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TAIL PROBABILITIES OF SOME CONTINUOUS FUNCTIONALS
OF GAUSSIAN PROCESSES

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1. Let \( X = \{X(t), 0 \leq t \leq 1\} \) be a path continuous Gaussian process with mean zero, and let \( T \) be a real continuous functional on \( C[0,1] \) such that \( T(cx) = c^p T(x) \) with \( p > 0 \) for any positive constant \( c \). In this note the following asymptotic estimate for the tail probabilities of \( T(X) \) is obtained:

\[
\lim_{\alpha \to \infty} (1/\alpha^{2/p}) \cdot \log P\{ T(X) > \alpha \} = -(1/2) b^2,
\]

where \( b^2 \) is a constant determined as the solution of certain extremal problem. For example, it is shown that if \( X \) is Brownian motion, then

\[
\lim_{\alpha \to \infty} (1/\alpha^{2/p}) \cdot \log P\{ \int_0^1 |X(t)|^p dt > \alpha \} = -(1/2) (C(p))^{-2/p},
\]

where \( p \geq 1 \) and

\[
c(p) = 2(p+2)(p/2)^{-1/2} / \int_0^1 (1-t^p)^{-1/2} dt P_{F_p}^{1/2},
\]

and also, if \( X \) is Brownian bridge, then the same formula holds with \( c(p) \) replaced by \( 2^{-p} c(p) \).

In his thesis [3] and also in [4], N. A. Marlow obtained a similar
asymptotic formula for tail probabilities of uniformly Hölder continuous, asymptotically homogeneous functionals \( F \) of path continuous Gaussian processes. His method of proof is to first estimate \( \log P\{ F(X) > \alpha \} \) in the finite-dimensional case by a Laplace asymptotic formula, and then to pass to the limit to obtain the function space version. Note also that H. P. McKean [5] obtained a similar asymptotic estimate for tail probabilities of multiple Wiener integrals.

Our method is different from Marlow's and is based on the following Fredlin-Wentzell type estimates for Gaussian measures given in [7] and [2]. Let \( C = C[0,1] \) be the space of all continuous functions on \([0,1]\) with the supremum norm \( ||\cdot||_\infty \), and let \( \mathcal{A} \) be the \( \sigma \)-field of Borel subsets of \( C \).

Let \( \mu \) be a Gaussian measure on \((C, \mathcal{A})\) with mean zero and covariance function \( R(s,t) \), i.e., \( \int_C x(t)u(dx) = 0 \), for \( 0 \leq t \leq 1 \), and \( R(s,t) = \int_C x(s)x(t)u(dx) \), for \( 0 \leq s, t \leq 1 \), where \( x \in C \). Let \( H = H(R) \) be the reproducing kernel Hilbert space (RKHS) with reproducing kernel (r.k.) \( R \), whose norm is denoted by \( ||\cdot||_H \). Note that \( H \subset C \), since \( R \) is continuous.

**Theorem 1.** Let \( \phi \in H \). Then, for any \( \delta, h > 0 \), there is a number \( a_0 = a_0(\delta, h, ||\phi||_H) \) such that

\[
\mu\{ x \mid \| (x/\alpha) - \phi \|_\infty < \delta \} \geq \mu\{ x \mid \| x - \alpha \phi \|_\infty < \delta \} \\
\geq \exp\left[-(\alpha^2/2)(||\phi||_H^2 + h)\right]
\]

for all \( \alpha \geq a_0 \).

**Theorem 2.** Let \( K_r = \{ \phi \in H \mid ||\phi||_H \leq r \} \) and let \( d(x, K_r) \) be the distance from \( x \in C \) to \( K_r \) in the sup norm \( ||\cdot||_\infty \). Then, for any \( \delta, h > 0 \),
there is a number $a_0 = a_0(\delta, h, r)$ such that

$$\mu\{ x \mid d(x/a, K_r) > \delta \} \leq \exp[-(a^2/2)(r^2 - h)]$$

for all $a \geq a_0$.

