Difference approximation of evolution equations
and generation of nonlinear semigroups

by Yoshikazu Kobayashi

Department of Mathematics, Waseda University

We consider the following nonlinear evolution equation

\[(d/dt) u(t) \in Au(t), \quad 0 < t < T,\]

where \(A\) is a (multi-valued) quasi-dissipative operator. In this note, we construct the solution of the evolution equation (DE) by the method of difference approximation. In addition, we give a generation theorem of nonlinear semigroups through the difference approximations.

1. Preliminaries. Let \(X\) be a real Banach space. For the multi-valued operator \(A\), we use the following notations:

\[D(A) = \{x \in X; A x \neq \emptyset \}, \quad R(A) = \bigcup_{x \in D(A)} \{y; y \in Ax\},\]

and \(\|Ax\| = \inf \{\|y\|; y \in Ax\}\) for \(x \in D(A)\).

We identify the multi-valued operator \(A\) with its graph, so that we write \([x, y] \in A\) if \(y \in Ax\).

Let \(F\) be the duality map in \(X\). Then we set

\[\langle y, x \rangle_i = \inf \{\langle y, f \rangle; f \in F(x)\}\]

and \(\langle y, x \rangle_s = -\langle -y, x \rangle_i = -\langle y, -x \rangle_i\) for \(x, y \in X\).

Let \(A \subseteq X \times X\). \(A\) is said to be dissipative if for any \([x_i, y_i] \in A\) (\(i = 1, 2\)),

\[\langle y_1 - y_2, x_1 - x_2 \rangle_i \leq 0.\]
According to Takahashi [9], we introduce the following notion
as a generalization of that of dissipative operator.

**Definition 1.** Let \( A \subseteq X \times X \). \( A \) is said to be *quasi-dissipative* if for any \([x_{i}, y_{i}] \in A \) \((i = 1, 2)\),

\[
\langle y_{1}, x_{1} - x_{2} \rangle_{i} + \langle y_{2}, x_{2} - x_{1} \rangle_{i} \leq 0.
\]

The following example shows that quasi-dissipative operators are not always dissipative.

**Example (I. Miyadera).** Let \( X = \mathbb{R}^{2} \) with maximum norm. Let \( x_{1} = (1, 1) \) and \( x_{2} = (0, 0) \). We set \( D(A) = \{x_{1}, x_{2}\} \), \( Ax_{1} = \{(\alpha, \beta); \alpha \leq 0 \text{ or } \beta < 0\} \) and \( Ax_{2} = \{(\alpha, \beta); \alpha > 0 \text{ or } \beta > 0\} \). Then \( A \) is quasi-dissipative in \( X \) but \( A - \omega \) is not dissipative in \( X \) for any real \( \omega \). In addition, \( R(I - \lambda A) \supseteq D(A) \) for any \( \lambda > 0 \).

For the quasi-dissipative operator, we have the following.

**Lemma 1.** Let \( A \subseteq X \times X \). Then the followings are equivalent.

(i) \( A \) is quasi-dissipative;

(ii) for any \([x_{i}, y_{i}] \in A \) \((i = 1, 2)\) and \( \lambda, \mu > 0 \),

\[
(\lambda + \mu)\|x_{1} - x_{2}\| \leq \lambda\|x_{1} - x_{2} - \mu y_{1}\| + \mu\|x_{2} - x_{1} - \lambda y_{2}\|;
\]

(iii) for any \([x_{i}, y_{i}] \in A \) \((i = 1, 2)\) and \( \lambda > 0 \),

\[
2\|x_{1} - x_{2}\| \leq \|x_{1} - x_{2} - \lambda y_{1}\| + \|x_{2} - x_{1} - \lambda y_{2}\|.
\]

We can verify Lemma 1 similarly as the proof of Kato's lemma [4].

