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New results concerning monotone operators
and nonlinear semigroups

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Our purpose is to describe here some recent developments in three different directions.
In §I we discuss a property of the range \( R(A+B) \) of the sum of two monotone operators. Surprisingly, it turns out that in "many" cases \( R(A+B) \) is "almost" equal to \( R(A)+R(B) \). A number of applications to nonlinear partial differential equations are given.
In §II we prove some estimates showing that \( (I+ta)^{-1} \) and \( S(t) \) have the same modulus of continuity at \( t=0 \) \( (S(t) \text{ denotes the semigroup generated by } -A) \). Next we present some consequences.
In §III we give a very general form of the convergence theorem of Trotter - Kato - Neveu type for nonlinear semigroups.

§I \( "R(A+B) \cong R(A)+R(B)" \) and applications

Let \( H \) be a real Hilbert space and let \( A \) and \( B \) be maximal monotone operators such that \( A+B \) is again maximal monotone.
We say that two subsets \( K_1 \) and \( K_2 \) of \( H \) are almost equal \( (K_1 \approx K_2) \) if \( K_1 \) and \( K_2 \) have the same closure and the same interior. We prove here, under various assumptions, that
\( R(A + B) \cong R(A) + R(B) \); we discuss here only the simplest forms (for more elaborate results see [7]).

**Theorem 1** Suppose A and B are subdifferentials of convex functions. Then \( R(A + B) \cong R(A) + R(B) \).

**Proof** First we prove that \( \overline{R(A + B)} = \overline{R(A) + R(B)} \); it is sufficient to verify that \( R(A) + R(B) \subseteq \overline{R(A + B)} \). Given \( f \in R(A) + R(B) \), there exist \( \xi \in D(A) \) and \( \eta \in D(B) \) such that \( f \in A\xi + B\eta \). The equation

\[
(1) \quad \varepsilon u_\varepsilon + Au_\varepsilon + Bu_\varepsilon \ni f
\]

has a unique solution \( u_\varepsilon \). The conclusion follows provided we show that \( \varepsilon u_\varepsilon \to 0 \) as \( \varepsilon \to 0 \). Let \( x \in D(A) \cap D(B) \) be fixed. Since A and B are cyclically monotone (see [21]) we have

\[
(2) \quad (Au_\varepsilon, u_\varepsilon - x) + (Ax, x - \xi) + (A\xi, \xi - u_\varepsilon) \geq 0
\]

\[
(3) \quad (Bu_\varepsilon, u_\varepsilon - x) + (Bx, x - \eta) + (B\eta, \eta - u_\varepsilon) \geq 0
\]

and therefore by adding (2) and (3) we obtain

\[
(f - \varepsilon u_\varepsilon, u_\varepsilon - x) + C - (f, u_\varepsilon) \geq 0,
\]

where \( C \) is independent of \( \varepsilon \). Hence

\[
\varepsilon |u_\varepsilon|^2 - \varepsilon (u_\varepsilon, x) \leq C
\]

and therefore \( \sqrt{\varepsilon} |u_\varepsilon| \) remains bounded as \( \varepsilon \to 0 \).

Next we prove that \( \text{Int}[R(A) + R(B)] = \text{Int}[R(A + B)] \). It is sufficient to check that \( \text{Int}[R(A) + R(B)] \subseteq R(A + B) \). Let \( f \in \text{Int}[R(A) + R(B)] \), so that a ball \( B(f, \rho) \) is contained in \( R(A) + R(B) \). For every \( h \in H \) with \( |h| < \rho \), there exist \( \xi \)
and \( \eta \) (depending on \( h \)) such that \( f + h \in A \xi + B \eta \). Going back to (2) and (3) and adding them we obtain now

\[
(f - \varepsilon u_\varepsilon, u_\varepsilon - x) + C(h) - (f + h, u_\varepsilon) \geq 0
\]

where \( C(h) \) depends on \( h \), but is independent of \( \varepsilon \).

Hence \( (h, u_\varepsilon) \leq C(h) \) for every \( h \in H \) with \( |h| < \rho \). It follows from the uniform boundedness principle that \( \{u_\varepsilon\} \) remains bounded as \( \varepsilon \to 0 \). Passing to the limit in (1) we conclude by standard methods that \( f \in R(A + B) \).

**Theorem 2** We suppose now that only \( A \) is the subdifferential of a convex function, but \( D(B) \subset D(A) \). Then \( R(A + B) \approx R(A) + R(B) \).

**Proof** We proceed as in the proof of Theorem 1.

First let \( f \in R(A + B) \) i.e. \( f \in A \xi + B \eta \); let \( u_\varepsilon \) be the solution of (1). We have

\[
\begin{align*}
(4) \quad (Au_\varepsilon, u_\varepsilon - \eta) &+ (A \eta, \eta - \xi) + (A \xi, \xi - u_\varepsilon) \geq 0 \\
(5) \quad (Bu_\varepsilon, u_\varepsilon - \eta) &+ (B \eta, \eta - u_\varepsilon) \geq 0.
\end{align*}
\]

By adding (4) and (5) we obtain

\[
(f - \varepsilon u_\varepsilon, u_\varepsilon - \eta) + C - (f, u_\varepsilon) \geq 0
\]

and hence

\[
\varepsilon |u_\varepsilon|^2 - \varepsilon (u_\varepsilon, \eta) \leq C'.
\]

Next suppose \( f \in \text{Int}[R(A) + R(B)] \); we obtain now, as in the proof of Theorem 1

\[
(f - \varepsilon u_\varepsilon, u_\varepsilon - \eta) + C(h) - (f + h, u_\varepsilon) \geq 0
\]

i.e. \( (h, u_\varepsilon) \leq C'(h) \).

**Theorem 3** Suppose \( A \) is a subdifferential of a convex
function $\varphi$ and let $B$ be a maximal monotone operator such that

$$\varphi((I+\lambda B)^{-1}x) \leq \varphi(x) \quad \forall \lambda > 0, \forall x \in D(\varphi).$$

Then $R(A+B) \simeq R(A) + R(B)$.

