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New results concerning monotone operators

and nonlinear semigroups

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Our purpose is to describe here some recent developments in three different directions.

In §I we discuss a property of the range \( R(A+B) \) of the sum of two monotone operators. Surprisingly, it turns out that in "many" cases \( R(A+B) \) is "almost" equal to \( R(A)+R(B) \). A number of applications to nonlinear partial differential equations are given.

In §II we prove some estimates showing that \((I+tA)^{-1}\) and \(S(t)\) have the same modulus of continuity at \( t=0 \) (\(S(t)\) denotes the semigroup generated by \(-A\)). Next we present some consequences.

In §III we give a very general form of the convergence theorem of Trotter - Kato - Neveu type for nonlinear semigroups.

§I \( "R(A+B) \simeq R(A)+R(B)" \) and applications

Let \( H \) be a real Hilbert space and let \( A \) and \( B \) be maximal monotone operators such that \( A+B \) is again maximal monotone.

We say that two subsets \( K_1 \) and \( K_2 \) of \( H \) are almost equal \((K_1 \simeq K_2)\) if \( K_1 \) and \( K_2 \) have the same closure and the same interior. We prove here, under various assumptions, that
\( R(A + B) \cong R(A) + R(B) \); we discuss here only the simplest forms (for more elaborate results see [7]).

**Theorem 1** Suppose \( A \) and \( B \) are subdifferentials of convex functions. Then \( R(A + B) \cong R(A) + R(B) \).

**Proof** First we prove that \( \overline{R(A + B)} = \overline{R(A) + R(B)} \); it is sufficient to verify that \( R(A) + R(B) \subset \overline{R(A + B)} \). Given \( f \in R(A) + R(B) \), there exist \( \xi \in D(A) \) and \( \eta \in D(B) \) such that \( f \in A\xi + B\eta \). The equation

\[
(1) \quad \epsilon u_\epsilon + Au_\epsilon + Bu_\epsilon \ni f
\]

has a unique solution \( u_\epsilon \). The conclusion follows provided we show that \( \epsilon u_\epsilon \to 0 \) as \( \epsilon \to 0 \). Let \( x \in D(A) \cap D(B) \) be fixed. Since \( A \) and \( B \) are cyclically monotone (see [21]) we have

\[
(2) \quad (Au_\epsilon, u_\epsilon - x) + (Ax, x - \xi) + (A\xi, \xi - u_\epsilon) \geq 0
\]

\[
(3) \quad (Bu_\epsilon, u_\epsilon - x) + (Bx, x - \eta) + (B\eta, \eta - u_\epsilon) \geq 0
\]

and therefore by adding (2) and (3) we obtain

\[
(f - \epsilon u_\epsilon, u_\epsilon - x) + C - (f, u_\epsilon) \geq 0,
\]

where \( C \) is independent of \( \epsilon \). Hence

\[
\epsilon |u_\epsilon|^2 - \epsilon (u_\epsilon, x) \leq C'
\]

and therefore \( \sqrt{\epsilon} |u_\epsilon| \) remains bounded as \( \epsilon \to 0 \).

Next we prove that \( \text{Int}[R(A) + R(B)] = \text{Int}[R(A + B)] \). It is sufficient to check that \( \text{Int}[R(A) + R(B)] \subset R(A + B) \). Let \( f \in \text{Int}[R(A) + R(B)] \), so that a ball \( B(f, \rho) \) is contained in \( R(A) + R(B) \). For every \( h \in H \) with \( |h| < \rho \), there exist \( \xi \)
and \( \eta \) (depending on \( h \)) such that \( f+h \in A \xi + B \eta \). Going back to (2) and (3) and adding them we obtain now

\[
(f - \xi u_\xi, u_\xi - x) + C(h) - (f+h, u_\xi) \geq 0
\]

where \( C(h) \) depends on \( h \), but is independent of \( \varepsilon \).

Hence \( (h, u_\xi) \leq C(h) \) for every \( h \in H \) with \( |h| < \rho \). It follows from the uniform boundedness principle that \( \{u_\varepsilon\} \)
remains bounded as \( \varepsilon \to 0 \). Passing to the limit in (1) we conclude by standard methods that \( f \in R(A+B) \).

**Theorem 2** We suppose now that only \( A \) is the subdifferential of a convex function, but \( D(B) \subseteq D(A) \). Then \( R(A+B) \approx R(A)+R(B) \).

**Proof** We proceed as in the proof of Theorem 1.

First let \( f \in R(A+B) \) i.e. \( f \in A \xi + B \eta \); let \( u_\varepsilon \) be the solution of (1). We have

\[
(4) \quad (Au_\varepsilon, u_\varepsilon - \eta) + (A \eta, \eta - \xi) + (A \xi, \xi - u_\varepsilon) \geq 0
\]

\[
(5) \quad (Bu_\varepsilon, u_\varepsilon - \eta) + (B \eta, \eta - u_\varepsilon) \geq 0.
\]

By adding (4) and (5) we obtain

\[
(f - \varepsilon u_\varepsilon, u_\varepsilon - \eta) + C - (f, u_\varepsilon) \geq 0
\]

and hence

\[
\varepsilon |u_\varepsilon|^2 - \varepsilon (u_\varepsilon, \eta) \leq C'.
\]

Next suppose \( f \in \text{Int}[R(A)+R(B)] \); we obtain now, as in the proof of Theorem 1

\[
(f - \varepsilon u_\varepsilon, u_\varepsilon - \eta) + C(h) - (f+h, u_\varepsilon) \geq 0
\]

i.e. \( (h, u_\varepsilon) \leq C'(h) \).

**Theorem 3** Suppose \( A \) is a subdifferential of a convex
function $\varphi$ and let $B$ be a maximal monotone operator such that

$$(6) \quad \varphi((I+\lambda B)^{-1}x) \leq \varphi(x) \quad \forall \lambda > 0, \forall x \in D(\varphi).$$

Then $R(A+B) \simeq R(A) + R(B)$.