For the proofs see [7] or [2]. From Theorems 1 and 2 we obtain the following

**Theorem 3.** Let $T$ be a real continuous functional on $C$ such that $T(cx) = c^p T(x)$ with $p > 0$ for any positive constant $c$ and $T(\phi) > 0$ for some $\phi \in H$. Then

$$\lim_{a \to \infty} (1/a^{2/p}) \cdot \log \mu\{ x \mid T(x) > a \} = -(1/2)b^2,$$

where $b^2 = \inf \{ ||\phi||_H^2 \mid T(\phi) > 1 \} = \sup \{ r^2 \mid \sup\{T(\phi)\mid \phi \in K_r\} < 1 \}$.

**Proof.** Let $D = \{ x \mid T(x) > 1 \}$. $D$ is open and its closure $\overline{D} = \{ x \mid T(x) \geq 1 \}$. For any $\phi \in H \cap D$, there is a $\delta > 0$ such that $||x - \phi||_\infty < \delta$ implies $x \in D$. Hence, using Theorem 1, we obtain

$$\mu\{ x \mid T(x) > a \} = \mu\{ x \mid T(x/a^{1/p}) > 1 \}$$

$$\geq \mu\{ x \mid ||(x/a^{1/p}) - \phi||_\infty < \delta \}$$

$$\geq \exp[-(a^{2/p}/2)(||\phi||_H^2 + h)]$$

for any $h > 0$, if $a$ is sufficiently large. Thus, for any $\phi \in H \cap D$,

$$\liminf_{a \to \infty} (1/a^{2/p}) \cdot \log \mu\{ x \mid T(x) > a \} \geq -(1/2)||\phi||_H^2.$$
and hence,

\[
\liminf_{\alpha \to \infty} (1/\alpha^{2/p}) \cdot \log \mu( x \mid T(x) > \alpha ) \\
\geq -(1/2) \cdot \inf \{ \| \phi \|_{H}^2 \mid T(\phi) > 1 \}.
\]

Since \( K_\alpha \) is compact in \( C \) (see, e.g. [6]) and \( T \) is continuous, there is a number \( r > 0 \) such that \( \sup\{ T(\phi) \mid \phi \in K_\alpha \} < 1 \), and for any such a number \( r \), there is a \( \delta > 0 \) such that \( d(K_\alpha, D) > \delta \), where \( d(K_\alpha, D) \) is the distance between \( K_\alpha \) and \( D \). If \( T(x) > \alpha \), then \( x/\alpha^{1/p} \in D \), and by Theorem 2,

\[
\mu( x \mid T(x) > \alpha ) \leq \mu( x \mid d(x/\alpha^{1/p}, K_\alpha) > \delta ) \\
\leq \exp[-(a^{2/p}/2)(r^2 - h)]
\]

for any \( h > 0 \), if \( a \) is sufficiently large. Therefore,

\[
\limsup_{\alpha \to \infty} (1/\alpha^{2/p}) \cdot \log \mu( x \mid T(x) > \alpha ) \leq -(1/2)r^2,
\]

and hence

\[
\limsup_{\alpha \to \infty} (1/\alpha^{2/p}) \cdot \log \mu( x \mid T(x) > \alpha ) \\
\leq -(1/2) \cdot \sup\{ r^2 \mid \sup\{ T(\phi) \mid \phi \in K_\alpha \} > 1 \}.
\]

It is easy to see that \( \inf\{ \| \phi \|_{H}^2 \mid T(\phi) > 1 \} = \inf\{ \| \phi \|_{H}^2 \mid T(\phi) \geq 1 \} \) (in fact, \( = \inf\{ \| \phi \|_{H}^2 \mid T(\phi) = 1 \} \)), and if \( r^2 < \inf\{ \| \phi \|_{H}^2 \mid T(\phi) \geq 1 \} \), then \( \sup\{ T(\phi) \mid \phi \in K_\alpha \} < 1 \), and so \( \sup\{ r^2 \mid \sup\{ T(\phi) \mid \phi \in K_\alpha \} < 1 \} = \inf\{ \| \phi \|_{H}^2 \mid T(\phi) > 1 \} \). This completes the proof.