**Remark** We know (see [4]) that (6) implies that $A+B$ is maximal monotone.

**Proof** Let $f \in R(A) + R(B)$ and let $u_\varepsilon$ be the solution of (1). It follows easily from (6) that $\varepsilon|u_\varepsilon|$, $|A u_\varepsilon|$ and $|B u_\varepsilon|$ remain bounded as $\varepsilon \to 0$. Next we have

$$(Au_\varepsilon - A \xi, u_\varepsilon - \xi) \geq 0$$

and

$$(Bu_\varepsilon - B \eta, u_\varepsilon - \eta) \geq 0.$$  

Hence, by adding (7) and (8) we obtain

$$(f - \varepsilon u_\varepsilon, u_\varepsilon) - (f, u_\varepsilon) + C \geq 0$$

i.e. $\varepsilon|u_\varepsilon|^2 \leq C$. Suppose now that $f \in \text{Int}[R(A) + R(B)]$, with the same argument as above we have

$$(f - \varepsilon u_\varepsilon, u_\varepsilon) - (f + h, u_\varepsilon) + C(h) \geq 0$$

i.e. $(h, u_\varepsilon) \leq C(h)$ for $|h| < \rho$.

**Some applications**

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial \Omega$. Let $\beta : \mathbb{R} \to \mathbb{R}$ be a monotone nondecreasing continuous function such that $\beta(0) = 0$. Consider the equation (for a given $f \in L^2(\Omega)$):

$$-\Delta u + \beta(u) = f \text{ on } \Omega, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial \Omega.$$  

**Theorem 4** A necessary condition for the existence of a
solution of (9) is that \( \frac{1}{|\Omega|} \int_{\Omega} f(x)dx \in R(\beta) \). A sufficient condition is that \( \frac{1}{|\Omega|} \int_{\Omega} f(x)dx \in \text{Int } R(\beta) \).

**Proof** The necessary condition is clear by integrating (9) on \( \Omega \). In order to prove the sufficient condition we apply Theorem 1 in \( H = L^2(\Omega) \) with

\[
A = -\Delta, \quad D(A) = \left\{ u \in H^2(\Omega) ; \quad \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega \right\}
\]

\[
B = \beta, \quad D(B) = \left\{ u \in L^2(\Omega) ; \quad \beta(u) \in L^2(\Omega) \right\}.
\]

Both \( A \) and \( B \) are subdifferentials of convex functions; also \( A + B \) is maximal monotone. It is well known that \( R(A) = \left\{ f \in L^2(\Omega) ; \int_{\Omega} f(x)dx = 0 \right\} \). Finally if \( \frac{1}{|\Omega|} \int_{\Omega} f(x)dx \in \text{Int } R(\beta) \), then \( f \in \text{Int}[R(A) + R(B)] \). Indeed for \( g \in L^2(\Omega) \) we have

\[
g = (g - \frac{1}{|\Omega|} \int_{\Omega} g(x)dx) + \frac{1}{|\Omega|} \int_{\Omega} g(x)dx.
\]

And so it is clear that \( g \in R(A) + R(B) \) as soon as

\[
\left| \frac{1}{|\Omega|} \int_{\Omega} g(x)dx - \frac{1}{|\Omega|} \int_{\Omega} f(x)dx \right| \leq |\Omega|^{-\frac{1}{2}} \| f - g \|_{L^2} \]

is small enough.

**Remark** Theorem 4 is related to a number of results of Schatzman [22], Hess [13], Landesman - Lazer [17], Nirenberg [19] etc. The method used in the proofs of Theorems 1 - 3 can be easily extended to include most results known about "semi coercive" problems.

Let \( \mathcal{H} \) be a Hilbert space and let \( \varphi \) be a convex function on \( \mathcal{H} \). Given \( f \in L^2(0, T; \mathcal{H}) \) consider the equation
\[ \frac{du}{dt} + A \varphi(u) \in f \text{ on } (0, T), \quad u(0) = u(T). \]

**Theorem 5** A necessary condition for the existence of a solution of (10) is that \( \frac{1}{T} \int_0^T f(t) dt \in R(A \varphi) \). A sufficient condition is that \( \frac{1}{T} \int_0^T f(t) dt \in \text{Int } R(A \varphi) \).

**Proof** Since \( R(A \varphi) \) is convex, the necessary condition follows from the integration of (10). For the sufficient condition we apply Theorem 3 in \( H = L^2(0, T; \mathcal{H}) \) with \( A = A \varphi \) i.e. \( f \in Au \) provided \( f, u \in H \) and \( f(t) \in A \varphi(u(t)) \) a.e. and with \( B = \frac{d}{dt} \), \( D(B) = \{ u \in H, \quad \frac{du}{dt} \in H \text{ and } u(0) = u(T) \} \). It is well known that \( A \) is a subdifferential of a convex function in \( H \), that \( B \) is maximal monotone and that (6) holds. The assumption
\[ \frac{1}{T} \int_0^T f(t) dt \in \text{Int } R(A \varphi) \] implies that \( f \in \text{Int}[R(A) + R(B)] \).

Indeed, note that \( R(B) = \{ f \in H; \quad \int_0^T f(t) dt = 0 \} \). For \( g \in H \) we can write
\[ g = (g - \frac{1}{T} \int_0^T g(t) dt) + \frac{1}{T} \int_0^T g(t) dt \in R(A) + R(B) \]
provided \( \| g - f \|_H \) is small enough.

**Theorem 6** Let \( H \) be a Hilbert space and let \( K \) be a maximal monotone operator in \( H \) with \( D(K) = H \). Let \( F \) be the subdifferential of a convex function on \( H \) with \( D(F) = H \). Then \( R(I + KF) = H \).

**Proof** Given \( f \in H \) we want to solve \( u + KFu = f \) i.e.
$-K^{-1}(f - u) + Fu \ni 0$. We apply Theorem 2 with $A = F$ and $Bu = -K^{-1}(f - u)$ so that $B$ is maximal monotone; it follows that $R(A + B) \ni R(A) + R(B)$. However $R(B) = -D(K) = H$ and therefore $R(A + B) = H$.

Remark Results related to Theorem 6 were obtained in [6].

§ II.1 Comparative behavior of $(I + tA)^{-1}$ and $S(t)$ near $t = 0$

1. The Hilbert space case

Suppose $H$ is a Hilbert space and let $A$ be a maximal monotone operator; let $S(t)$ be the semigroup generated by $-A$ in the sense of Kato-Komura (see e.g. [23] or [4]).

