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On certain nonlinear parabolic variational inequalities in Hilbert spaces

By

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1. Introduction. Let H be a (real) Hilbert space and T be a fixed positive number. Let $\{\phi_t;\ 0 \le t \le T\}$ be a family of proper l.s.c. (lower semicontinuous) convex functions on H. Assume that for each $v \in L^2(0, T; H)$ the function $t \to \phi_t(v(t))$ is measurable on $(0, T)$. Then for any given $u_o \in H$ and $f \in L^2(0, T; H)$ we consider the Cauchy problem:

$$\text{(E)} \qquad (d/dt)u(t) + \partial\phi_t(u(t)) \ni f(t) \qquad \text{on } [0, T],$$

$$\text{(I)} \qquad u(0) = u_o,$$

where for each t, $\partial\phi_t$ is the subdifferential of ϕ_t. This kind of Cauchy problem has been studied by many mathematicians; for instance, we can recall results of Brézis [4], Watanabe [10], Moreau [8], Péralba [9], Attouch-Damlamian [2], Attouch-Bénilan-Damlamian-Picard [1] and the author [5].

In [4] Brézis treated the case of

$$\phi_t = \phi + I_{K(t)},$$

where ϕ is a time-independent proper l.s.c. convex function on H, K(t) is a closed convex subset of H with parameter t and $I_{K(t)}$ is the indicator function of K(t). Also, Watanabe [10] and

Attouch-Damlamian [2] dealt with this Cauchy problem. But they required that the effective domain $D(\phi_t)$ of ϕ_t is invariant with respect to the time t. By the effective domain of ϕ_t we mean the set of all $x \in H$ such that $\phi_t(x) < \infty$. In this paper we are going to treat the case where the effective domain of ϕ_t may change with the time t.

As is easily seen, the evolution equation (E) is translated into the following parabolic variational inequality:

$$(V) \quad \begin{cases} \int_0^T (u'(t) - f(t), u(t) - v(t))dt \leq \Phi(v) - \Phi(u) \\ \text{whenever } v \in D(\Phi) \equiv \{v \in L^2(0, T; H); \phi_t(v(t) \in L^1(0, T)\}, \end{cases}$$

where Φ is a function on $L^2(0, T; H)$ given by

$$\Phi(v) = \begin{cases} \int_0^T \phi_t(v(t))dt & \text{if } v \in D(\Phi), \\ \infty & \text{otherwise.} \end{cases}$$

Therefore we consider the Cauchy problem for this parabolic variational inequality (V) instead of (E).

2. <u>Formulation of a problem $P[\phi_t, f, u_o]$</u>. Let us formulate a problem precisely. Denote by D_o the effective domain of ϕ_0, and by D the closure of D_o in H. Then, given $u_o \in D$ and $f \in L^2(0, T; H)$ we formulate the problem $P[\phi_t, f, u_o]$ to find a function $u \in C([0, T]; H)$ such that

(a) $u(0) = u_o$;

(b) $u \in D(\Phi)$ (and hence $\phi_t(u(t)) < \infty$ for a.e. $t \in [0, T]$);

(c) $u' = (d/dt)u \in L^2(0, T; H)$;

(d) (V) holds.

Such a function u is called a strong solution of $P[\phi_t, f, u_o]$, while a function $u \in C([0, T]; H)$ is often called a weak solution of $P[\phi_t, f, u_o]$, if conditions (a), (b) and the following (e) are satisfied:

$$\text{(e)} \quad \begin{cases} \int_0^T (v' - f, u - v)dt - \frac{1}{2}\|u_o - v(0)\|^2 \\ \leq \Phi(v) - \Phi(u) \quad \text{whenever } v \in D(\Phi) \text{ and } v' \in L^2(0, T; H). \end{cases}$$

Before stating a sufficient condition for a strong or weak solution of $P[\phi_t, f, u_o]$ to exist, we consider a simple example.