Remark. Note that \( \sup\{ T(\phi) \mid \phi \in K_D \} = 1 \). Since \( \sup\{ T(\phi) \mid \phi \in K_D \} = B^p \cdot \sup\{ T(\phi) \mid \phi \in K_1 \} \), we have \( B^2 = ( \sup\{ T(\phi) \mid \phi \in K_1 \} )^{-2/p} \).
2. In what follows we consider several examples for which the values of \( b^2 \) can be explicitly given by evaluating \( \sup \{ T(\phi) \mid \phi \in K_1 \} \).

(i) Let \( X \) be a path continuous Gaussian process with mean zero and covariance function \( R(s, t) \). Then

\[
\lim_{a \to \infty} \frac{1}{a} \cdot \log P \left( \int_0^1 x^2(t) \, dt > a \right) = -1/(2\lambda_1),
\]

where \( \lambda_1 \) is the largest eigenvalue of the covariance operator \( R \) with kernel \( R(s, t) \) on \( L^2[0,1] \).

This is a known result, and so we just indicate briefly how it can be derived from Theorem 3. In this case \( T(x) = \int_0^1 x^2(t) \, dt = ||x||^2_2 \) and \( p = 2 \).

Let \( \{ \lambda_1 \} \) and \( \{ \psi_1 \} \) be the eigenvalues and the corresponding normalized eigenfunctions of \( R \). Then \( \{ \phi_1 = \lambda_1^{1/2} \psi_1 \} \) is a complete orthonormal system in \( H(R) \). It can be shown that \( ||\phi||^2_2 \leq \lambda_1 ||\phi||^2_H \) for any \( \phi \in H(R) \). Hence

\[
\sup \{ T(\phi) \mid \phi \in K_1 \} \leq \lambda_1.
\]

Since \( ||\phi_1||^2_2 = \lambda_1 \), we have \( \sup \{ T(\phi) \mid \phi \in K_1 \} = \lambda_1 \), and hence the result.

(ii) Let \( \mu \) be the Wiener measure and let \( T(x) = \int_0^1 |x(t)|^p \, dt \), \( p \geq 1 \).

The RKHS \( H(R) \) associated with the Wiener measure is the space of all absolutely continuous functions \( \phi \) on \([0,1]\) such that \( \phi(0) = 0 \) and \( d\phi/dt \in L^2[0,1] \), and \( (\phi, \psi)_H = \int_0^1 (d\phi/dt)(d\psi/dt) \, dt \), where \( (\cdot, \cdot)_H \) denotes the inner product of \( H(R) \). V. Strassen ([8], p.220) proved that \( \sup \{ T(\phi) \mid \phi \in K_1 \} = c(p) \), where

\[
c(p) = 2(p+2)^{(p/2)-1}/(\int_0^1 (1-t)^p \, dt)^{p/2}.
\]

We thus obtain the result for Brownian motion stated at the beginning of this note. In particular, \( C(1) = 3^{-1/2} \) and \( c(2) = 4/\pi^2 \). The case \( p = 1 \) has been previously obtained by Marlow [3] by a different method, and the
case $p = 2$ is of course a particular case of (i). If $p$ in an integer, then the same formula holds for $T(x) = \int_0^1 (x(t))^p \, dt$.

(iii) Let $\mu$ be the Wiener measure and let
\[ T(x) = \int_0^1 |x(t)|^2 \, dt / \int_0^1 |x(t)| \, dt. \]

Then $\sup\{ T(\phi) | \phi \in \mathcal{X}_1 \} = 2q$, where $0 < q < 1$ is the largest solution of
\[ (1-q)^{1/2} \sin((1-q)^{1/2}/q) + \cos((1-q)^{1/2}/q) = 0 \]
(see [8], p.222). Hence, if $X$ is Brownian motion, then
\[ \lim_{a \to \infty} (1/a^2) \cdot \log P\{ \int_0^1 |X(t)|^2 \, dt / \int_0^1 |X(t)| \, dt > a \} = -1/(8q)^2. \]

(iv) Let $X$ be Brownian bridge. We shall show that
\[ \lim_{a \to \infty} (1/a^{2/p}) \cdot \log P\{ \int_0^1 |X(t)|^p \, dt > a \} = -2 (c(p))^{-2/p}, \quad p \geq 1, \]
where $c(p)$ is the same as in (ii).