For $x \in \overline{D(A)}$ and $y \in D(A)$ we have

$$|x - S(t)x| \leq 2|x - y| + |y - S(t)y| \leq 2|x - y| + t|A^0y|.$$  

Choosing $y = J_\lambda x = (I + \lambda A)^{-1}x$ we get

$$|x - S(t)x| \leq (2 + \frac{t}{\lambda})|x - J_\lambda x|$$

and in particular, for $\lambda = t$, we obtain

$$|x - S(t)x| \leq 3|x - J_t x|.$$  

In case $A = \partial \varphi$ we can show (see [5]) that

$$|x - J_t x| \leq (1 + \frac{1}{\sqrt{2}})|x - S(t)x|$$

(the best constants are not known).

For general monotone operators an inequality of the kind (13) does not hold (consider for example in $H = \mathbb{R}^2$, $A$ = a rotation
by $\pi/2$). However one can obtain a "substitute" for (13) in the general case as follows:

**Theorem 7** Let $A$ be a general maximal monotone operator, then we have

$$|x - J_t x| \leq \frac{2}{t} \int_0^t |x - S(\tau)x| \, d\tau, \quad \forall x \in \overline{D(A)}, \quad \forall t > 0. \tag{14}$$

**Remark** It is clear that the constant 2 in (14) can not be improved. Otherwise we would have for $x \in D(A)$, $|x - J_t x| \leq \frac{C}{t} \int_0^t |A^0 x| \, d\tau$ and as $t \to 0$, $|A^0 x| \leq \frac{C}{2} |A^0 x|$ with $C < 2$.

**Proof** Clearly, it is sufficient to prove (14) for $x \in D(A)$. Let $u(t) = S(t)x$; by the monotonicity of $A$, we have for $v \in D(A)$

$$\langle Av + \frac{du}{dt}(t), v - u(t) \rangle \geq 0. \tag{15}$$

Integrating (15) on $(0, t)$ we obtain

$$\frac{1}{2} |u(t) - v|^2 - \frac{1}{2} |x - v|^2 \leq \int_0^t \langle Av, v - u(\tau) \rangle \, d\tau = t\langle Av, v - x \rangle + \int_0^t \langle Av, x - u(\tau) \rangle \, d\tau. \tag{16}$$

Thus $\frac{1}{2} |u(t) - v|^2 - \frac{1}{2} |x - v|^2 \leq t\langle Av, v - x \rangle + |Av| \int_0^t |x - u(\tau)| \, d\tau$.

Choosing $v = J_t x$ we get

$$\frac{1}{2} |u(t) - J_t x|^2 - \frac{1}{2} |x - J_t x|^2 \leq -|x - J_t x|^2 + \frac{|x - J_t x|}{t} \int_0^t |x - u(\tau)| \, d\tau,$$

and (14) follows.
Remark Combining (12) and (14) we see that $|x - J_t x|$ and $|x - S(t)x|$ have the same modulus of continuity at $t = 0$.

Also, using Hardy's inequality we can deduce that for $1 \leq \alpha > 0$ and $1 \leq p \leq \infty$

$$\left\| \frac{x - S(t)x}{t^\alpha} \right\|_{L^p_*} \leq 3 \left\| \frac{x - J_t x}{t^\alpha} \right\|_{L^p_*}$$

and

$$\left\| \frac{x - J_t x}{t^\alpha} \right\|_{L^p_*} \leq \frac{2}{1 + \alpha} \left\| \frac{x - S(t)x}{t^\alpha} \right\|_{L^p_*}$$

where $L^p_* = L^p([0, 1], H; \frac{dt}{t})$. These inequalities are useful in the study of nonlinear interpolation classes (see [3]).

In a "similar spirit" we have the following

**Theorem 8** Let $A$ be a general maximal monotone operator.

For $x \in D(A)$, $\lambda > 0$ and $t > 0$ we set

$$y_{\lambda, t} = (I + \frac{\lambda}{t}(I - S(t)))^{-1} x.$$ 

Then

$$|y_{\lambda, t} - J_\lambda x|^2 \leq |x - J_\lambda x| \frac{2}{t} \int_0^t |x - S(\tau)x| d\tau.$$ 

**Remark** Let $\omega(t) = \sup_{0 \leq \tau \leq t} |x - S(\tau)x|$. By a result of Kato [14] (see also [4] Lemma 4.2) we know that for every integer $n$

$$|y_{\lambda, t} - y_{\lambda, t/n}|^2 \leq 2 \omega(t) |y_{\lambda, t/n} - x|.$$ 

Using the fact that $y_{\lambda, s} \to J_\lambda x$ as $s \to 0$ (see e.g. [4] Proposition 4.1) we obtain as $n \to \infty$

$$|y_{\lambda, t} - J_\lambda x|^2 \leq 2 \omega(t) |J_\lambda x - x|.$$ 

Such an inequality follows also directly from (17).
\textbf{Proof} \ We apply (16) with $x$ replaced by $y_{\lambda, t}$ and $v$ by $J_{\lambda}x$. Thus

\begin{equation}
\frac{1}{2} \left| S(t)y_{\lambda, t} - J_{\lambda}x \right|^2 - \frac{1}{2} \left| y_{\lambda, t} - J_{\lambda}x \right|^2 \\
\leq \int_0^t \left( \frac{x - J_{\lambda}x}{\lambda}, J_{\lambda}x - S(\tau)y_{\lambda, t} \right) \, d\tau.
\end{equation}

However, $S(\tau)y_{\lambda, t} = (1 + \frac{\tau}{\lambda})y_{\lambda, t} - \frac{\tau}{\lambda}x$ and so

\begin{equation}
\left| S(t)y_{\lambda, t} - J_{\lambda}x \right|^2 \geq \left| y_{\lambda, t} - J_{\lambda}x \right|^2 + \frac{2\tau}{\lambda} (y_{\lambda, t} - J_{\lambda}x, y_{\lambda, t} - x).
\end{equation}

On the other hand

\begin{equation}
(x - J_{\lambda}x, J_{\lambda}x - S(\tau)y_{\lambda, t}) = -\left| x - J_{\lambda}x \right|^2 + (x - J_{\lambda}x, x - S(\tau)y_{\lambda, t}) \\
\leq -\left| x - J_{\lambda}x \right|^2 + |x - J_{\lambda}x|(|x - S(\tau)x| + |x - y_{\lambda, t}|).
\end{equation}