Let \( X_{0} \subseteq X \). A one parameter family \( \{T(t); t \geq 0\} \) of operators from \( X_{0} \) into itself is called (nonlinear) *contraction semigroup* on \( X_{0} \) if it has the following properties:

(i) \( \|T(t)x - T(t)y\| \leq \|x - y\| \) for \( x, y \in X_{0} \) and \( t \geq 0 \);

(ii) \( T(0)x = x \) for \( x \in X_{0} \) and \( T(t+s) = T(t)T(s) \) for \( t, s \geq 0 \);

(iii) for each \( x \in X_{0} \), \( T(t)x \) is strongly continuous in \( t \geq 0 \).
2. Cauchy problems and difference approximation.

Let $A$ be a quasi-dissipative operator in $X$. Let $x_0 \in X$ and $T > 0$. Then we treat the following Cauchy problem for the evolution equation (DE):

$$
\begin{align*}
(CP;x_0) & \quad \begin{cases} 
(d/dt) u(t) \in Au(t) & \text{for } t \in (0,T), \\
u(0) = x_0.
\end{cases}
\end{align*}
$$

For the Cauchy problem $(CP;x_0)$, we consider the following type of difference approximation:

$$
(DS;x_0) \quad \begin{cases}
\left\| \frac{x_k^n - x_{k-1}^n}{t_k^n - t_{k-1}^n} - y_k^n \right\| \leq \varepsilon_k^n, \quad k = 1, 2, \ldots, N_n; \quad n \geq 1, \\
x_0^n = x_0,
\end{cases}
$$

where for each $n$, $[x_k^n, y_k^n] \in A$ (for $k = 1, 2, \ldots, N_n$) and $\{t_k^n\}$ represents the partition of $[0,T]$ such that $0 = t_0^n < t_1^n < \cdots < t_{N_n-1}^n < T = t_N^n$ and

$$
\delta_n = \max_{1 \leq k \leq n} (t_k^n - t_{k-1}^n) \to 0 \text{ as } n \to \infty.
$$

The $\varepsilon_k^n$ may be referred as an error bound which occurs at the $k$-th step of the $n$-th approximation of the difference approximation $(DS;x_0)$.

**Definition 2.** Let $u_n(t)$ be a sequence in $L^\infty(0,T;X)$. We say that $u_n(t)$ is a (backward) **DS-approximate solution** of the Cauchy problem $(CP;x_0)$ if there exists a difference approximation $(DS;x_0)$ satisfying the following:

(i) $u_n(0) = x_0^n = x_0$ , $n \geq 1$;

(ii) $u_n(t) = x_k^n$ for $t \in (t_{k-1}^n, t_k^n \cap (0,T], \quad k = 1, 2, \ldots, N_n; \quad n \geq 1$;

(iii) $\sum_{k=1}^{N_n} \varepsilon_k^n (t_k^n - t_{k-1}^n) \to 0$ as $n \to \infty$.

Then we have

**Theorem I.** Let $x_0 \in \mathcal{D}(A)$ and $u_n(t)$ be a DS-approximate solution of $(CP;x_0)$ on $[0,T]$. Then there exists a $u(t) \in \mathcal{C}([0,T];X)$ satisfying the following:
(i) \[ u(t) = \lim_{n \to \infty} u^n_n(t) \quad \text{for } t \in [0,T], \]
and the convergence is uniform on [0,T];

(ii) \[ u(t) \in \overline{D(A)} \quad \text{for } t \in [0,T] \text{ and } u(0) = x_0; \]

(iii) for any DS-approximate solution \( \hat{u}_n(t) \) of (CP;\( x_0 \)),

\[ u(t) = \lim_{n \to \infty} \hat{u}_n(t) \quad \text{for } t \in [0,T]. \]

Remarks. 1) Kenmochi-Oharu [5] and Takahashi [9], [10] studied the convergence (i) under the additional condition, which is called the stability condition by them. Our result is an extension of their results.

2) By Benilan's method [2], we find that the limiting function \( u(t) \) is the unique integral solution of the Cauchy problem (CP;\( x_0 \)).

The proof of Theorem I is based on the following.