The covariance function of Brownian bridge is
\[ R(s,t) = \begin{cases} s(1-t), & \text{for } s \leq t, \\ t(1-s), & \text{for } s \geq t, \end{cases} \]
\[ = \int_0^1 \varrho(u,s) \varrho(u,t) \, du, \]
where
\[ \varrho(u,t) = \begin{cases} l-t, & \text{for } u \leq t, \\ -t, & \text{for } u > t. \end{cases} \]

Hence the RKHS $H(R)$ with r.k. $R$ is isometrically isomorphic to the closed subspace $M$ of $L^2[0,1]$, spanned by $\{ \varrho(u,t), \quad 0 \leq t \leq 1 \}$, and any function $\phi$ in $H(R)$ has a representation $\phi(t) = \int_0^1 m(u) \varrho(u,t) \, du$ with $m \in M$. Note that
\[ \mathbb{M}^l, \text{i.e., } \int_0^1 m(u)\,du = 0 \text{ for all } m \in \mathbb{M}, \text{ since } \int_0^1 q(u,t)\,du = 0 \text{ for all } t \in [0,1] \text{ if } \int_0^1 n(u)\,du = 0 \text{ and } \int_0^1 n(u)q(u,t)\,du = 0 \text{ for all } t \in [0,1], \text{ then } n = 0. \text{ Hence } \phi(t) = \int_0^1 m(u)q(u,t)\,du = \int_0^t m(u)\,du, \text{ which shows that } \phi \text{ is absolutely continuous. Therefore, } H(\mathbb{R}) \text{ is the space of all absolutely continuous functions } \phi \text{ on } [0,1] \text{ such that } \phi(0) = \phi(1) = 0 \text{ and } \phi' = d\phi/dt \in L^2[0,1], \text{ and } (\phi, \psi)_H = \int_0^1 \phi'\psi'\,dt. \]

As in Strassen's proof [8] for Brownian motion case, we shall evaluate
\[ \sup\{T(\phi) \mid \phi \in K_1\} = \sup\{ \int_0^1 |\phi(t)|^2\,dt \mid \phi(0) = \phi(1) = 0 \text{ and } \int_0^1 \phi'\,dt \leq 1 \} \]
by classical methods of the calculus of variations. Since \( K_1 \) is compact and \( T \) is continuous, there is a maximizing point \( \phi \) with \( ||\phi||_H^2 = \int_0^1 \phi'^2\,dt = 1 \).

We may assume \( \phi \geq 0 \), and \( \phi \) satisfies the equation
\[ \int_0^1 \phi P^{1-1} \psi\,dt = 2\lambda \int_0^1 \phi'^\psi\,dt, \text{ for any } \psi \in H(\mathbb{R}), \]
where \( \lambda > 0 \) is a Lagrange multiplier. Integrating by parts the left-hand side and noting that \( \psi' \perp 1 \), we obtain
\[ \int_0^1 (\int_L \phi P^{1-1}(s)\,ds - \int_0^1 [s\phi P^{1-1}(u)du]\,ds)\psi'(s)\,ds = 2\lambda \int_0^1 \phi'^\psi\,dt \]
for all \( \psi' \in M \). Therefore,
\[ (1) \int_L \phi P^{1-1}(s)\,ds - \int_0^1 [s\phi P^{1-1}(u)du]\,ds = 2\lambda \phi'(t), \text{ for } 0 \leq t \leq 1. \]

Since \( \phi \geq 0 \) and \( \lambda > 0 \), (1) shows that \( \phi'(0) \geq 0 \), \( \phi'(1) \leq 0 \) and \( \phi' \) is differentiable and monotone decreasing. Hence there is a point \( t_0 \) such that \( \phi'(t_0) = 0 \) and \( \phi'(t) \geq 0 \) or \( \leq 0 \) according as \( 0 \leq t \leq t_0 \) or \( t_0 \leq t \leq 1 \).