**Remark** We know (see [4]) that (6) implies that $A+B$ is maximal monotone.

**Proof** Let $f \in R(A)+R(B)$ and let $u_\varepsilon$ be the solution of (1). It follows easily from (6) that $\varepsilon |u_\varepsilon|$, $|Au_\varepsilon|$ and $|Bu_\varepsilon|$ remain bounded as $\varepsilon \to 0$. Next we have

$$(7) \quad (Au_\varepsilon - A\xi, u_\varepsilon - \xi) \geq 0$$

$$(8) \quad (Bu_\varepsilon - B\eta, u_\varepsilon - \eta) \geq 0.$$  

Hence, by adding (7) and (8) we obtain

$$(f - \varepsilon u_\varepsilon, u_\varepsilon) - (f, u_\varepsilon) + C \geq 0$$  

i.e. $\varepsilon |u_\varepsilon|^2 \leq C$. Suppose now that $f \in \text{Int}[R(A)+R(B)]$, with the same argument as above we have

$$(f - \varepsilon u_\varepsilon, u_\varepsilon) - (f+h, u_\varepsilon) + C(h) \geq 0$$  

i.e. $(h, u_\varepsilon) \leq C(h)$ for $|h| < \rho$.

**Some applications**

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial \Omega$. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a monotone nondecreasing continuous function such that $\beta(0) = 0$. Consider the equation (for a given $f \in L^2(\Omega)$):

$$(9) \quad -\Delta u + \beta(u) = f \text{ on } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega.$$  

**Theorem 4** A necessary condition for the existence of a...
solution of (9) is that \( \frac{1}{|\Omega|} \int_{\Omega} f(x)dx \in R(\beta) \). A sufficient condition is that \( \frac{1}{|\Omega|} \int_{\Omega} f(x)dx \in \text{Int } R(\beta) \).

**Proof**  The necessary condition is clear by integrating (9) on \( \Omega \). In order to prove the sufficient condition we apply Theorem 1 in \( H = L^2(\Omega) \) with

\[
A = -\Delta, \quad D(A) = \{ u \in H^2(\Omega); \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega \} \\
B = \beta, \quad D(B) = \{ u \in L^2(\Omega); \quad \beta(u) \in L^2(\Omega) \}.
\]

Both \( A \) and \( B \) are subdifferentials of convex functions; also \( A + B \) is maximal monotone. It is well known that \( R(A) = \{ f \in L^2(\Omega); \int_{\Omega} f(x)dx = 0 \} \). Finally if \( \frac{1}{|\Omega|} \int_{\Omega} f(x)dx \in \text{Int } R(\beta) \), then \( f \in \text{Int}[R(A) + R(B)] \). Indeed for \( g \in L^2(\Omega) \) we have

\[
g = (g - \frac{1}{|\Omega|} \int_{\Omega} g(x)dx) + \frac{1}{|\Omega|} \int_{\Omega} g(x)dx.
\]

And so it is clear that \( g \in R(A) + R(B) \) as soon as

\[
|\frac{1}{|\Omega|} \int_{\Omega} g(x)dx - \frac{1}{|\Omega|} \int_{\Omega} f(x)dx| \leq |\Omega|^{-\frac{1}{2}} \| f - g \|_{L^2} \text{ is small enough.}
\]

**Remark**  Theorem 4 is related to a number of results of Schatzman [22], Hess [13], Landesman - Lazer [17], Nirenberg [19] etc...

The method used in the proofs of Theorems 1-3 can be easily extended to include most results known about "semi coercive" problems.

Let \( \mathcal{H} \) be a Hilbert space and let \( \varphi \) be a convex function on \( \mathcal{H} \). Given \( f \in L^2(0, T; \mathcal{H}) \) consider the equation
(10) \[ \frac{du}{dt} + \partial \varphi(u) \ni f \text{ on } (0, T), \quad u(0) = u(T). \]

**Theorem 5** A necessary condition for the existence of a solution of (10) is that \( \frac{1}{T} \int_0^T f(t) dt \in R(\partial \varphi). \) A sufficient condition is that \( \frac{1}{T} \int_0^T f(t) dt \in \text{Int } R(\partial \varphi). \)

**Proof** Since \( R(\partial \varphi) \) is convex, the necessary condition follows from the integration of (10). For the sufficient condition we apply Theorem 3 in \( H = L^2(0, T; \mathcal{H}) \) with \( A = \partial \varphi \) i.e. \( f \in A u \)

provided \( f, u \in H \) and \( f(t) \in \partial \varphi(u(t)) \) a.e. and with \( B = \frac{du}{dt}, \)

\( D(B) = \{ u \in H, \; \frac{du}{dt} \in H \text{ and } u(0) = u(T) \}. \) It is well known that \( A \) is a subdifferential of a convex function in \( H, \) that \( B \) is maximal monotone and that (6) holds. The assumption

\[ \frac{1}{T} \int_0^T f(t) dt \in \text{Int } R(\partial \varphi) \]

implies that \( f \in \text{Int}[R(A)+R(B)]. \)

Indeed, note that \( R(B) = \{ f \in H; \int_0^T f(t) dt = 0 \}. \) For \( g \in H \)

we can write

\[ g = (g - \frac{1}{T} \int_0^T g(t) dt) + \frac{1}{T} \int_0^T g(t) dt \in R(A)+R(B) \]

provided \( \| g - f \|_H \) is small enough.

**Theorem 6** Let \( H \) be a Hilbert space and let \( K \) be a maximal monotone operator in \( H \) with \( D(K) = H. \) Let \( F \) be the subdifferential of a convex function on \( H \) with \( D(F) = H. \) Then \( R(I+KF) = H. \)

**Proof** Given \( f \in H \) we want to solve \( u+KFu = f \) i.e.
- $K^{-1}(f - u) + F u \ni 0$. We apply Theorem 2 with $A = F$ and $B = -K^{-1}(f - u)$ so that $B$ is maximal monotone; it follows that $R(A + B) \cong R(A) + R(B)$. However $R(B) = -D(K) = H$ and therefore $R(A + B) = H$.

**Remark** Results related to Theorem 6 were obtained in [6].

§ II.1 **Comparative behavior of $(I + tA)^{-1}$ and $S(t)$ near $t = 0$**

1. **The Hilbert space case**

Suppose $H$ is a Hilbert space and let $A$ be a maximal monotone operator; let $S(t)$ be the semigroup generated by $-A$ in the sense of Kato-Komura (see e.g. [23] or [4]).