Lemma 2. Let (DS;\( x_0 \)) and (DS;\( \hat{x}_0 \)) be two difference approximations as above of the Cauchy problems (CP;\( x_0 \)) and (CP;\( \hat{x}_0 \)) on [0,T], respectively. Let the notations with \( \hat{\cdot} \) represents the difference approximation (DS;\( \hat{x}_0 \)). Then

\[ \| x_i - \hat{x}_j \| \leq \| x_0 - u \| + \| x_0 - u \|
+ \left( t_{i-1}^m - t_i^m \right)^2 + \delta t_{i-1}^m + \hat{\delta} n t_{i-1}^m \right)^{1/2} \| Au \|
+ \sum_{k=1}^j \varepsilon_k (t_{k-1}^m - t_k^m) + \sum_{k=1}^j \varepsilon_k (t_{k-1}^n - t_k^n), \]

for \( 0 \leq i \leq N_m \) and \( 0 \leq j \leq N_n \) and \( u \in D(A) \).

Proof. Let \( u \in D(A) \) and \( v \in Au \). We set \( a_{i,j} = \| x_i^m - x_j^n \|, \)

\[ h_i^m = t_{i-1}^m - t_i^m \text{ and } h_j^n = t_{j-1}^n - t_j^n \text{ for } 0 \leq i \leq N_m \text{ and } 0 \leq j \leq N_n. \]

By (iii) of Lemma 1, we have

\[ \| x_k^m - u \| \leq \| x_k^m - h_k y_k^m - u \| + h_k \| v \|
\leq \| x_{k-1}^m - u \| + h_{k-1} \| v \| \]

for \( 1 \leq k \leq N_m \). Therefore, inductively, we have
\[ \| x_i^m - u \| \leq \| x_0^m - u \| + t_i^m \| v \| + \sum_{k=1}^{\infty} k^m h_k^m \]

or

\[ a_{i,0} \leq \| x_0 - u \| + \| x_0 - u \| + t_i^m \| Au \| + \sum_{k=1}^{\infty} k^m h_k^m. \]

This shows that (1) holds true for \((i,0)\) with \(0 \leq i \leq N_m\).
Similarly we have (1) for \((0,j)\) with \(0 \leq j \leq N_n\). Furthermore, by (ii) of Lemma 1, we have

\[ (h_i^m + h_j^m) a_{i,j} \leq h_j^n \| x_i^m - h_i^m x_i - x_i \| + h_i^n \| x_j^m - h_j^m x_j - x_j \| \]

\[ \leq h_j^n a_{i-1,j} + h_i^m a_{i-1,j} + h_i^n h_j^m (\varepsilon_i^m + \varepsilon_j) \]

for \(1 \leq i \leq N_m\) and \(1 \leq j \leq N_n\). Hence, using the Cauchy-
Schwarz' inequality, we can verify (1) for every \((i,j)\) by
the induction for \((i,j)\). Q.E.D.

Remark. Let \(A\) be a dissipative operator in \(X\) such that
\(R(I - \lambda A) \supseteq D(A)\) for \(\lambda > 0\). Then estimate (1) gives

\[ \| (I - \lambda A)^{-n} x - (I - \mu A)^{-m} x \| \leq \{ (\lambda - \mu)^2 + n\lambda^2 + m\mu^2 \}^{1/2} \| Au \| \]

for \(n,m \geq 1\), \(\lambda, \mu > 0\) and \(x \in D(A)\). This estimate is similar to
but different from that of Crandall-Liggett [3].

