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Difference approximation of evolution equations and generation of nonlinear semigroups

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We consider the following nonlinear evolution equation

(DE) $\qquad (d/dt)\, u(t) \in Au(t) \quad , \quad 0<t<T$,

where A is a (multi-valued) quasi-dissipative operator. In this note, we construct the solution of the evolution equation (DE) by the method of difference approximation. In addition, we give a generation theorem of nonlinear semigroups through the difference approximations.

1. Preliminaries. Let X be a real Banach space. For the multi-valued operator A, we use the following notations:

$$D(A)=\{x\in X;\ Ax\neq\phi\} \quad , \quad R(A)=\bigcup_{x\in D(A)}\{y;\ y\in Ax\} \quad ,$$

and $|||Ax||| = \inf\{\ \|y\|\ ;\ y\in Ax\}$ for $x\in D(A)$.

We identify the multi-valued operator A with its graph, so that we write $[x,y]\in A$ if $y\in Ax$.

Let F be the duality map in X. Then we set

$$<y\ ,\ x>_i = \inf\{\ <y\ ,\ f>;\ f\in F(x)\}$$

and $<y\ ,\ x>_s = -<-y\ ,\ x>_i = -<y\ ,\ -x>_i$ for $x,y\in X$.

Let $A\subset X\times X$. A is said to be dissipative if for any $[x_i,y_i]\in A$ $(i=1,2)$,

$$<y_1-y_2\ ,\ x_1-x_2>_i \le 0.$$

According to Takahashi [9], we introduce the following notion as a generalization of that of dissipative operator.

Definition 1. Let $A \subset X \times X$. A is said to be quasi-dissipative if for any $[x_i, y_i] \in A$ $(i=1,2)$,

$$< y_1 , x_1 - x_2 >_i + < y_2 , x_2 - x_1 >_i \leq 0.$$

The following example shows that quasi-dissipative operators are not always dissipative.

Example (I. Miyadera). Let $X=R^2$ with maximum norm. Let $x_1=(1,1)$ and $x_2=(0,0)$. We set $D(A)=\{x_1,x_2\}$, $Ax_1=\{(\alpha,\beta); \alpha\leq 0 \text{ or } \beta\leq 0\}$ and $Ax_2=\{(\alpha,\beta); \alpha\geq 0 \text{ or } \beta\geq 0\}$. Then A is quasi-dissipative in X but $A - \omega$ is not dissipative in X for any real ω. In addition, $R(I - \lambda A) \supset D(A)$ for any $\lambda > 0$.

For the quasi-dissipative operator, we have the following.

Lemma 1. Let $A \subset X \times X$. Then the followings are equivalent.

(i) A is quasi-dissipative;

(ii) for any $[x_i,y_i] \in A$ $(i=1,2)$ and $\lambda,\mu > 0$,

$$(\lambda + \mu)\|x_1 - x_2\| \leq \lambda\|x_1 - x_2 - \mu y_1\| + \mu\|x_2 - x_1 - \lambda y_2\|;$$

(iii) for any $[x_i,y_i] \in A$ $(i=1,2)$ and $\lambda > 0$,

$$2\|x_1 - x_2\| \leq \|x_1 - x_2 - \lambda y_1\| + \|x_2 - x_1 - \lambda y_2\|.$$

We can verify Lemma 1 similarly as the proof of Kato's lemma [4].

Let $X_0 \subset X$. A one parameter family $\{T(t); t \geq 0\}$ of operators from X_0 into itself is called (nonlinear) contraction semigroup on X_0 if it has the following properties:

(i) $\|T(t)x - T(t)y\| \leq \|x - y\|$ for $x,y \in X_0$ and $t \geq 0$;

(ii) $T(0)x=x$ for x X_0 and $T(t+s)=T(t)T(s)$ for $t,s \geq 0$;

(iii) for each $x \in X_0$, $T(t)x$ is strongly continuous in $t \geq 0$.

2. Cauchy problems and difference approximation.

Let A be a quasi-dissipative operator in X. Let $x_0 \in X$ and T > 0. Then we treat the following Cauchy problem for the evolution equation (DE):

$$(CP;x_0) \qquad \begin{cases} (d/dt)\ u(t) \in Au(t) & \text{for } t \in (0,T), \\ u(0) = x_0. \end{cases}$$

For the Cauchy problem $(CP;x_0)$, we consider the following type of difference approximation:

$$(DS;x_0) \qquad \begin{cases} \left\| \dfrac{x_k^n - x_{k-1}^n}{t_k^n - t_{k-1}^n} - y_k^n \right\| \le \varepsilon_k^n, \quad k=1,2,\cdots,N_n;\ n \ge 1, \\ x_0^n = x_0, \end{cases}$$

where for each n, $[x_k^n, y_k^n] \in A$ $(k=1,2,\cdots,N_n)$ and $\{t_k^n\}$ represents the partition of [0,T] such that $0=t_0^n<t_1^n<\cdots<t_{N_n-1}^n<T\le t_{N_n}^n$ and $\delta_n = \max_{1\le k\le n}(t_k^n - t_{k-1}^n) \to 0$ as $n \to \infty$. The ε_k^n may be refferred as an error bound which occurs at the k-th step of the n-th approximation of the difference approximation $(DS;x_0)$.