For $x \in \overline{D(A)}$ and $y \in D(A)$ we have

$$|x - S(t)x| \leq 2|x - y| + |y - S(t)y| \leq 2|x - y| + t|A^0 y| .$$

Choosing $y = J_{\lambda} x = (I + \lambda A)^{-1} x$ we get

$$|x - S(t)x| \leq (2 + \frac{t}{\lambda}) |x - J_{\lambda} x|$$

and in particular, for $\lambda = t$, we obtain

$$|x - S(t)x| \leq 3|x - J_{t} x| .$$

In case $A = \partial \varphi$ we can show (see [5]) that

$$|x - J_{t} x| \leq (1 + \frac{1}{\sqrt{2}}) |x - S(t)x|$$

(the best constants are not known).

For general monotone operators an inequality of the kind (13) does not hold (consider for example in $H = \mathbb{R}^2$, $A$ = a rotation
by $\pi/2$). However one can obtain a "substitute" for (13) in the general case as follows:

**Theorem 7** Let $A$ be a general maximal monotone operator; then we have

$$ |x - J_t x| \leq \frac{2}{t} \int_0^t |x - S(\tau)x| d\tau, \quad \forall x \in D(A), \quad \forall t > 0. \tag{14} $$

**Remark** It is clear that the constant 2 in (14) can not be improved. Otherwise we would have for $x \in D(A)$, $|x - J_t x| \leq \frac{C}{t} \int_0^t |A^\circ x| d\tau$ and as $t \to 0$, $|A^\circ x| \leq \frac{C}{2} |A^\circ x|$ with $C < 2$.

**Proof** Clearly, it is sufficient to prove (14) for $x \in D(A)$. Let $u(t) = S(t)x$; by the monotonicity of $A$, we have for $v \in D(A)$

$$ (Av + \frac{du}{dt}(t), v - u(t)) \geq 0. \tag{15} $$

Integrating (15) on $(0, t)$ we obtain

$$ \frac{1}{2} |u(t) - v|^2 - \frac{1}{2} |x - v|^2 \leq t(Av, v - u(\tau))d\tau = t(Av, v - x) + \int_0^t (Av, x - u(\tau))d\tau. \tag{16} $$

Thus $\frac{1}{2} u(t) - v|^2 - \frac{1}{2} |x - v|^2 \leq t(Av, v - x) + |Av| \int_0^t |x - u(\tau)|d\tau$.

Choosing $v = J_t x$ we get

$$ \frac{1}{2} |u(t) - J_t x|^2 - \frac{1}{2} |x - J_t x|^2 \leq -|x - J_t x|^2 + \frac{|x - J_t x|}{t} \int_0^t |x - u(\tau)|d\tau, $$

and (14) follows.
Remark Combining (12) and (14) we see that $|x - J_t x|$ and $|x - S(t) x|$ have the same modulus of continuity at $t = 0$.

Also, using Hardy's inequality we can deduce that for $1 > \alpha > 0$ and $1 \leq p \leq \infty$

$$\left\| \frac{x - J_t x}{t^\alpha} \right\|_{L^p} \leq 3 \left\| \frac{x - J_t x}{t^\alpha} \right\|_{L^p}$$ and

$$\left\| \frac{x - J_t x}{t^\alpha} \right\|_{L^p} \leq \frac{2}{1 + \alpha} \left\| \frac{x - S(t) x}{t^\alpha} \right\|_{L^p}$$

where $L^p = L^p([0, 1], H; \frac{dt}{t})$. These inequalities are useful in the study of nonlinear interpolation classes (see [3]).

In a "similar spirit" we have the following

**Theorem 8** Let $A$ be a general maximal monotone operator.

For $x \in \overline{D(A)}$, $\lambda > 0$ and $t > 0$ we set

$$y_{\lambda, t} = (I + \frac{\lambda}{t} (I - S(t)))^{-1} x.$$ 

Then

$$|y_{\lambda, t} - J_{\lambda} x|^2 \leq |x - J_{\lambda} x| \frac{2}{t} \int_0^t |x - S(\tau) x| d\tau.$$  

**Remark** Let $\omega(t) = \text{Sup}_{0 \leq \tau \leq t} |x - S(\tau) x|$. By a result of Kato [14] (see also [4] Lemma 4.2) we know that for every integer $n$

$$|y_{\lambda, t} - y_{\lambda, t/n}|^2 \leq 2 \omega(t) |y_{\lambda, t/n} x|.$$ 

Using the fact that $y_{\lambda, s} \rightarrow J_{\lambda} x$ as $s \rightarrow 0$ (see e.g. [4]

**Proposition 4.1** we obtain as $n \rightarrow \infty$

$$|y_{\lambda, t} - J_{\lambda} x|^2 \leq 2 \omega(t) |J_{\lambda} x - x|.$$  

Such an inequality follows also directly from (17).
Proof. We apply (16) with $x$ replaced by $y_{\lambda},t$ and $v$ by $J_{\lambda}x$. Thus

$$
\frac{1}{2} |S(t)y_{\lambda},t - J_{\lambda}x|^2 - \frac{1}{2} |y_{\lambda},t - J_{\lambda}x|^2 \\
\leq \int_0^t \left( \frac{x - J_{\lambda}x}{\lambda}, J_{\lambda}x - S(\tau)y_{\lambda},t \right) d\tau.
$$

However $S(t)y_{\lambda},t = (1 + \frac{t}{\lambda})y_{\lambda},t - \frac{t}{\lambda}x$ and so

$$
|S(t)y_{\lambda},t - J_{\lambda}x|^2 \geq |y_{\lambda},t - J_{\lambda}x|^2 + \frac{2t}{\lambda} (y_{\lambda},t - J_{\lambda}x, y_{\lambda},t - x).
$$

On the other hand

$$
(x - J_{\lambda}x, J_{\lambda}x - S(\tau)y_{\lambda},t) = - |x - J_{\lambda}x|^2 + (x - J_{\lambda}x, x - S(\tau)y_{\lambda},t)
\leq - |x - J_{\lambda}x|^2 + |x - J_{\lambda}x|(|x - S(\tau)x| + |x - y_{\lambda},t|).
$$