We deduce from (19), (20) and (21) that

\[ \frac{\tau}{\lambda} (y_{\lambda, t} - J_{\lambda}x, y_{\lambda, t} - x) \leq -\frac{\tau}{\lambda} |x - J_{\lambda}x|^2 + \frac{\tau}{\lambda} |x - J_{\lambda}x| |x - y_{\lambda, t}| \\
+ \frac{|x - J_{\lambda}x|}{\lambda} \int_0^t |x - S(\tau)x| \, d\tau. \]

Therefore

\[ |x - J_{\lambda}x|^2 + (y_{\lambda, t} - J_{\lambda}x, y_{\lambda, t} - x) \leq |x - J_{\lambda}x| |x - y_{\lambda, t}| \\
+ |x - J_{\lambda}x| \frac{\tau}{\lambda} \int_0^t |x - S(\tau)x| \, d\tau \]

i.e. $|a|^2 + (b-a, b) \leq |a| |b| + |x - J_{\lambda}x| \frac{\tau}{\lambda} \int_0^t |x - S(\tau)x| \, d\tau$

with $a = x - J_{\lambda}x$ and $b = x - y_{\lambda, t}$. Hence

\[ \frac{1}{2} |a-b|^2 = \frac{1}{2} |a|^2 + \frac{1}{2} |b|^2 - (a,b) \leq \\
- \frac{1}{2} |a|^2 - \frac{1}{2} |b|^2 + |a| |b| + |x - J_{\lambda}x| \frac{\tau}{\lambda} \int_0^t |x - S(\tau)x| \, d\tau \]

and

\[ \frac{1}{2} |a-b|^2 \leq |x - J_{\lambda}x| \frac{\tau}{\lambda} \int_0^t |x - S(\tau)x| \, d\tau. \]
II.2 The Banach space case

Let $X$ be a general Banach space and let $A$ be an $m$-accretive operator on $X$. Let $S(t)$ be the semigroup generated by $-A$ in the sense of Crandall-Liggett (see [10] or [23]). Clearly we have as in §II.1

$$
\|x - S(t)x\| \leq (2 + \frac{t}{\lambda})\|x - J_\lambda x\|.
$$

We don't know whether the exact analogue of (14) holds true. However we can prove the following

Theorem 9 For every $x \in D(A)$, $t > 0$ and $\lambda > 0$ we have

$$
\|x - J_\lambda x\| \leq (1 + \frac{\lambda}{t})\frac{2}{\lambda}\int_0^t \|x - S(\tau)x\| d\tau
$$

and in particular

$$
\|x - J_\tau x\| \leq \frac{4}{\tau}\int_0^t \|x - S(\tau)x\| d\tau.
$$

Proof As usual we denote for $x, y \in X$

$$
\tau(x, y) = \lim_{\lambda \downarrow 0} \frac{1}{\lambda} (\|x + \lambda y\| - \|x\|) = \inf_{\lambda > 0} \frac{1}{\lambda} (\|x + \lambda y\| - \|x\|).
$$

The analogue of (16) becomes now (see [10] or [2] for equivalent forms):

$$
\|S(t)x - v\| - \|v - x\| \leq \int_0^t \tau(v - S(s)x, Av) ds
$$

for every $v \in D(A)$.

However we have for every $\lambda > 0$

$$
\tau(v - S(s)x, Av) \leq \frac{1}{\lambda} (\|v - S(s)x + \lambda Av\| - \|v - S(s)x\|).
$$

If we choose in (26) $v = J_\lambda x$ we obtain
(27) \( \tau (J_{\lambda} x - S(s)x, A_{\lambda} x) \leq \frac{1}{\lambda} (\|x - S(s)x\| - \|J_{\lambda} x - S(s)x\|) \)

and by (25) we get

(28) \( \|S(t)x - J_{\lambda} x\| - \|J_{\lambda} x - x\| \leq \frac{1}{\lambda} \int_0^t (\|x - S(s)x\| - \|J_{\lambda} x - S(s)x\|) ds. \)

But \( -\|J_{\lambda} x - S(s)x\| \leq \|x - S(s)x\| - \|x - J_{\lambda} x\| \) and therefore (28) leads to

\[-\|x - S(s)x\| \leq \frac{1}{\lambda} \int_0^t \|x - S(s)x\| ds + \frac{1}{\lambda} \int_0^t (\|x - S(s)x\| ds - \frac{t}{\lambda} \|x - J_{\lambda} x\| \]

i.e.

(29) \( \|x - J_{\lambda} x\| \leq \frac{\lambda}{t} \|x - S(t)x\| + \frac{2}{\lambda} \int_0^t \|x - S(s)x\| ds. \)

Finally note that

(30) \( \|x - S(t)x\| \leq \frac{2}{\lambda} \int_0^t \|x - S(s)x\| ds; \)

indeed

\[ \|S(t)x - \frac{1}{t} \int_0^t S(s)x ds\| \leq \frac{1}{t} \int_0^t \|S(t)x - S(s)x\| ds \]

\[ \leq \frac{1}{t} \int_0^t \|S(t-s)x - x\| ds = \frac{1}{t} \int_0^t \|S(s)x - x\| ds, \]

and so

\[ \|x - S(t)x\| \leq \|x - \frac{1}{t} \int_0^t S(s)x ds\| + \frac{1}{t} \int_0^t \|S(s)x - x\| ds \leq \frac{2}{\lambda} \int_0^t \|x - S(s)x\| ds. \]

Combining (29) and (30) we obtain (23).

**Remarks:**

1) I would like to thank Prof. M. Crandall, Y. Konishi and I. Miyadera for stimulating discussions concerning Theorem 9.