Definition 2. Let $u_n(t)$ be a sequence in $L^\infty(0,T;X)$. We say that $u_n(t)$ is a (backward) DS-approximate solution of the Cauchy problem $(CP;x_0)$ if there exists a difference approximation $(DS;x_0)$ satisfying the following:

(i) $u_n(0) = x_0^n = x_0$, $n \ge 1$;

(ii) $u_n(t) = x_k^n$ for $t \in (t_{k-1}^n, t_k^n] \cap (0,T]$, $k=1,2,\cdots,N_n; n\ge 1$;

(iii) $\sum_{k=1}^{N_n} \varepsilon_k^n (t_k^n - t_{k-1}^n) \to 0$ as $n \to \infty$.

Then we have

Theorem I. Let $x_0 \in \overline{D(A)}$ and $u_n(t)$ be a DS-approximate solution of $(CP;x_0)$ on [0,T]. Then there exists a $u(t) \in C([0,T];X)$ satisfying the following:

(i) $$u(t) = \lim_{n\to\infty} u_n(t) \quad \text{for } t \in [0,T],$$

and the convergence is uniform on [0,T];

(ii) $u(t) \in \overline{D(A)}$ for $t \in [0,T]$ and $u(0)=x_0$;

(iii) for any DS-approximate solution $\hat{u}_n(t)$ of $(CP;x_0)$,

$$u(t) = \lim_{n\to\infty} \hat{u}_n(t) \quad \text{for } t \in [0,T].$$

Remarks. 1) Kenmochi-Oharu [5] and Takahashi [9], [10] studied the convergence (i) under the additional condition, which is called the stability condition by them. Our result is an extension of their results.

2) By Benilan's method [2], we find that the limiting function u(t) is the unique integral solution of the Cauchy problem $(CP;x_0)$.

The proof of Theorem I is based on the following.

Lemma 2. Let $(DS;x_0)$ and $(DS;\hat{x}_0)$ be two difference approximations as above of the Cauchy problems $(CP;x_0)$ and $(CP;\hat{x}_0)$ on [0,T], respectively. Let the notations with $\hat{}$ represents the difference approximation $(DS;\hat{x}_0)$. Then

(1) $$\|x_i^m - \hat{x}_j^n\| \le \|x_0 - u\| + \|\hat{x}_0 - u\| + \{(t_i^m - \hat{t}_j^n)^2 + \delta_m t_i^m + \hat{\delta}_n \hat{t}_j^n\}^{1/2} |||Au||| + \sum_{k=1}^{i} \varepsilon_k^m (t_k^m - t_{k-1}^m) + \sum_{k=1}^{j} \hat{\varepsilon}_k^n (\hat{t}_k^n - \hat{t}_{k-1}^n),$$

for $0 \le i \le N_m$, $0 \le j \le \hat{N}_n$ and $u \in D(A)$.

Proof. Let $u \in D(A)$ and $v \in Au$. We set $a_{i,j} = \|x_i^m - \hat{x}_j^n\|$, $h_i^m = t_i^m - t_{i-1}^m$ and $\hat{h}_j^n = \hat{t}_j^n - \hat{t}_{j-1}^n$ for $0 \le i \le N_m$ and $0 \le j \le \hat{N}_n$. By (iii) of Lemma 1, we have

$$\|x_k^m - u\| \le \|x_k^m - h_k^m y_k^m - u\| + h_k^m\|v\| \le \|x_{k-1}^m - u\| + h_k^m \varepsilon_k^m + h_k^m\|v\|$$

for $1 \le k \le N_m$. Therefore, inductively, we have

$$\|x_i^m - u\| \le \|x_0^m - u\| + t_i^m\|v\| + \sum_{k=1}^{i} \varepsilon_k^m h_k^m$$

or

$$a_{i,0} \le \|x_0 - u\| + \|\hat{x}_0 - u\| + t_i^m |||Au||| + \sum_{k=1}^{i} \varepsilon_k^m h_k^m .$$