We deduce from (19), (20) and (21) that

$$
\frac{t}{\lambda}(y_{\lambda},t - J_{\lambda}x, y_{\lambda},t - x) \leq - \frac{t}{\lambda} |x - J_{\lambda}x|^2 + \frac{2t}{\lambda} |x - J_{\lambda}x| |x - y_{\lambda},t|
+ \frac{|x - J_{\lambda}x|}{\lambda} \int_0^t |x - S(\tau)x| d\tau.
$$

Therefore

$$
|x - J_{\lambda}x|^2 + (y_{\lambda},t - J_{\lambda}x, y_{\lambda},t - x) \leq |x - J_{\lambda}x| |x - y_{\lambda},t|
+ |x - J_{\lambda}x| \frac{1}{t} \int_0^t |x - S(\tau)x| d\tau.
$$

i.e. $|a|^2 + (b,a, b) \leq |a| |b| + |x - J_{\lambda}x| \frac{1}{t} \int_0^t |x - S(\tau)x| d\tau$

with $a = x - J_{\lambda}x$ and $b = x - y_{\lambda},t$. Hence

$$
\frac{1}{2} |a - b|^2 = \frac{1}{2} |a|^2 + \frac{1}{2} |b|^2 - (a,b) \leq
- \frac{1}{2} |a|^2 - \frac{1}{2} |b|^2 + |a| |b| + |x - J_{\lambda}x| \frac{1}{t} \int_0^t |x - S(\tau)x| d\tau
$$

and

$$
\frac{1}{2} |a - b|^2 \leq |x - J_{\lambda}x| \frac{1}{t} \int_0^t |x - S(\tau)x| d\tau.
$$
II.2 The Banach space case

Let $X$ be a general Banach space and let $A$ be an $m$-accretive operator on $X$. Let $S(t)$ be the semigroup generated by $-A$ in the sense of Crandall-Liggett (see [10] or [23]). Clearly we have as in § II.1

\begin{equation}
\|x - S(t)x\| \leq (2 + \frac{t}{\lambda}) \|x - J_\lambda x\|.
\end{equation}

We don't know whether the exact analogue of (14) holds true. However we can prove the following

Theorem 9 For every $x \in \overline{D(A)}$, $t > 0$ and $\lambda > 0$ we have

\begin{equation}
\|x - J_\lambda x\| \leq (1 + \frac{\lambda}{\lambda + t}) \int_0^t \|x - S(\tau)x\| d\tau.
\end{equation}

and in particular

\begin{equation}
\|x - J_\lambda x\| \leq \frac{4}{\lambda} \int_0^t \|x - S(\tau)x\| d\tau.
\end{equation}

Proof As usual we denote for $x, y \in X$

\[ \tau(x, y) = \lim_{\lambda \to 0} \frac{1}{\lambda} \left( \|x + \lambda y\| - \|x\| \right) = \inf_{\lambda > 0} \frac{1}{\lambda} \left( \|x + \lambda y\| - \|x\| \right). \]

The analogue of (16) becomes now (see [10] or [2] for equivalent forms):

\begin{equation}
\|S(t)x - v\| - \|v - x\| \leq \int_0^t \tau(v - S(s)x, Av) ds
\end{equation}

for every $v \in D(A)$.

However we have for every $\lambda > 0$

\begin{equation}
\tau(v - S(s)x, Av) \leq \frac{1}{\lambda} \left( \|v - S(s)x + \lambda Av\| - \|v - S(s)x\| \right).
\end{equation}

If we choose in (26) $v = J_\lambda x$ we obtain
(27) \[ \tau(J_x - S(s)x, A_x) \leq \frac{1}{\lambda} (\|x - S(s)x\| - \|J_x - S(s)x\|) \]

and by (25) we get

(28) \[ \|S(t)x - J_x x\| - \|J_x x - x\| \leq \frac{1}{\lambda} \int_0^t (\|x - S(s)x\| - \|J_x x - S(s)x\|) ds. \]

But \[ -\|J_x x - S(s)x\| \leq \|x - S(s)x\| - \|x - J_x x\| \] and therefore (28) leads to

\[ -\|x - S(s)x\| \leq \frac{1}{\lambda} \int_0^t \|x - S(s)x\| ds + \frac{1}{\lambda} \int_0^t (\|x - S(s)x\| ds - \frac{t}{\lambda} \|x - J_x x\| \]

i.e.

(29) \[ \|x - J_x x\| \leq \frac{\lambda}{t} \|x - S(t)x\| + \frac{2}{t} \int_0^t \|x - S(s)x\| ds. \]

Finally note that

(30) \[ \|x - S(t)x\| \leq \frac{2}{t} \int_0^t \|x - S(s)x\| ds; \]

indeed

\[ \|S(t)x - \frac{1}{t} \int_0^t S(s)x ds\| \leq \frac{1}{t} \int_0^t \|S(t)x - S(s)x\| ds \]

\[ \leq \frac{1}{t} \int_0^t \|S(t) - S(s)x - x\| ds = \frac{1}{t} \int_0^t \|S(s)x - x\| ds, \]

and so

\[ \|x - S(t)x\| \leq \|x - \frac{1}{t} \int_0^t S(s)x ds\| + \frac{1}{t} \int_0^t \|S(s)x - x\| ds \leq \frac{2}{t} \int_0^t \|x - S(s)x\| ds. \]

Combining (29) and (30) we obtain (23).

Remarks:

1) I would like to thank Prof. M. Crandall, Y. Konishi and I. Miyadera for stimulating discussions concerning Theorem 9.

After our first result was obtained \((\|x - J_t x\| \leq \frac{2}{t} \int_0^{2t} \|x - S(\tau)x\| d\tau),\)

I. Miyadera showed that \(\|x - J_t x\| \leq \frac{6}{t} \int_0^t \|x - S(\tau)x\| d\tau\) and
Y. Konishi got \(|x - J_t x| \leq \frac{4}{t} \int_0^t |x - S(\tau)x| d\tau|.