After our first result was obtained \( \|x - J_{\lambda} x\| \leq \frac{2}{\lambda} \int_0^{2t} \|x - S(\tau)x\| d\tau \),

I. Miyadera showed that \( \|x - J_{\lambda} x\| \leq \frac{6}{\lambda} \int_0^t \|x - S(\tau)x\| d\tau \) and
Y. Konishi got \[ \|x - J_t x\| \leq \frac{4}{7} \int_0^t \|x - S(\tau)x\| d\tau. \]

2) Using (22) and (23) one can prove directly the following result of M. Crandall [9]:

\[ \lim_{t \downarrow 0} \sup_{\lambda} \|x - S(t)x\| = \lim_{\lambda \downarrow 0} \frac{\|x - J_\lambda x\|}{\lambda}. \]

Indeed let \( \alpha = \lim_{t \downarrow 0} \sup_{t} \|x - S(t)x\| \); and so \( \forall \varepsilon > 0 \exists \delta > 0 \) such that \( 0 < t < \delta \)

\[ \|x - S(t)\| \leq t(\alpha + \varepsilon). \]

From (23) we have for \( 0 < t < \delta \) and every \( \lambda > 0 \)

\[ \|x - J_\lambda x\| \leq (1 + \frac{\lambda}{t}) \frac{2}{t} (\alpha + \varepsilon) \int_0^t \tau d\tau = (\lambda + t)(\alpha + \varepsilon). \]

It follows that \( \|x - J_\lambda x\| \leq \lambda(\alpha + \varepsilon) \) for every \( \lambda > 0 \) and \( \varepsilon > 0 \). Next let \( \beta = \lim_{\lambda \downarrow 0} \frac{\|x - J_\lambda x\|}{\lambda} \); and so \( \forall \varepsilon > 0 \exists \delta > 0 \) such that for \( 0 < \lambda < \delta \)

\[ \|x - J_\lambda x\| \leq \lambda(\beta + \varepsilon). \]

From (22) we get for \( 0 < \lambda < \delta \) and every \( t > 0 \)

\[ \|x - S(t)x\| \leq (2 + \frac{\lambda}{\lambda}) \lambda(\beta + \varepsilon) = (t + 2\lambda)(\beta + \varepsilon). \]

Hence \( \|x - S(t)x\| \leq t\beta \) for every \( t > 0 \).

3) In general for \( x \in \overline{D(A)} \), \( \frac{\|x - S(t)x\|}{\|x - J_t x\|} \) does not necessarily converge to 1 as \( t \to 0 \).

Consider for example in \( H = \mathbb{R} \), \( Au = \frac{-1}{u} \) for \( u > 0 \) and \( Au = \phi \) for \( u \leq 0 \). In this case \( J_t 0 = \sqrt{t} \) and \( S_t 0 = \sqrt{2t} \) (slightly more complicated examples were built previously by A. Plant and L. Veron).

4) In view of the example built by Crandall-Liggett in [11]
one can not expect to extend Theorem 8 to Banach spaces (or even to $\mathbb{R}^3$ with some Banach norm) since $y_{\lambda,t}$ does not necessarily converge to a limit as $t \to 0$.

II.3 An application to the characterization of compact semi-groups.

Let $A$ be an $m$-accretive operator in a general Banach space $X$ and let $S(t)$ be the semigroup generated by $-A$.

Theorem 10. The following properties are equivalent.

(31) For every $t > 0$, $S(t)$ is compact i.e. $S(t)$ maps bounded sets of $\overline{D(A)}$ into compact sets of $X$

\begin{align*}
(32a) \quad & \text{For every } \lambda > 0, \quad (I + \lambda A)^{-1} \quad \text{is compact i.e.} \\
& \text{maps bounded sets of } X \quad \text{into compact sets of } X \\
(32b) \quad & \text{For every bounded set } B \text{ in } \overline{D(A)} \quad \text{and every } t_0 > 0 \\
& \text{the mappings } t \mapsto S(t)x \quad \text{are equicontinuous at } t = t_0 \\
& \text{as } x \in B.
\end{align*}

Remarks

1) Theorem 10 is due to A. Pazy [20] in the linear case and to Y. Konishi [15] in the nonlinear Hilbert case (his proof relies on a consequence of (18) and could not be extended to Banach spaces)

2) It is obvious that (32a) is equivalent to

(32a') \quad (I + A)^{-1} \quad \text{is compact}

and also to

(32a'') \quad \text{For every } M > 0 \text{ the set }
\\{ x \in D(A); \| x \| \leq M \text{ and } \| y \| \leq M \text{ for some } y \in Ax \} \\

is relatively compact in X.

Proof (31) \implies (32a)

Let $\lambda$ be fixed and let $x \in X$; we have for every $t \geq 0$
\[
\| J_\lambda x - S(t)J_\lambda x \| \leq t\| A_\lambda x \| = \frac{t}{\lambda} \| x - J_\lambda x \|. 
\]

Let $B$ be a bounded set in $X$; given $\varepsilon > 0$, choose $t_0$ so small that
\[
\frac{t_0}{\lambda} \| x - J_\lambda x \| < \varepsilon/2 \text{ for } x \in B.
\]

Since $J_\lambda(B)$ is bounded in $D(A)$, it follows from (31) that $S(t_0)J_\lambda(B)$ is relatively compact. Thus $S(t_0)J_\lambda(B)$ can be covered by a finite union $\bigcup_i B(x_i, \varepsilon/2)$. Hence $J_\lambda(B) \subseteq \bigcup_i B(x_i, \varepsilon)$ and consequently $J_\lambda(B)$ is precompact.

(31) \implies (32b)

Using (31) we have only to prove that the mappings $t \mapsto S(t)x$ are equicontinuous at $t = \frac{t_0}{2}$ as $x \in K$, $K$ compact
\[
( K = S(\frac{t_0}{2})B). \text{ This follows directly from the fact that for each fixed } x, \ t \mapsto S(t)x \text{ is continuous and that } x \mapsto S(t)x \text{ is a contraction.}
\]

(32a) + (32b) \implies (31)

Fix a $t_0 > 0$ and let $B$ be a bounded set in $D(A)$. By (32b), for every $\varepsilon > 0$ there exists $\delta > 0$ such that
\[
\| S(t)x - S(t_0)x \| < \varepsilon \text{ for } |t - t_0| \leq \delta \text{ and } x \in B.
\]

We deduce from (23) that for $x \in B$ and $\lambda > 0$,
\[
\| S(t_0)x - J_\lambda S(t_0)x \| \leq (1 + \frac{\lambda}{\varepsilon}) \frac{2}{t} \int_0^t \| S(t_0)x - S(t + t_0)x \| d \tau
\]
\[ \leq (1 + \frac{\lambda}{t}) 2\varepsilon \quad \text{for every } 0 < t \leq \delta. \]

In particular for \( 0 < \lambda \leq \delta \) and \( x \in B \) we have

\[ \|S(t_0)x - J_{\lambda}S(t_0)x\| \leq 4\varepsilon. \]

Since \( J_{\delta}S(t_0)B \) is relatively compact it can be covered by a
finite union \( \bigcup_i B(x_i, \varepsilon) \). Hence \( S(t_0)B \) can also be covered
by a finite union of balls of radius \( 5\varepsilon \) and thus \( S(t_0)B \) is
precompact.