Differentiating (1), multiplying with \( \phi' \) and integrating again, we have
\[ (2) \phi^2(t) + \lambda \phi'^2(t) = \phi^2(1) + \lambda \phi'^2(1) = \lambda \phi'^2(1). \]
Hence \(|\phi'(1)| > 0\) and

\[
\phi'(t) = \begin{cases} 
(\phi^2(1) - (1/\lambda)\phi^2(t))^{1/2} & \text{for } 0 \leq t \leq t_0, \\
-(\phi^2(1) - (1/\lambda)\phi^2(t))^{1/2} & \text{for } t_0 \leq t \leq 1.
\end{cases}
\]

Therefore, noting that \(\phi(0) = \phi(1) = 0\), we get, for \(0 \leq t \leq t_0\),

\[
(3) \quad t = \int_0^{\phi(t)} |\phi'(1)|^{-1}(1 - v^P/(\lambda\phi^2(1)))^{-1/2} \, du \\
= \lambda^{1/P} |\phi'(1)|^{(2/p) - 1} \int_0^{\phi(t)/(\lambda\phi^2(1))} \left(1 - v^P\right)^{-1/2} \, dv,
\]

and, for \(t_0 \leq t \leq 1\),

\[
(4) \quad t - 1 = -\lambda^{1/P} |\phi'(1)|^{(2/p) - 1} \frac{1}{\lambda\phi^2(1)} \int_0^{\phi(t)/(\lambda\phi^2(1))} \left(1 - v^P\right)^{-1/2} \, dv.
\]

Put \(t = t_0\) in (3) and (4). Then \(t_0 = 1 - t_0\), and so \(t_0 = 1/2\). Put \(t = 1/2\) in (2) and (3). Then \(\phi^P(1/2) = \lambda\phi^2(1)\) and

\[
(5) \quad 1/2 = \lambda^{1/P} |\phi'(1)|^{(2/p) - 1} \int_0^{1/2} \left(1 - v^P\right)^{-1/2} \, dv.
\]

Integrating (2) and noting \(\int_0^1 \phi^2 \, dt = 1\), we have

\[
(6) \quad \int_0^1 \phi^P(t) \, dt = \lambda (\phi^2(1) - 1)
\]

Using (3) and (4), we obtain

\[
\int_0^{1/2} \phi^P(t) \, dt = \lambda^{1+(1/p)} |\phi'(1)|^{1+(2/p)} \int_0^{1/2} \left(1 - v^P\right)^{-1/2} \, dv \\
= \int_0^{1/2} \phi^P(t) \, dt.
\]

Hence

\[
(7) \quad \int_0^1 \phi^P(t) \, dt = 2\lambda^{1+(1/p)} |\phi'(1)|^{1+(2/p)} \int_0^{1} \left(1 - v^P\right)^{-1/2} \, dv \\
= (p+2)^{-1} \lambda^{1+(1/p)} |\phi'(1)|^{1+(2/p)} \int_0^{1} \left(1 - v^P\right)^{-1/2} \, dv.
\]
Eliminating $\lambda$ and $|\phi'(1)|$ from (5), (6) and (7), we obtain

$$\int_0^1 \phi'(t) dt = 2^{-P(p+2)}(p/2)^{-1/2} \int_0^1 (1 - t^P)^{-1/2} dt \frac{P}{P+2} = 2^{-P} c(p).$$

Thus $b^2 = 4(c(p))^{-2/p}$.

Remark. If $p$ is an integer, $T(x) = \int_0^1 |x(t)|^p dt$ can be replaced by $T(x) = \int_0^1 x^p(t) dt$. The above result $\sup\{T(\phi)| \phi \in K_1\} = 2^{-P} c(p)$ can be used to obtain an iterated logarithm result for the functional $T$ of empirical distributions (cf. H. Finkelstein [1]). Finkelstein discusses only the case $p = 2$, which can be obtained as a particular case of (i).

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