<u>Example</u>. Let us take $H = L^2(0, 1)$ and consider a function β as follws:

$$\beta(r) = \begin{cases} r & \text{if } r < 0, \\ \tan r & \text{if } 0 \leq r < \pi/2, \\ \infty & \text{if } r \geq \pi/2. \end{cases}$$

Define proper l.s.c. convex functions ϕ^1 and ϕ^2 on $L^2(0, 1)$ by the following:

$$\phi^1(v) = \frac{1}{2}\|v\|^2,$$

$$\phi^2(v) = \int_0^1 \int_0^{v(x)} \beta(r)drdx.$$

Then we set

$$\phi_t(v) = \begin{cases} \phi^1(v) & \text{if } t \in [0, \pi/2), \\ \phi^2(v) & \text{if } t \in [\pi/2, 2]. \end{cases}$$

and consider the Cauchy problem:

$$(*)\begin{cases}(a) & \int_0^2 (u', u - v)dt \leq \Phi(v) - \Phi(u) \quad \text{for all } v \in D(\Phi),\\ (b) & u(0) = u_o \in L^2(0, 1).\end{cases}$$

Clearly, the inequality (a) is equivalent to the evolution equation

$$u' + \partial\phi_t(u) = 0 \qquad \text{on } [0, 2].$$

If this Cauchy problem (*) has a strong solution u, then we have

$$u(t) = u_o e^{-t} \qquad \text{on } [0, \pi/2],$$

because $\partial\phi_t$ is the identity for any $t \in [0, \pi/2)$. Moreover, the function u must satisfy

$$(**)\begin{cases} u' + \partial\phi^2(u) = 0 & \text{on } [\pi/2, 2],\\ u(\pi/2) = u_o e^{-\pi/2} \ (\in D(\phi^2)), & \end{cases}$$

that is, u is a strong solution of the Cauchy problem (**) on $[\pi/2, 2]$. Therefore $u_o e^{-\pi/2}$ must be contained in the effective domain $D(\phi^2)$ of ϕ^2. But this is impossible if u_o is sufficiently large, because

$$D(\phi^2) \subset \{\rho \in L^2(0, 1);\ \rho(x) < \pi/2 \text{ a.e. } x \in (0, 1)\}.$$

Thus for a sufficiently large initial data, the Cauchy problem (*) cannot have a strong or even weak solution. Such a phenomenon arises from the fact that the effective domain of ϕ_t undergoes a change from a large set into a small set suddenly at the time $\pi/2$, so we can say about the problem $P[\phi_t, f, u_o]$ that in oder for a strong solution to exist the effective domain of ϕ_t should move smoothly with the time in a sense, in particular when the effective domain of ϕ_t is decreasing.

In this note, we require the following assumption on the time-dependence of the family $\{\phi_t\}$:

Assumption. For each $t \in [0, T]$, $x \in H$ with $\phi_t(x) < \infty$ and $s \in [t, T]$, there is an element $\tilde{x} \in H$ such that

$$\|\tilde{x} - x\| \leq \text{const.}|t - s|,$$

$$\phi_s(\tilde{x}) \leq \phi_t(x) + \text{const.}|t - s|(1 + \|x\|^2 + |\phi_t(x)|),$$

where these constants are independent of t, x, s and $\tilde{x}$.

By the way, the family $\{\phi_t\}$ in the Example does not satisfy the Assumption at $t = \pi/2$. If we exchange ϕ^1 for ϕ^2 in the Example, the family $\{\phi_t\}$ given by this exchange satisfies the Assumption. More generally, if $\phi_t(x)$ is a decreasing function in t, then the Assumption is trivially satisfied.