**Remark** Suppose \( H \) is a Hilbert space, \( \varphi \) is a convex func-
tion on \( H \) and let \( A = \partial \varphi \). In this case (31) is equivalent to
(32a) since (32b) is satisfied automatically. Indeed we have

\[ |S(t)x - S(t_0)x| = |S(t - \frac{t_0}{2})y - S(\frac{t_0}{2})y| \leq |t - t_0| |A^o y| \]

where \( y = S(\frac{t_0}{2})x \). On the other hand (see e.g. [4] Théorème
3.2) we know that

\[ |A^o S(\frac{t_0}{2})x| \leq |A^o v| + \frac{2}{t_0} |x - v| \quad \text{for every } v \in D(A). \]

Therefore the mappings \( t \mapsto S(t)x \) are equicontinuous at \( t = t_0 \)
as \( x \) remains bounded.

In this case property (32a) is also equivalent to

(32a'') For every \( M \) the set

\[ \{ x \in D(\varphi); \ |x| \leq M \text{ and } \varphi(x) \leq M \} \]
is relatively compact in \( H \).

Indeed (32a'') \( \implies \) (32a''):

Let \( E = \{ x \in D(A); \ |x| \leq M \text{ and } |A^o x| \leq M \}; \) for a fixed \( v_0 \in D(\varphi) \) we have
\( \varphi(v_0) - \varphi(x) \geq (A^o x, v_0 - x) \)

and so \( \varphi(x) \leq \varphi(v_0) + M(|v_0| + M) = M' \) when \( x \in E \).

Conversely (32a) \( \implies (32a'''') \):

Let
\[
F = \{ x \in D(\varphi); \quad |x| \leq M \quad \text{and} \quad \varphi(x) \leq M \};
\]

for \( x \in F \) we have
\[
\varphi(x) - \varphi(J_\lambda x) \geq (A_\lambda x, x - J_\lambda x) = \frac{1}{\lambda} |x - J_\lambda x|^2.
\]

Therefore, since \( \varphi \) is bounded below by some affine function, we get for \( x \in F \),
\[
\frac{1}{\lambda} |x - J_\lambda x|^2 \leq M + C_1 |J_\lambda x| + C_2 \leq M + C_1 |x - J_\lambda x| + C_1 M + C_2.
\]

Thus
\[
|x - J_\lambda x| \leq \sqrt{\frac{\lambda(C_3 \lambda + C_4)}{\lambda_0 (C_3 \lambda_0 + C_4)}} \quad \text{for} \quad x \in F.
\]

Given \( \varepsilon > 0 \) we choose \( \lambda_0 > 0 \) so small that \( \sqrt{\frac{\lambda_0 (C_3 \lambda_0 + C_4)}{\lambda (C_3 \lambda + C_4)}} < \varepsilon \). Since \( J_\lambda (F) \) is relatively compact, it can be covered by a finite union \( \bigcup_i B(x_i, \varepsilon) \) and then \( F \subset \bigcup_i B(x_i, 2\varepsilon) \).

§ III. A convergence theorem for nonlinear semigroups

Let \( H \) be a Hilbert space; let \( \{A_n\}_{n \geq 1} \) and \( A \) be maximal monotone operators. Let \( \{S_n(t)\}_{n \geq 1} \) and \( S(t) \) be the corresponding semigroups.

Our next result is a nonlinear version of the Theorem of Trotter-Kato-Neveu. A number of related results have been obtained previously by Miyadera-Oharu [18], Brezis-Pazy [8], Benilan [1], Goldstein [12], Kurtz [16] etc...

**Theorem 11.** The following properties are equivalent.
\( \forall x \in \mathcal{D}(A), \ \forall \lambda > 0 \ (I + \lambda A_n)^{-1} x \rightarrow (I + \lambda A)^{-1} x \) 

\( \forall x \in \mathcal{D}(A) \ \exists x_n \in \mathcal{D}(A_n) \) such that \( x_n \rightarrow x \) and 
\( A_n^o x_n \rightarrow A^o x \) 

\( \forall x \in \mathcal{D}(A) \ \exists x_n \in \mathcal{D}(A_n) \) such that \( x_n \rightarrow x \) and \( \forall t \geq 0 \) 
\( S_n(t)x_n \rightarrow S(t)x \).

In addition the convergence in (33) (resp. (35)) is uniform for bounded \( \lambda \) (resp. bounded \( t \)).

The proof of Theorem 11 is divided into four parts

Part A \( (33) \Rightarrow (34) \)

Part B \( (34) \Rightarrow (33) \)

Part C \( (33) \Rightarrow (35) \)

Part D \( (35) \Rightarrow (33) \).

Part A \( (33) \Rightarrow (34) \)

Let \( x \in \mathcal{D}(A) \); given \( \varepsilon > 0 \) there is a \( \lambda > 0 \) such that
\[
|x - (I + \lambda A)^{-1} x| < \varepsilon / 2 \\
|A_n^o x - A^o x| < \varepsilon / 2.
\]

Next, by (33) there is an integer \( N \) such that for \( n \geq N \)
\[
| (I + \lambda A_n)^{-1} x - (I + \lambda A)^{-1} x | < \varepsilon / 2 \\
| (A_n^o x - A^o x) | < \varepsilon / 2.
\]

Combining these estimates we see that given \( \varepsilon > 0 \) there is an integer \( N(\varepsilon) \) and sequences \( u_n(\varepsilon) = (I + \lambda A_n)^{-1} x \) and
\[ f_n(\varepsilon) = (A_n^o x) \] such that \( [u_n(\varepsilon), f_n(\varepsilon)] \in \mathcal{G}(A_n) \) and for \( n \geq N(\varepsilon) \), \( |u_n(\varepsilon) - x| < \varepsilon \), \( |f_n(\varepsilon) - A^o x| < \varepsilon \). Let \( N_k = N(\frac{1}{k}) \); we can always assume that \( N_k \) is increasing to \( \infty \).
We define the sequences \( x_n \) and \( g_n \) by \( x_n = u_n(\frac{1}{k}) \) and \( g_n = f_n(\frac{1}{k}) \) for \( N_k \leq n < N_{k+1} \). Therefore \( [x_n, g_n] \in G(A_n) \) and for \( N_k \leq n < N_{k+1} \) we have \( |x_n - x| < \frac{1}{k} \) and \( |g_n - A^o x| < \frac{1}{k} \).