2) Using (22) and (23) one can prove directly the following result of M. Crandall [9]:

\[
\lim_{t \downarrow 0} \sup_{\tau} \frac{|x - S(t)x|}{t} = \lim_{\lambda \downarrow 0} \frac{|x - J_\lambda x|}{\lambda}.
\]

Indeed let \( \alpha = \lim_{t \downarrow 0} \sup_{\tau} \frac{|x - S(t)x|}{t} \); and so \( \forall \varepsilon > 0 \exists \delta > 0 \) such that \( 0 < t < \delta \)

\[|x - S(t)| \leq t(\alpha + \varepsilon).\]

From (23) we have for \( 0 < t < \delta \) and every \( \lambda > 0 \)

\[|x - J_\lambda x| \leq (1 + \frac{\lambda}{t})^2 (\alpha + \varepsilon) \int_0^t \tau d\tau = (\lambda + t)(\alpha + \varepsilon).\]

It follows that \( |x - J_\lambda x| \leq \lambda(\alpha + \varepsilon) \) for every \( \lambda > 0 \) and \( \varepsilon > 0 \). Next let \( \beta = \lim_{\lambda \downarrow 0} \frac{|x - J_\lambda x|}{\lambda} \); and so \( \forall \varepsilon > 0 \exists \delta > 0 \) such that for \( 0 < \lambda < \delta \)

\[|x - J_\lambda x| \leq \lambda(\beta + \varepsilon).\]

From (22) we get for \( 0 < \lambda < \delta \) and every \( t > 0 \)

\[|x - S(t)x| \leq (2 + \frac{\lambda}{t}) \lambda(\beta + \varepsilon) = (t + 2\lambda)(\beta + \varepsilon).\]

Hence \( |x - S(t)x| \leq t \beta \) for every \( t > 0 \).

3) In general for \( x \in \overline{D(A)} \), \( \frac{|x - S(t)x|}{|x - J_t x|} \) does not necessarily converge to 1 as \( t \to 0 \).

Consider for example in \( H = \mathbb{R} \), \( Au = -\frac{1}{u} \) for \( u > 0 \) and \( Au = \frac{1}{u} \) for \( u \leq 0 \). In this case \( J_t 0 = \sqrt{t} \) and \( S_t 0 = \sqrt{2t} \) (slightly more complicated examples were built previously by A. Plant and L. Veron).

4) In view of the example built by Crandall - Liggett in [11]
one can not expect to extend Theorem 8 to Banach spaces (or even to \( \mathbb{R}^3 \) with some Banach norm) since \( y_{\lambda,t} \) does not necessarily converge to a limit as \( t \to 0 \).

II.3 An application to the characterization of compact semi-groups.

Let \( A \) be an \( m \)-accretive operator in a general Banach space \( X \) and let \( S(t) \) be the semigroup generated by \(-A\).

Theorem 10. The following properties are equivalent.

(31) For every \( t > 0 \), \( S(t) \) is compact i.e. \( S(t) \) maps bounded sets of \( \overline{D(A)} \) into compact sets of \( X \)

\[
\begin{align*}
(32a) \text{ For every } \lambda > 0, \ (I + \lambda A)^{-1} \text{ is compact i.e.} \\
\text{maps bounded sets of } X \text{ into compact sets of } X \\
(32b) \text{ For every bounded set } B \text{ in } \overline{D(A)} \text{ and every } t_0 > 0 \\
\text{the mappings } t \mapsto S(t)x \text{ are equicontinuous at } t = t_0 \\
\text{as } x \in B.
\end{align*}
\]

Remarks

1) Theorem 10 is due to A. Pazy [20] in the linear case and to Y. Konishi [15] in the nonlinear Hilbert case (his proof relies on a consequence of (18) and could not be extended to Banach spaces)

2) It is obvious that (32a) is equivalent to

\[(32a') \quad (I+A)^{-1} \text{ is compact}\]

and also to

\[(32a'') \quad \text{For every } M > 0 \text{ the set }\]
\[ \{ x \in D(A); \| x \| \leq M \text{ and } \| y \| \leq M \text{ for some } y \in Ax \} \]
is relatively compact in \( X \).

**Proof** (31) \( \Longrightarrow \) (32a)

Let \( \lambda \) be fixed and let \( x \in X \); we have for every \( t \geq 0 \)
\[ \| J_\lambda x - S(t)J_\lambda x \| \leq t \| A \lambda x \| = \frac{t}{\lambda} \| x - J_\lambda x \|. \]
Let \( B \) be a bounded set in \( X \); given \( \varepsilon > 0 \), choose \( t_0 \) so small that
\[ \frac{t_0}{\lambda} \| x - J_\lambda x \| < \varepsilon / 2 \quad \text{for } x \in B. \]
Since \( J_\lambda(B) \) is bounded in \( D(A) \), it follows from (31) that
\( S(t_0)J_\lambda(B) \) is relatively compact. Thus \( S(t_0)J_\lambda(B) \) can be covered by a finite union \( \bigcup_i B(x_i, \varepsilon / 2) \). Hence \( J_\lambda(B) \subseteq \bigcup_i B(x_i, \varepsilon) \) and consequently \( J_\lambda(B) \) is precompact.

(31) \( \Longrightarrow \) (32b)

Using (31) we have only to prove that the mappings \( t \mapsto S(t)x \)
are equicontinuous at \( t = \frac{t_0}{2} \) as \( x \in K, K \) compact
\( (K = S(\frac{t_0}{2})B) \). This follows directly from the fact that for
each fixed \( x \), \( t \mapsto S(t)x \) is continuous and that \( x \mapsto S(t)x \)
is a contraction.