3. Main results. Under the Assumption mentioned in the previous section, we establish the following existence theorem.

Theorem 1. i) If $u_o \in D_o$ and $f \in L^2(0, T; H)$, then $P[\phi_t, f, u_o]$ has a unique strong solution u such that $t \to \phi_t(u(t))$ is bounded on $[0, T]$.

ii) If $u_o \in D$ and $f \in L^2(0, T; H)$, then $P[\phi_t, f, u_o]$ has a unique weak solution u such that for any positive number δ,

$$u' \in L^2(\delta, T; H),$$

$$t \to \phi_t(u(t)) \text{ is bounded on } [\delta, T].$$

So far as a weak solution is concerned, we see the following:

Let u_o be any element of D and f be any function in $L^2(0, T; H)$. Then a function $u \in L^2(0, T; H)$ is a weak solution of $P[\phi_t, f, u_o]$ if and only if there are sequenses $\{f_n\} \subset L^2(0, T; H)$, $\{u_{o,n}\} \subset D$ and $\{u_n\} \subset C([0, T]; H)$ such that each u_n is a strong solution of $P[\phi_t, f_n, u_{o,n}]$ and

$$f_n \to f \quad \text{in } L^2(0, T; H),$$

$$u_{o,n} \to u_o \quad \text{in } H,$$

$$u_n \to u \quad \text{in } L^2(0, T; H)$$

as $n \to \infty$.

Moreover, for any given $u_o \in D$, define a multivalued operator M_{u_o} from $L^2(0, T; H)$ into itself by the following:

$$f \in M_{u_o}(u) \Longleftrightarrow u \text{ is a weak solution of } P[\phi_t, f, u_o].$$

Then we see that $f \in M_{u_o}(u)$ if and only if $u \in D(\Phi)$ and (e) holds, and have an interesting result about the operator M_{u_o}.

Theorem 2. For each $u_o \in D$, M_{u_o} is a maximal monotone operator in $L^2(0, T; H)$.

Remark. In particular, when ϕ_t is time-independent, Theorem 2 was proved by Brézis [3].

Remark. Detail proofs of Theorems 1 and 2 are found in [6] and [7], respectively.

4. Construction of a strong solution. Finally we state how to construct a strong solution of $P[\phi_t, f, u_o]$. Here we

employ a finite difference method with respect to t.

For each positive integer N we set

$$\varepsilon_N = T/N \text{ and } f_{N,n} = \varepsilon_N^{-1}\int_{\varepsilon_N(n-1)}^{\varepsilon_N n} f(t)dt,\ n = 1, 2, \ldots, N,$$

and successively define a sequence $\{u_{N,n}\}_{n=1}^{N}$ as follows:

$$u_{N,0} = u_o,$$

$$(***)\quad (u_{N,n} - u_{N,n-1})/\varepsilon_N + \partial\phi_{\varepsilon_N n}(u_{N,n}) \ni f_{N,n},\ n=1, 2, \ldots, N;$$

when the element $u_{N,n-1}$ in the (n-1)-th step is defined, the next element $u_{N,n}$ is chosen so that the relation (***) is satisfied. In fact, such an element $u_{N,n}$ exists, since $\partial\phi_{\varepsilon_N n}$ is maximal monotone in H.

Now, we put

$$\left.\begin{aligned} u_N(t) &= u_{N,n} \\ \nabla_N u_N(t) &= (u_{N,n} - u_{N,n-1})/\varepsilon_N \end{aligned}\right\} \text{if } t \in [\varepsilon_N(n-1), \varepsilon_N n),\ n = 1, 2, \ldots, N$$

to obtain two sequences $\{u_N\}_{N=1}^{\infty}$ and $\{\nabla_N u_N\}_{N=1}^{\infty}$ of simple functions. If $u_o \in D_o$ and $f \in L^2(0, T; H)$, we can show by using the Assumption that $\{u_N\}$ is bounded in $L^\infty(0, T; H)$ and $\{\nabla_N u_N\}$ is bounded in $L^2(0, T; H)$. So we can choose a weakly* convergent subsequence $\{u_{N_k}\}$ and a weakly convergent subsequence $\{\nabla_{N_k} u_{N_k}\}$:

$$u_{N_k} \to u \quad \text{weakly* in } L^\infty(0, T; H)$$

and

$$\nabla_{N_k} u_{N_k} \to v \quad \text{weakly in } L^2(0, T; H).$$

Then we have u' = v and can show that the limit u is the required strong solution.

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