( N \) we set

\[
\varepsilon_N = T/N \quad \text{and} \quad f_{N,n} = \varepsilon_N^{-1} \int_{\varepsilon_N(n-1)}^{\varepsilon_N n} f(t) \, dt, \quad n = 1, 2, \ldots, N,
\]

and successively define a sequence \( \{ u_{N,n} \}_{n=1}^N \) as follows:

\[
u_{N,0} = u_0,
\]

\[
(***) \quad (u_{N,n} - u_{N,n-1})/\varepsilon_N + \exists \phi_{\varepsilon_N n}(u_{N,n}) \exists f_{N,n}, \quad n = 1, 2, \ldots, N;
\]

when the element \( u_{N,n-1} \) in the \((n-1)\)-th step is defined, the next element \( u_{N,n} \) is chosen so that the relation (***') is satisfied. In fact, such an element \( u_{N,n} \) exists, since \( \exists \phi_{\varepsilon_N n} \) is maximal monotone in \( H \).

Now, we put

\[
\begin{align*}
u_N(t) &= u_{N,n} \quad \text{if } t \in [\varepsilon_N(n-1), \varepsilon_N n),
\v_n u_N(t) &= (u_{N,n} - u_{N,n-1})/\varepsilon_N \quad \text{if } n = 1, 2, \ldots, N
\end{align*}
\]

to obtain two sequences \( \{ u_N \}_{N=1}^\infty \) and \( \{ \nabla_N u_N \}_{N=1}^\infty \) of simple functions.

If \( u_0 \in D_0 \) and \( f \in L^2(0, T; H) \), we can show by using the Assumption that \( \{ u_N \} \) is bounded in \( L^\infty(0, T; H) \) and \( \{ \nabla_N u_N \} \) is bounded in \( L^2(0, T; H) \). So we can choose a weakly* convergent subsequence \( \{ u_{N_k} \} \) and a weakly convergent subsequence \( \{ \nabla_{N_k} u_{N_k} \} \):

\[
u_{N_k} \rightharpoonup u \quad \text{weakly* in } L^\infty(0, T; H)
\]

and

\[
\nabla_{N_k} u_{N_k} \rightharpoonup v \quad \text{weakly in } L^2(0, T; H).
\]

Then we have \( u' = v \) and can show that the limit \( u \) is the required strong solution.
ACKNOWLEDGEMENT. The author would like to express his hearty thanks to Professor H. Brézis who kindly gave the author many valuable advices about Theorems 1 and 2.

References