Proof of Theorem I. Let \((DS; x_0)\) be the corresponding
difference approximation to \(u_n(t)\). Then by Lemma 2, we have

\[ \| x_i^m - x_j^n \| \leq 2 \| x_0 - u_p \| \]

\[ + \{ (t_i^m - t_j^n)^2 + \delta_m t_i^m + \delta_n t_j^n \}^{1/2} \| Au_p \| \]

\[ + \sum_{k=1}^{N_m} k^m (t_k^m - t_{k-1}^m) + \sum_{k=1}^{N_n} k^m (t_k^n - t_{k-1}^n) \]

for \(0 \leq i \leq N_m\) and \(0 \leq j \leq N_n\), where \(\{ u_p \} \subseteq D(A)\) is a sequence
such that \(u_p \to x_0\) as \(p \to \infty\). This estimate shows that there
exists

\[ u(t) = \lim_{n \to \infty} x_k^n \quad \text{as} \quad t_k^n \to t, \ n \to \infty, \]

\[ = \lim_{n \to \infty} u_n(t) \]

for every \(t \in [0,T]\). Furthermore, by (2), we have
\[ \| u(t) - u(s) \| \leq 2\| x_0 - u_p \| + |t - s| \| Au_p \| \]
for \( t, s \in [0, T] \). This shows that \( u(t) \) is continuous on \([0, T]\).

The property (ii) is evident. Let \( \hat{u}_n(t) \) be a DS-approximate solution of \( (CP; x_0) \) with \( \hat{x}_0 \in \overline{D(A)} \). And let us set
\[
\hat{u}(t) = \lim_{n \to \infty} \hat{u}_n(t) \quad \text{for} \ t \in [0, T].
\]
Then by the estimate (1), we have
\[
\| u(t) - \hat{u}(t) \| \leq \| x_0 - \hat{x}_0 \| \quad \text{for} \ t \in [0, T].
\]
Especially, we have (iii). Q.E.D.

By Theorem I, we define the following.

**Definition 3.** Let \( u(t) \in C([0, T]; X) \) and \( x_0 \in \overline{D(A)} \). We say that \( u(t) \) is a (backward) DS-limit solution of the Cauchy problem \( (CP; x_0) \) on \([0, T]\) if there exists a (backward) DS-approximate solution \( u_n(t) \) of \( (CP; x_0) \) on \([0, T]\), such that \( u_n(t) \) converges to \( u(t) \), uniformly for \( t \in [0, T] \).

In the proof of Theorem I, we obtained the following.

**Corollary.** Let \( u(t), \hat{u}(t) \) be two DS-limit solutions of \( (CP) \) on \([0, T]\). Then
\[
\| u(t) - \hat{u}(t) \| \leq \| u(0) - \hat{u}(0) \| \quad \text{for} \ t \in [0, T].
\]

3. **Generation of semigroups**

By Theorem I and the Corollary, we have a generation theorem of semigroups.

**Definition 4.** Let \( A \) be a quasi-dissipative operator in \( X \). We say that \( A \) has the property (\( \mathcal{Q} \)) if for any \( x \in \overline{D(A)} \) and \( T > 0 \), there exists a DS-approximate solution of the Cauchy problem \( (CP; x) \) on \([0, T]\).

**Theorem II.** Let \( A \) be a quasi-dissipative operator in \( X \), having the property (\( \mathcal{Q} \)). Then there exists a contraction
semigroup \( \{T(t); \ t \geq 0\} \) on \( \overline{D(A)} \) such that for any \( x \in \overline{D(A)} \) and \( T > 0 \), \( u(t) = T(t)x \) is the unique DS-limit solution of the Cauchy problem \((CP;x_0)\) on \([0,T]\).

**Proof.** Let \( x \in \overline{D(A)} \) and \( T > 0 \). Then, by Theorem I, there exists the unique DS-limit solution \( u(t) \) of \( (CP;x) \) on \([0,T]\). By its corollary, we can extend the solution \( u(t) \) onto \([0,\omega)\). Then we define \( T(t):\overline{D(A)} \to \overline{D(A)} \) by

\[
T(t)x = u(t) \quad \text{for} \ t \geq 0.
\]

Using Theorem I and its corollary, we can verify that \( \{T(t); \ t \geq 0\} \) is the desired contraction semigroup. Q.E.D.