(32a) + (32b) \( \Longrightarrow \) (31)

Fix a \( t_0 > 0 \) and let \( B \) be a bounded set in \( D(A) \). By (32b),
for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that
\[ \| S(t)x - S(t_0)x \| < \varepsilon \quad \text{for } |t - t_0| < \delta \text{ and } x \in B. \]
We deduce from (23) that for \( x \in B \) and \( \lambda > 0 \),
\[ \| S(t_0)x - J_\lambda S(t_0)x \| \leq (1 + \frac{\lambda}{c}) \frac{2}{t} \int_0^t \| S(S(t_0)x - S(t + t_0)x) \| d \tau \]
\[ \leq (1 + \frac{\lambda}{t}) 2 \varepsilon \quad \text{for every } 0 < t \leq T. \]

In particular for \( 0 < \lambda \leq T \) and \( x \in B \) we have

\[ \| S(t_0)x - J_\lambda S(t_0)x \| \leq 4 \varepsilon. \]

Since \( J_\lambda S(t_0)B \) is relatively compact it can be covered by a finite union \( \bigcup_i B(x_i, \varepsilon) \). Hence \( S(t_0)B \) can also be covered by a finite union of balls of radius \( 5 \varepsilon \) and thus \( S(t_0)B \) is precompact.

**Remark** Suppose \( H \) is a Hilbert space, \( \varphi \) is a convex function on \( H \) and let \( A = \partial \varphi \). In this case (31) is equivalent to (32a) since (32b) is satisfied automatically. Indeed we have

\[ |S(t)x - S(t_0)x| = |S(t - \frac{t_0}{2})y - S(\frac{t_0}{2})y| \leq |t - t_0| |A^\circ y| \]

where \( y = S(\frac{t_0}{2})x \). On the other hand (see e.g. [4] Théorème 3.2) we know that

\[ |A^\circ S(\frac{t_0}{2})x| \leq |A^\circ v| + \frac{2}{t_0} |x - v| \quad \text{for every } v \in D(A). \]

Therefore the mappings \( t \mapsto S(t)x \) are equicontinuous at \( t = t_0 \) as \( x \) remains bounded.

In this case property (32a) is also equivalent to (32a'')

For every \( M \) the set

\[ \{ x \in D(\varphi); \ |x| \leq M \text{ and } \varphi(x) \leq M \} \]

is relatively compact in \( H \).

Indeed (32a'') \( \iff \) (32a''):

Let \( E = \{ x \in D(A); \ |x| \leq M \text{ and } |A^\circ x| \leq M \} \); for a fixed \( v_0 \in D(\varphi) \) we have
\[ \varphi(v_0) - \varphi(x) \geq (A^*x, v_0 - x) \]

and so \( \varphi(x) \leq \varphi(v_0) + M(|v_0| + M) = M' \) when \( x \in E \).

Conversely (32a) \( \Rightarrow \) (32a'''):

Let

\[ F = \{ x \in D(\varphi); \ |x| \leq M \text{ and } \varphi(x) \leq M \}; \]

for \( x \in F \) we have

\[ \varphi(x) - \varphi(J_{\lambda}x) \geq (A_{\lambda}x, x - J_{\lambda}x) = \frac{1}{\lambda} |x - J_{\lambda}x|^2. \]

Therefore, since \( \varphi \) is bounded below by some affine function, we get for \( x \in F \),

\[ \frac{1}{\lambda} |x - J_{\lambda}x|^2 \leq M + C_1 |J_{\lambda}x| + C_2 \leq M + C_1 |x - J_{\lambda}x| + C_1 M + C_2. \]

Thus \( |x - J_{\lambda}x| \leq \sqrt{\lambda(C_3 \lambda + C_4)} \) for \( x \in F \).

Given \( \varepsilon > 0 \) we choose \( \lambda_0 > 0 \) so small that \( \sqrt{\lambda_0 (C_3 \lambda_0 + C_4)} < \varepsilon \). Since \( J_{\lambda_0}(F) \) is relatively compact, it can be covered by a finite union \( \bigcup_i B(x_i, \varepsilon) \) and then \( F \subset \bigcup_i B(x_i, 2\varepsilon) \).

\section{III. A convergence theorem for nonlinear semigroups}

Let \( H \) be a Hilbert space; let \( \{A_n\}_{n \geq 1} \) and \( A \) be maximal monotone operators. Let \( \{S_n(t)\}_{n \geq 1} \) and \( S(t) \) be the corresponding semigroups.

Our next result is a nonlinear version of the Theorem of Trotter-Kato-Neveu. A number of related results have been obtained previously by Miyadera-Oharu [18], Brezis-Pazy [8], Benilan [1], Goldstein [12], Kurtz [16] etc...

\textbf{Theorem II.} The following properties are equivalent.
(33) $\forall x \in D(A), \ \forall \lambda > 0 \ (I + \lambda A_n)^{-1}x \rightarrow (I + \lambda A)^{-1}x$

(34) $\forall x \in D(A) \ \exists x_n \in D(A_n) \text{ such that } x_n \rightarrow x \text{ and } A^\circ x_n \rightarrow A^\circ x$

(35) $\forall x \in D(A) \ \exists x_n \in D(A_n) \text{ such that } x_n \rightarrow x \text{ and } \forall t \geq 0 \ S_n(t)x_n \rightarrow S(t)x$.

In addition, the convergence in (33) (resp. (35)) is uniform for bounded $\lambda$ (resp. bounded $t$).

The proof of Theorem 11 is divided into four parts

Part A (33) $\Rightarrow$ (34)

Part B (34) $\Rightarrow$ (33)

Part C (33) $\Rightarrow$ (35)

Part D (35) $\Rightarrow$ (33).

Part A (33) $\Rightarrow$ (34)

Let $x \in D(A)$; given $\varepsilon > 0$ there is a $\lambda > 0$ such that

$$|x - (I + \lambda A)^{-1}x| < \varepsilon/2$$

$$|A^\circ x - A^\circ x| < \varepsilon/2.$$  

Next, by (33) there is an integer $N$ such that for $n \geq N$

$$|(I + \lambda A_n)^{-1}x - (I + \lambda A)^{-1}x| < \varepsilon/2$$

$$|(A_n)_{\lambda^\circ}x - A_{\lambda^\circ}x| < \varepsilon/2.$$  

Combining these estimates we see that given $\varepsilon > 0$ there is an integer $N(\varepsilon)$ and sequences $u_n(\varepsilon) = (I + \lambda A_n)^{-1}x$ and $f_n(\varepsilon) = (A_n)_{\lambda^\circ}x$ such that $[u_n(\varepsilon), f_n(\varepsilon)] \in G(A_n)$ and for $n \geq N(\varepsilon)$, $|u_n(\varepsilon) - x| < \varepsilon$, $|f_n(\varepsilon) - A_{\lambda^\circ}x| < \varepsilon$. Let $N_k = N(1/k)$; we can always assume that $N_k$ is increasing to $\infty$.  