This shows that (1) holds true for $(i,0)$ with $0 \le i \le N_m$. Similarly we have (1) for $(0,j)$ with $0 \le j \le \hat{N}_n$. Furthermore, by (ii) of Lemma 1, we have

$$(h_i^m + \hat{h}_j^n)\; a_{i,j} \le \hat{h}_j^n \|x_i^m - h_i^m y_i^m - \hat{x}_j^n\| + h_i^m \|\hat{x}_j^n - \hat{h}_j^n \hat{y}_j^n - x_i^m\|$$
$$\le \hat{h}_j^n\, a_{i-1,j} + h_i^m\, a_{i,j-1} + h_i^m \hat{h}_j^n (\varepsilon_i^m + \hat{\varepsilon}_j^n)$$

for $1 \le i \le N_m$ and $1 \le j \le \hat{N}_n$. Hence, using the Cauchy-Schwarz'inequality, we can verify (1) for every (i,j) by the induction for (i,j). Q.E.D.

Remark. Let A be a dissipative operator in X such that $R(I - \lambda A) \supset D(A)$ for $\lambda > 0$. Then estimate (1) gives

$$\|(I - \lambda A)^{-n}x - (I - \mu A)^{-m}x\| \le \{(n\lambda - m\mu)^2 + n\lambda^2 + m\mu^2\}^{1/2} |||Au|||$$

for $n,m \ge 1$, $\lambda,\mu > 0$ and $x \in D(A)$. This estimate is similar to but different from that of Crandall-Liggett [3].

Proof of Theorem I. Let $(DS;x_0)$ be the corresponding difference approximation to $u_n(t)$. Then by Lemma 2, we have

$$(2)\quad \|x_i^m - x_j^n\| \le 2\|x_0 - u_p\| + \{(t_i^m - t_j^n)^2 + \delta_m t_i^m + \delta_n t_j^n\}^{1/2} |||Au_p||| + \sum_{k=1}^{N_m} \varepsilon_k^m (t_k^m - t_{k-1}^m) + \sum_{k=i}^{N_n} \varepsilon_k^n (t_k^n - t_{k-1}^n)$$

for $0 \le i \le N_m$ and $0 \le j \le N_n$, where $\{u_p\} \subset D(A)$ is a sequence such that $u_p \to x_0$ as $p \to \infty$. This estimate shows that there exists

$$u(t) = \lim x_k^n \qquad \text{as } t_k^n \to t,\ n \to \infty,$$
$$= \lim_{n\to\infty} u_n(t)$$

for every $t \in [0,T]$. Furthermore, by (2), we have

$$\|u(t) - u(s)\| \leq 2\|x_0 - u_p\| + |t - s| \,|||Au_p|||$$

for $t,s \in [0,T]$. This shows that $u(t)$ is continuous on $[0,T]$. The property (ii) is evident. Let $\hat{u}_n(t)$ be a DS-approximate solution of $(CP;\hat{x}_0)$ with $\hat{x}_0 \in \overline{D(A)}$. And let us set

$$\hat{u}(t) = \lim_{n\to\infty}\hat{u}_n(t) \quad \text{for } t \in [0,T].$$

Then by the estimate (1), we have

$$\|u(t) - \hat{u}(t)\| \leq \|x_0 - \hat{x}_0\| \quad \text{for } t \in [0,T].$$

Especially, we have (iii). Q.E.D.

By Theorem I, we define the following.

Definition 3. Let $u(t) \in C([0,T];X)$ and $x_0 \in \overline{D(A)}$. We say that $u(t)$ is a (backward) DS-limit solution of the Cauchy problem $(CP;x_0)$ on $[0,T]$ if there exists a (backward) DS-approximate solution $u_n(t)$ of $(CP;x_0)$ on $[0,T]$, such that $u_n(t)$ converges to $u(t)$, uniformly for $t \in [0,T]$.

In the proof of Theorem I, we obtained the following.

Corollary. Let $u(t)$, $\hat{u}(t)$ be two DS-limit solutions of (CP) on $[0,T]$. Then

$$\|u(t) - \hat{u}(t)\| \leq \|u(0) - \hat{u}(0)\| \quad \text{for } t \in [0,T].$$

3. Generation of semigroups

By Theorem I and the Corollary , we have a generation theorem of semigroups.

Definition 4. Let A be a quasi-dissipative operator in X. We say that A has the property ($\mathcal{D}$) if for any $x \in \overline{D(A)}$ and $T > 0$, there exists a DS-approximate solution of the Cauchy problem $(CP;x)$ on $[0,T]$.

Theorem II . Let A be a quasi-dissipative operator in X, having the property ($\mathcal{D}$). Then there exists a contraction semigroup $\{T(t);\ t \geq 0\}$ on $\overline{D(A)}$ such that for any $x \in \overline{D(A)}$ and $T > 0$, $u(t)=T(t)x$ is the unique DS-limit solution of the Cauchy problem $(CP;x_0)$ on $[0,T]$.