Consequently \( x_n \rightarrow x \) and \( g_n \rightarrow A^o x \); we are going to prove now that \( A^o_{n_j} x_{n_j} \rightarrow A^o x \). Indeed \( |A^o_{n_j} x_{n_j}| \leq |g_n| \) and thus for a subsequence we get \( A^o_{n_j} x_{n_j} \rightarrow h \). Let \( v \in D(A) \); by the monotonicity of \( A_n \) we have

\[
((A_n)\lambda v - A^o_{n_j} x_{n_j}, (1+\lambda A_n)^{-1}v - x_n) \geq 0.
\]

At the limit as \( n_j \rightarrow \infty \) we obtain

\[
(A_{\lambda v}^o h, (I+\lambda A)^{-1}v - x) \geq 0.
\]

Next we pass to the limit as \( \lambda \rightarrow 0 \):

\[
(A^o v - h, v - x) \geq 0 \quad \forall v \in D(A).
\]

Therefore \( h \in Ax \) (see e.g. [4] Proposition 2.7). Since on the other hand \( |h| \leq |A^o x| \) we have \( h = A^o x \). By the uniqueness of the limit, and the fact that \( \limsup |A^o_{n_j} x_{n_j}| \leq |A^o x| \) we conclude that \( A^o_{n_j} x_{n_j} \rightarrow A^o x \).

**Part B** \( (34) \Rightarrow (33) \)

Without loss of generality we may assume that \( \lambda = 1 \). Let \( x \in D(A) \) and let \( u_n = (I+A_n)^{-1}x \). Given \( y \in D(A) \), let \( y_n \in D(A_n) \) be the sequence given by (34) so that \( y_n = (I+A_n)^{-1}(y_n + A^o_n y_n) \). Therefore \( |u_n - y_n| \leq |x - y_n - A^o y_n| \) and thus \( u_n \) is bounded. For a subsequence \( u_{n_j} \rightarrow u \); by the monotonicity of \( A_n \) we have

\[
(x - u_n - A^o y_n, u_n - y_n) \geq 0.
\]

Passing to the limit in (36) we obtain
(37) \[(x - u - A\ast y, u - y) \geq 0 \quad \forall y \in D(A).\]

In (37) we choose \( y = (I + \lambda A)^{-1}u \) and so
\[(x - u, u - J\lambda u) \geq \lambda (A\ast J\lambda u, A\lambda u) \geq 0.\]

As \( \lambda \to 0 \) we see that
\[(x - u, u - \text{Proj}_{D(A)} u) \geq 0.\]

On the other hand since \( x \in D(A) \) we have
\[(\text{Proj}_{D(A)} u - x, u - \text{Proj}_{D(A)} u) \geq 0\]
and consequently \( u = \text{Proj}_{D(A)} u \) i.e. \( u \in D(A) \). Going back to
(37) we deduce now from [4] Proposition 2.7 that \( x - u \in Au \) i.e.
\( u = (I+A)^{-1}x \). By the uniqueness of the limit we have in fact
\( u_n \to (I+A)^{-1}x \).

It follows from (36) that for every \( y \in D(A) \)
\[\limsup |u_n|^2 \leq (x, u-y) + (u, y) + (A\ast y, y-u).\]

In particular if we take \( y = u \) we get
\[\limsup |u_n|^2 \leq |u|^2 \quad \text{and thus} \quad u_n \to u.\]

The convergence in (33) is uniform in \( \lambda \) as \( \lambda \) remains bounded:

Without loss of generality we may assume that \( x \in D(A) \) and let
\( x_n \in D(A_n) \) with \( x_n \to x \) and \( A_n x_n \to A\ast x \). We have
\[|((I+\lambda A_n)^{-1}x_n - (I+\mu A_n)^{-1}x_n| \leq |\lambda - \mu| |A_n x_n|.\]

Therefore the functions \( f_n(\lambda) = (I+\lambda A_n)^{-1}x_n \) are uniformly
lipschitz continuous on \([0, +\infty)\). Since they converge simply to
\((I+\lambda A)^{-1}x \) as \( n \to +\infty \), we conclude that the convergence is
uniform in \( \lambda \) as \( \lambda \) remains in a bounded interval.

Part C \((33) \Rightarrow (35)\)

Without loss of generality we may assume that \( x \in D(A) \). By (34)
we have a sequence \( x_n \in D(A_n) \) such that \( x_n \to x \) and \( A_n^\circ x_n \to A^\circ x \). We are going to prove that \( S_n(t)x_n \to S(t)x \). It is known (see e.g. [4] Corollaire 4.4) that

\[
|S_n(t)x_n - (I + \frac{t}{k} A_n)^{-k} x_n| \leq \frac{2t}{\sqrt{k}} |A_n^\circ x_n| \leq \frac{2tM}{\sqrt{k}}
\]

and

\[
|S(t)x - (I + \frac{t}{k} A)^{-k} x| \leq \frac{2t}{\sqrt{k}} |A^\circ x| \leq \frac{2tM}{\sqrt{k}}
\]

where \( M = \operatorname{Sup}_n |A_n^\circ x_n| \). Given \( \varepsilon > 0 \), we first fix \( k \) large enough so that \( \frac{2tM}{\sqrt{k}} < \varepsilon \). Next observe, by induction, that for every integer \( N \) and for every sequence \( u_n \to u \) with \( u \in \overline{D(A)} \), then \( (I + \lambda A_n)^{-N} u_n \to (I + \lambda A_n)^{-N} u \), as \( n \to +\infty \). Thus

\[
|S_n(t)x_n - S(t)x| \leq 2\varepsilon + |(I + \frac{t}{k} A_n)^{-k} x_n - (I + \frac{t}{k} A)^{-k} x| \leq 3\varepsilon
\]

provided \( n \) is large enough.