4. **Existence of difference approximation**

In this section, we give a sufficient condition that a quasi-dissipative operator has the property \((\mathcal{Z})\). Let \( A \) be a quasi-dissipative operator in \( X \). We add the following condition on \( A \):

\((R)_t\) for any \( x \in \overline{D(A)} \), there exist sequences \( \delta_n \to 0 \) and \( [x_n, y_n] \in A \) (\( n \geq 1 \)) such that

\[
\lim_{n \to \infty} \delta_n^{-1} \| x_n - x - \delta_n y_n \| = 0.
\]

Then we have

**Theorem III.** Let \( A \) be a quasi-dissipative operator in \( X \), satisfying the condition \((R)_t\). Then \( A \) has the property \((\mathcal{Z})\). Thus \( A \) generates a contraction semigroup on \( \overline{D(A)} \), in the sense of Theorem II.

**Remarks.** 1) This theorem implies the fundamental result of Crandall-Liggett [3]: a part of the results of Martin [7] on ordinary differential equations; and the results of Webb [11] and Barbu [1] on the continuous perturbations of \( m \)-dissipative
operators. The details will be treated in [6].

2) Yorke announces in [12] that he obtained a similar result.

**Proof of Theorem III.** Let $x_0 \in \overline{D(A)}$ and $\varepsilon_n \downarrow 0$. Let $n$ be fixed. Then for each $x \in \overline{D(A)}$, we set

$$\delta_n(x) = \sup \{ \delta : 0 < \delta \leq \varepsilon_n \text{ and there exists } [x_\delta, y_\delta] \subseteq A \text{ such that } ||x_\delta - x - \delta y_\delta|| \leq \delta \cdot \varepsilon_n \}.$$ 

Then $\delta_n(x)$ are positive by the assumption. Therefore, inductively, we can choose $h_k > 0$ and $[x_k^n, y_k^n] \subseteq A$, for $k = 1, 2, \ldots$, so that they satisfy the following:

(i) $x_0^n = x_0$;

(ii) $(1/2^n) \delta_n(x_{k-1}^n) < h_k \leq \varepsilon_n$ for $k = 1, 2, \ldots$;

(iii) $||x_k^n - x_{k-1}^n - h_k y_k^n|| \leq h_k \cdot \varepsilon_n$ for $k = 1, 2, \ldots$.

Then we set $t_i^n = \sum_{k=1}^i h_k^n$. We may show that $t_i^n \to \infty$ as $i \to \infty$.

For the purpose, we use the following estimate:

$$(3) \quad ||x_i^n - x_j^n|| \leq (t_i^n - t_j^n)||y_k^n|| + \varepsilon_n(t_i^n - t_j^n) + \varepsilon_n(t_j^n - t_k^n)$$

for any $i \geq j \geq k \geq 1$. This estimate may be verified by the induction for $(i, j)$ with $i \geq j \geq k$ for each fixed $k \geq 1$, by using Lemma 1 as in the proof of Lemma 2.

Now, suppose that $t_i^n + s_0 \not< \infty$ as $i \to \infty$, for contradiction.

Then by (3), we see that there exists $u_0 \in \overline{D(A)}$ such that $x_i^n \to u_0$ as $i \to \infty$. By the assumption, we can choose $\delta > 0$ and $[u_\delta, v_\delta] \subseteq A$ such that $0 < \delta \leq \varepsilon_n$ and

$$||u_\delta - u_0 - \delta v_\delta|| \leq \delta \varepsilon_n/2.$$ 

Since $\delta_n(x_i^n) \to 0$ and $x_i^n \to u_0$ as $i \to \infty$, there exists $i_0$ such that $\delta_n(x_i^n) < \delta$ and $||x_i^n - u_0|| \leq \delta \cdot \varepsilon_n/2$ for $i \geq i_0$. Then we have

$$||u_\delta - u_0 - \delta v_\delta|| \leq \delta \varepsilon_n \text{ for } i \geq i_0,$$

which is contrary to the definition of $\delta_n(x_i^n)$. Q.E.D.
Remark. The construction and properties of the DS-limit solution of evolution equations will be studied more systematically and generally in [6].

Acknowledgement. The author would like to express his hearty thanks to Prof. I. Miyadera and Mr. T. Takahashi for their valuable advices.

References.