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We define the sequences $x_n$ and $g_n$ by $x_n = u_n(\frac{1}{k})$ and $g_n = f_n(\frac{1}{k})$ for $N_k \leq n < N_{k+1}$. Therefore $[x_n, g_n] \in G(A_n)$ and for $N_k \leq n < N_{k+1}$ we have $|x_n - x| < \frac{1}{k}$ and $|g_n - A^o x| < \frac{1}{k}$.

Consequently $x_n \rightarrow x$ and $g_n \rightarrow A^o x$; we are going to prove now that $A_n^o x_n \rightarrow A^o x$. Indeed $|A_n^o x_n| \leq |g_n|$ and thus for a subsequence we get $A_{n_j}^o x_{n_j} \rightarrow h$. Let $v \in D(A)$; by the monotonicity of $A_n$ we have

$$((A_n)\lambda v - A_n^o x_n, (1 + \lambda A_n)^{-1} v - x_n) \geq 0.$$ 

At the limit as $n_j \rightarrow \infty$ we obtain

$$(A_n^0 v - h, (1 + \lambda A)^{-1} v - x) \geq 0.$$ 

Next we pass to the limit as $\lambda \rightarrow 0$:

$$(A^o v - h, v - x) \geq 0 \quad \forall v \in D(A).$$ 

Therefore $h \in Ax$ (see e.g. [4] Proposition 2.7). Since on the other hand $|h| \leq |A^o x|$ we have $h = A^o x$. By the uniqueness of the limit, and the fact that $\limsup |A_n^o x_n| \leq |A^o x|$ we conclude that $A_n^o x_n \rightarrow A^o x$.

Part B (34) $\Rightarrow$ (33)

Without loss of generality we may assume that $\lambda = 1$. Let $x \in D(A)$ and let $u_n = (I + A_n)^{-1} x$. Given $y \in D(A)$, let $y_n \in D(A_n)$ be the sequence given by (34) so that $y_n = (I + A_n)^{-1} (y_n + A_n^o y_n)$. Therefore $|u_n - y_n| \leq |x - y_n - A_n^o y_n|$ and thus $u_n$ is bounded. For a subsequence $u_{n_j} \rightarrow u$; by the monotonicity of $A_n$ we have

$$(x - u_n - A_n^o y_n, u_n - y_n) \geq 0.$$ 

Passing to the limit in (36) we obtain
In (37) we choose $y = (I + \lambda A)^{-1}u$ and so

$$ (x - u, u - J\lambda u) \geq \lambda (A^0 J u, A\lambda u) \geq 0. $$

As $\lambda \to 0$ we see that

$$ (x - u, u - \text{Proj}_{\overline{D(A)}} u) \geq 0. $$

On the other hand since $x \in \overline{D(A)}$ we have

$$ (\text{Proj}_{\overline{D(A)}} u - x, u - \text{Proj}_{\overline{D(A)}} u) \geq 0 $$

and consequently $u = \text{Proj}_{\overline{D(A)}} u$ i.e. $u \in \overline{D(A)}$. Going back to (37) we deduce now from [4] Proposition 2.7 that $x - u \in Au$ i.e. $u = (I+A)^{-1}x$. By the uniqueness of the limit we have in fact $u_n \to (I+A)^{-1}x$.

It follows from (36) that for every $y \in D(A)$

$$ \lim \sup \left| u_n \right|^2 \leq (x, u-y) + (u, y) + (A^0 y, y-u). $$

In particular if we take $y = u$ we get

$$ \lim \sup \left| u_n \right|^2 \leq |u|^2 \text{ and thus } u_n \to u. $$

The convergence in (33) is uniform in $\lambda$ as $\lambda$ remains bounded:

Without loss of generality we may assume that $x \in D(A)$ and let $x_n \in D(A_n)$ with $x_n \to x$ and $A^0 x_n \to A^0 x$. We have

$$ \left| (I + \lambda A_n)^{-1} x_n - (I + \lambda A_n)^{-1} x_n \right| \leq |\lambda - \mu| |A^0 x_n|. $$

Therefore the functions $f_n(\lambda) = (I + \lambda A_n)^{-1} x_n$ are uniformly Lipschitz continuous on $[0, +\infty)$. Since they converge simply to $(I + \lambda A)^{-1} x$ as $n \to +\infty$, we conclude that the convergence is uniform in $\lambda$ as $\lambda$ remains in a bounded interval.

Part C (33) $\Rightarrow$ (35)

Without loss of generality we may assume that $x \in D(A)$. By (34)
we have a sequence \( x_n \in D(A_n) \) such that \( x_n \to x \) and \( A_n^* x_n \to A^* x \). We are going to prove that \( S_n(t)x_n \to S(t)x \). It is known (see e.g. [4] Corollaire 4.4) that

\[
|S_n(t)x_n - (I + \frac{t}{k} A_n)^{-k} x_n| \leq \frac{2t}{\sqrt{k}} |A_n^* x_n| \leq \frac{2tM}{\sqrt{k}}
\]

and

\[
|S(t)x - (I + \frac{t}{k} A)^{-k} x| \leq \frac{2t}{\sqrt{k}} |A^* x| \leq \frac{2tM}{\sqrt{k}}
\]

where \( M = \sup_n |A_n^* x_n| \). Given \( \varepsilon > 0 \), we first fix \( k \) large enough so that \( \frac{2Mt}{\sqrt{k}} < \varepsilon \). Next observe, by induction, that for every integer \( N \) and for every sequence \( u_n \to u \) with \( u \in \overline{D(A)} \), then \((I + \lambda A_n)^{-N} u_n \to (I + \lambda A_n)^{-N} u\), as \( n \to +\infty \). Thus

\[
|S_n(t)x_n - S(t)x| \leq 2\varepsilon + |(I + \frac{t}{k} A_n)^{-k} x_n - (I + \frac{t}{k} A)^{-k} x| \leq 3\varepsilon
\]

provided \( n \) is large enough.