Finally (35) holds true uniformly in \( t \) as \( t \) remains bounded since (33) holds true uniformly in \( \lambda \) as \( \lambda \) remains bounded.

**Part D** \((35) \Rightarrow (33)\)

The proof relies on the following

**Lemma 1** Suppose (35) holds. Let \( f_n \in \overline{D(A_n)} \) be such that \( f_n \to f \) and \( f \in \overline{D(A)} \). Then \( \forall \lambda > 0, \forall t > 0 \)

\[
u_n = (I + \frac{\lambda}{t}(I - S_n(t)))^{-1} f_n \to u = (I + \frac{\lambda}{t}(I - S(t)))^{-1} f.
\]

**Proof of Lemma 1** By (35) there exists a sequence \( x_n \in \overline{D(A_n)} \) such that \( x_n \to u \) and \( S_n(t)x_n \to S(t)u \). Writing the monotonicity of \( I - S_n(t) \) we have

\[
((u_n - S_n(t)u_n) - (x_n - S_n(t)x_n), u_n - x_n) \geq 0
\]
and therefore

\[
\left( \frac{u - u_n}{\lambda} + \delta_n, u_n - x_n \right) \geq 0
\]

where

\[
\delta_n = \frac{f_n - f}{\lambda} + \frac{u - x_n}{t} + \frac{S_n(t)x_n - S(t)u}{t}
\]

and \( \delta_n \to 0 \).

Hence

\[
\frac{1}{\lambda} |u_n - u|^2 \leq |\delta_n||u_n - u| + |\delta_n||u - x_n| + \frac{1}{\lambda} |u - u_n||u - x_n|
\]

and consequently \( u_n \to u \) as \( n \to \infty \).

**Lemma 2.** Let \( x_n \in D(A_n) \) be a sequence such that \( x_n \to x \) with \( x \in D(A) \) and \( S_n(t)x_n \to S(t)x \) for every \( t \geq 0 \). Then for every \( T \) there exists a constant \( K \) such that \( |(I + \lambda A_n)^{-1} x_n| \leq K \) and \( |S_n(t)x_n| \leq K \) for every \( 0 < \lambda < T \), for every \( 0 < t < T \) and every \( n \).

**Proof of Lemma 2.** Let \( M = \sup_{0 \leq t \leq 1} |S(t)x| \) and let

\[
E_n = \{ t \in [0, 1]; |S_p(t)x_p| \leq M + 1 \text{ for every } p \geq n \}
\]

Clearly \( E_n \) is closed and \( \bigcup_{n=1}^{\infty} E_n = [0, 1] \); it follows from Baire's theorem that \( \text{Int} E_N \neq \emptyset \) for some \( N \). Let \( [t_0, t_0 + h] \subset E_N \) so that

\[
|S_p(t)x_p| \leq M + 1 \text{ for } n \geq N \text{ and } t_0 \leq t \leq t_0 + 1.
\]

It follows from Theorem 9 that

\[
|S_n(t_0)x_n - (I + \lambda A_n)^{-1} S_n(t_0)x_n| \leq (1 + \frac{\lambda}{h}) \frac{2}{h} \int_0^h |S_n(t_0)x_n - S_n(t_0 + \tau)x_n| d\tau.
\]

Choosing \( n \geq N \) we get

\[
|(I + \lambda A_n)^{-1} x_n| \leq |x_n - S_n(t_0)x_n| + |S_n(t_0)x_n| + \frac{2}{h} (1 + \frac{\lambda}{h}) 2(M + 1) h
\]

- 22 -
\[ \leq |x_n| + 2(M+1) + 4(1+\frac{\lambda}{n})(M+1). \]

We conclude by using the fact that
\[ |x_n - S_n(t)x_n| \leq 3|x_n - (I + tA_n)^{-1}x_n|. \]

Proof of (35) \( \Rightarrow \) (33). In what follows \( \lambda \) is fixed. Using Theorem 8 we get
\[
| (I + \frac{\lambda}{t}(I - S_n(t)))^{-1}x_n - (I + \lambda A_n)^{-1}x_n |^2
\leq |x_n - (I + \lambda A_n)^{-1}x_n| \cdot \frac{2}{t} \int_0^t |x_n - S_n(\tau)x_n| \, d\tau
\]
and
\[
| (I + \frac{\lambda}{t}(I - S(t)))^{-1}x - (I + \lambda A)^{-1}x |^2
\leq |x - (I + \lambda A)^{-1}x| \cdot \frac{2}{t} \int_0^t |x - S(\tau)x| \, d\tau.
\]
Let \( P = 2|x - (I + \lambda A)^{-1}x| + 2 \sup_n |x_n - (I + \lambda A_n)^{-1}x_n| < \infty \) (by Lemma 2). We have
\[
\frac{1}{t} \int_0^t |x_n - S_n(\tau)x_n| \, d\tau \leq |x_n - x| + \frac{1}{t} \int_0^t |x - S(\tau)x| \, d\tau + \frac{1}{t} \int_0^t |S(\tau)x - S_n(\tau)x_n| \, d\tau
\]
and so
\[
| (I + \lambda A_n)^{-1}x_n - (I + \lambda A)^{-1}x | \leq |(I + \frac{\lambda}{t}(I - S_n(t)))^{-1}x_n - (I + \frac{\lambda}{t}(I - S(t)))^{-1}x | + \sqrt{P|x_n - x| + 2 \frac{P}{t} \int_0^t |x - S(\tau)x| \, d\tau + \frac{P}{t} \int_0^t |S(\tau)x - S_n(\tau)x_n| \, d\tau}
\]
\[ = X_1 + X_2 + X_3 + X_4. \]

Given \( \epsilon > 0 \) we choose first \( t > 0 \) small enough so that \( X_4 < \epsilon \)
and then we choose \( n \) large enough so that \( X_1 + X_3 + X_4 < \epsilon \) (we use here Lemma 1 to make \( X_1 \) small and Lemma 2 combined with Lebesgue's Theorem to make \( X_2 \) small).
References


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