Proof. Let $x \in \overline{D(A)}$ and $T > 0$. Then, by Theorem I, there exists the unique DS-limit solution $u(t)$ of $(CP;x)$ on $[0,T]$. By its corollary, we can extend the soluton $u(t)$ onto $[0,\infty)$. Then we define $T(t):\overline{D(A)} \to \overline{D(A)}$ by

$$T(t)x = u(t) \quad \text{for } t \geq 0.$$

Using Theorem I and its corollary, we can verify that $\{T(t);\ t \geq 0\}$ is the desired contraction semigroup. Q.E.D.

4. Existence of difference approximation

In this section, we give a sufficient condition that a quasi-dissipative operator has the property $(\mathcal{D})$. Let A be a quasi-dissipative operator in X. We add the following condition on A:

$(R)_t$ for any $x \in \overline{D(A)}$, there exist sequences $\delta_n \downarrow 0$ and $[x_n, y_n] \in A$ $(n \geq 1)$ such that

$$\lim_{n\to\infty} \delta_n^{-1} \| x_n - x - \delta_n y_n \| = 0.$$

Then we have

Theorem III. Let A be a quasi-dissipative operator in X, satisfying the condition $(R)_t$. Then A has the property $(\mathcal{D})$. Thus A generates a contraction semigroup on $\overline{D(A)}$, in the sense of Theorem II.

Remarks. 1) This theorem implies the fundamental result of Crandall-Liggett [3]: a part of the results of Martin [7] on ordinary differential equations; and the results of Webb [11] and Barbu [1] on the continuous perturbations of m-dissipative operators. The details will be treated in [6].

2) Yorke announces in [12] that he obtained a similar result.

Proof of Theorem III. Let $x_0 \in \overline{D(A)}$ and $\varepsilon_n \downarrow 0$. Let n be fixed. Then for each $x \in \overline{D(A)}$, we set

$$\delta_n(x) = \sup\{\ \delta\ ;\ 0 < \delta \leq \varepsilon_n \text{ and there exists } [x_\delta, y_\delta] \in A \text{ such that } \|x_\delta - x - \delta y_\delta\| \leq \delta\ \varepsilon_n\}.$$

Then $\delta_n(x)$ are positive by the assumption. Therefore, inductively, we can choose $h_k^n > 0$ and $[x_k^n, y_k^n] \in A$, for $k=1,2,\cdots$, so that they satisfy the following:

(i) $\quad x_0^n = x_0$;

(ii) $\quad (1/2)\ \delta_n(x_{k-1}^n) < h_k^n \leq \varepsilon_n$, for $k=1,2,\cdots$;

(iii) $\quad \|x_k^n - x_{k-1}^n - h_k^n y_k^n\| \leq h_k^n\ \varepsilon_n$, for $k=1,2,\cdots$.

Then we set $t_i^n = \sum_{k=1}^{i} h_k^n$. We may show that $t_i^n \to \infty$ as $i \to \infty$. For the purpose, we use the following estimate:

(3) $\quad \|x_i^n - x_j^n\| \leq (t_i^n - t_j^n)\|y_k^n\| + \varepsilon_n(t_i^n - t_k^n) + \varepsilon_n(t_j^n - t_k^n)$

for any $i \geq j \geq k \geq 1$. This estimate may be verified by the induction for (i,j) with $i \geq j \geq k$ for each fixed $k \geq 1$, by using Lemma 1 as in the proof of Lemma 2.

Now, suppose that $t_i^n \to s_0 < +\infty$ as $i \to \infty$, for contradiction. Then by (3), we see that there exists $u_0 \in \overline{D(A)}$ such that $x_i^n \to u_0$ as $i \to \infty$. By the assumption, we can choose $\delta > 0$ and $[u_\delta, v_\delta] \in A$ such that $0 < \delta \leq \varepsilon_n$ and

$$\|u_\delta - u_0 - \delta v_\delta\| \leq \delta\varepsilon_n/2.$$

Since $\delta_n(x_i^n) \to 0$ and $x_i^n \to u_0$ as $i \to \infty$, there exists i_0 such that $\delta_n(x_i^n) < \delta$ and $\|x_i^n - u_0\| \leq \delta\ \varepsilon_n/2$ for $i \geq i_0$. Then we have

$$\|u_\delta - u_0 - \delta v_\delta\| \leq \delta\ \varepsilon_n \text{ for } i \geq i_0,$$

which is contrary to the definition of $\delta_n(x_i^n)$. Q.E.D.

Remark. The construction and properties of the DS-limit solution of evolution equations will be studied more systematically and generally in [6].

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