Finally (35) holds true uniformly in \( t \) as \( t \) remains bounded since (33) holds true uniformly in \( \lambda \) as \( \lambda \) remains bounded.

**Part D** (35) \( \Rightarrow \) (33)

The proof relies on the following

**Lemma 1** Suppose (35) holds. Let \( f_n \in \overline{D(A_n)} \) be such that \( f_n \to f \) and \( f \in \overline{D(A)} \). Then \( \forall \lambda > 0, \forall t > 0 \)

\[
u_n = (I + \frac{\lambda}{t}(I - S_n(t)))^{-1} f_n \to u = (I + \frac{\lambda}{t}(I - S(t)))^{-1} f.
\]

**Proof of Lemma 1** By (35) there exists a sequence \( x_n \in \overline{D(A_n)} \) such that \( x_n \to u \) and \( S_n(t)x_n \to S(t)u \). Writing the monotonicity of \( I - S_n(t) \) we have

\[
((u_n - S_n(t)u_n) - (x_n - S_n(t)x_n), u_n - x_n) \geq 0
\]
and therefore

\[
\left( \frac{u - u_n}{\lambda} + S_n, u_n - x_n \right) \geq 0
\]

where \( S_n = \frac{f_n - f}{\lambda} + \frac{u - x_n}{t} + \frac{S_n(t)x_n - S(t)u}{t} \) and \( S_n \to 0 \).

Hence

\[
\frac{1}{\lambda} |u_n - u|^2 \leq |S_n| |u_n - u| + |S_n| |u - x_n| + \frac{1}{\lambda} |u - u_n| |u - x_n|,
\]

and consequently \( u_n \to u \) as \( n \to \infty \).

**Lemma 2.** Let \( x_n \in D(A_n) \) be a sequence such that \( x_n \to x \) with \( x \in D(A) \) and \( S_n(t)x_n \to S(t)x \) for every \( t \geq 0 \). Then for every \( T \) there exists a constant \( K \) such that \( |(I + \lambda A_n)^{-1} x_n| \leq K \) and \( |S_n(t)x_n| \leq K \) for every \( 0 < \lambda < T \), for every \( 0 < t < T \) and every \( n \).

**Proof of Lemma 2** Let \( M = \text{Sup} |S(t)x| \) and let

\[
E_n = \left\{ t \in [0, 1]; \ |S_p(t)x_p| \leq M + 1 \text{ for every } p \geq n \right\}.
\]

Clearly \( E_n \) is closed and \( \bigcup_{n=1}^{\infty} E_n = [0, 1] \); it follows from Baire's theorem that \( \text{Int } E_N \neq \emptyset \) for some \( N \). Let \( [t_0, t_0 + h] \subset E_n \) so that

\[
|S_p(t)x_p| \leq M + 1 \text{ for } n \geq N \text{ and } t_0 \leq t \leq t_0 + 1.
\]

It follows from Theorem 9 that

\[
|S_n(t_0)x_n - (I + \lambda A_n)^{-1} S_n(t_0)x_n| \leq \left( 1 + \frac{\lambda}{h} \right) 2 \int_0^h |S_n(t_0)x_n - S_n(t_0 + \tau)x_n| d\tau.
\]

Choosing \( n \geq N \) we get

\[
|S_n(t_0)x_n - (I + \lambda A_n)^{-1} x_n| \leq |x_n - S_n(t_0)x_n| + |S_n(t_0)x_n| + \frac{2}{h} (1 + \lambda) 2(M + 1) h.
\]
\[ \leq |x_n| + 2(M + 1) + 4(1 + \frac{\lambda}{n})(M + 1). \]

We conclude by using the fact that
\[ |x_n - S_n(t)x_n| \leq 3|x_n - (I + \tau A_n)^{-1}x_n|. \]

**Proof of (35) \Rightarrow (33)** In what follows \( \lambda \) is fixed. Using Theorem 8 we get
\[ \left| (I + \frac{\lambda}{t}(I - S_n(t)))^{-1}x_n - (I + \lambda A_n)^{-1}x_n \right|^2 \leq |x_n - (I + \lambda A_n)^{-1}x_n| \left\{ \frac{1}{t} \int_0^t |x_n - S_n(\tau)x_n| d\tau \right\}, \]

and
\[ \left| (I + \frac{\lambda}{t}(I - S(t)))^{-1}x - (I + \lambda A)^{-1}x \right|^2 \leq |x - (I + \lambda A)^{-1}x| \left\{ \frac{1}{t} \int_0^t |x - S(\tau)x| d\tau \right\}. \]

Let \( P = 2|x - (I + \lambda A)^{-1}x| + 2\sup_n |x_n - (I + \lambda A_n)^{-1}x_n| < \infty \) (by Lemma 2). We have
\[ \frac{1}{t} \int_0^t |x_n - S_n(\tau)x_n| d\tau \leq |x_n - x| + \frac{1}{t} \int_0^t |x - S(\tau)x| d\tau + \frac{1}{t} \int_0^t |S(\tau)x - S_n(\tau)x_n| d\tau \]

and so
\[
\left| (I + \lambda A_n)^{-1}x_n - (I + \lambda A)^{-1}x \right| \leq \left| (I + \frac{\lambda}{t}(I - S_n(t)))^{-1}x_n - (I + \frac{\lambda}{t}(I - S(t)))^{-1}x \right| \\
+ \sqrt{P|x_n - x|} + 2\sqrt{\frac{P}{t} \int_0^t |x - S(\tau)x| d\tau} + \sqrt{\frac{P}{t} \int_0^t |S(\tau)x - S_n(\tau)x_n| d\tau}
\]

\[ = X_1 + X_2 + X_3 + X_4. \]

Given \( \varepsilon > 0 \) we choose first \( t > 0 \) small enough so that \( X_3 < \varepsilon \) and then we choose \( n \) large enough so that \( X_1 + X_3 + X_4 < \varepsilon \) (we use here Lemma 1 to make \( X_1 \) small and Lemma 2 combined with Lebesgue's Theorem to make \( X_2 \) small).
References


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