<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>タイトル</td>
</tr>
<tr>
<td>著者</td>
<td>著者</td>
</tr>
<tr>
<td>註釈</td>
<td>註釈</td>
</tr>
<tr>
<td>インデックス</td>
<td>インデックス</td>
</tr>
<tr>
<td>一覧</td>
<td>一覧</td>
</tr>
<tr>
<td>プリント版</td>
<td>プリント版</td>
</tr>
<tr>
<td>URL</td>
<td>URL</td>
</tr>
<tr>
<td>右</td>
<td>右</td>
</tr>
<tr>
<td>Type</td>
<td>Type</td>
</tr>
<tr>
<td>Textversion</td>
<td>Textversion</td>
</tr>
<tr>
<td>部</td>
<td>部</td>
</tr>
<tr>
<td>部</td>
<td>部</td>
</tr>
<tr>
<td>部</td>
<td>部</td>
</tr>
<tr>
<td>部</td>
<td>部</td>
</tr>
<tr>
<td>部</td>
<td>部</td>
</tr>
</tbody>
</table>

タイトル: タイトル
著者: 著者
注釈: 註釈
インデックス: インデックス
一覧: 一覧
プリント版: プリント版
URL: URL
右: 右
Type: Type
Textversion: Textversion
TAIL PROBABILITIES OF SOME CONTINUOUS FUNCTIONALS
OF GAUSSIAN PROCESSES

Hiroshi Oodaira
Yokohama National University

1. Let $X = \{X(t), \ 0 \leq t \leq 1\}$ be a path continuous Gaussian process with mean zero, and let $T$ be a real continuous functional on $C[0,1]$ such that $T(cx) = c^p T(x)$ with $p > 0$ for any positive constant $c$. In this note the following asymptotic estimate for the tail probabilities of $T(X)$ is obtained:

$$\lim_{\alpha \to \infty} \frac{1}{\alpha^{2/p}} \log P\{ T(X) > \alpha \} = -(1/2)b^2,$$

where $b^2$ is a constant determined as the solution of certain extremal problem. For example, it is shown that if $X$ is Brownian motion, then

$$\lim_{\alpha \to \infty} \frac{1}{\alpha^{2/p}} \log P\{ \int_0^1 |X(t)|^p dt > \alpha \} = -(1/2)(c(p))^{-2/p},$$

where $p \geq 1$ and

$$c(p) = 2(p+2)(p/2)^{-1/(p/2)} \int_0^1 (1-t^p)^{-1/2} dt, \quad p > 0,$$

and also, if $X$ is Brownian bridge, then the same formula holds with $c(p)$ replaced by $2^{-p}c(p)$.

In his thesis [3] and also in [4], N. A. Marlow obtained a similar
asymptotic formula for tail probabilities of uniformly Hölder continuous, asymptotically homogeneous functionals $F$ of path continuous Gaussian processes. His method of proof is to first estimate $\log P\{ F(X) > \alpha \}$ in the finite-dimensional case by a Laplace asymptotic formula, and then to pass to the limit to obtain the function space version. Note also that H. P. McKean [5] obtained a similar asymptotic estimate for tail probabilities of multiple Wiener integrals.

Our method is different from Marlow's and is based on the following Fredlin-Wentzell type estimates for Gaussian measures given in [7] and [2]. Let $C = C[0,1]$ be the space of all continuous functions on $[0,1]$ with the supremum norm $\| \cdot \|_{\infty}$, and let $\mathcal{A}$ be the $\sigma$-field of Borel subsets of $C$. Let $\mu$ be a Gaussian measure on $(C, \mathcal{A})$ with mean zero and covariance function $R(s,t)$, i.e., $\int_C x(t) \mu(dx) = 0$, for $0 \leq t \leq 1$, and $R(s,t) = \int_C x(s)x(t) \mu(dx)$, for $0 \leq s, t \leq 1$, where $x \in C$. Let $H = H(R)$ be the reproducing kernel Hilbert space (RKHS) with reproducing kernel (r.k.) $R$, whose norm is denoted by $\| \cdot \|_H$. Note that $H \subset C$, since $R$ is continuous.

**Theorem 1.** Let $\phi \in H$. Then, for any $\delta, h > 0$, there is a number $a_0 = a_0(\delta, h, \| \phi \|_H)$ such that

$$
\mu\left\{ x \mid \| (x/\alpha) - \phi \|_{\infty} < \delta \right\} \geq \mu\left\{ x \mid \| x - \alpha \phi \|_{\infty} < \delta \right\} \geq \exp\left\{ -(\alpha^2/2)(\| \phi \|_H^2 + h) \right\}
$$

for all $\alpha \geq a_0$.

**Theorem 2.** Let $K_r = \{ \phi \in H \mid \| \phi \|_H \leq r \}$ and let $d(x, K_r)$ be the distance from $x \in C$ to $K_r$ in the sup norm $\| \cdot \|_{\infty}$. Then, for any $\delta, h > 0$,
there is a number \( a_0 = a_0(\delta, h, r) \) such that

\[
\mu\{ x \mid d(x/a, K_r) > \delta \} \leq \exp[-(a^2/2)(r^2 - h)]
\]

for all \( a \geq a_0 \).

For the proofs see [7] or [2]. From Theorems 1 and 2 we obtain the following

**Theorem 3.** Let \( T \) be a real continuous functional on \( C \) such that \( T(cx) = c^p T(x) \) with \( p > 0 \) for any positive constant \( c \) and \( T(\phi) > 0 \) for some \( \phi \in H \).

Then

\[
\lim_{a \to \infty} (1/a^{2/p}) \log \mu\{ x \mid T(x) > a \} = -(1/2)b^2,
\]

where \( b^2 = \inf \{ \| \phi \|^2_H \mid T(\phi) > 1 \} = \sup \{ r^2 \mid \sup\{ T(\phi) \mid \phi \in K_r \} < 1 \} \).

**Proof.** Let \( D = \{ x \mid T(x) > 1 \} \). \( D \) is open and its closure \( \overline{D} = \{ x \mid T(x) \geq 1 \} \). For any \( \phi \in H \cap D \), there is a \( \delta > 0 \) such that \( \| x - \phi\|_\infty < \delta \) implies \( x \in D \). Hence, using Theorem 1, we obtain

\[
\mu\{ x \mid T(x) > a \} = \mu\{ x \mid T(x/a^{1/p}) > 1 \}
\]

\[
\geq \mu\{ x \mid \| (x/a^{1/p}) - \phi\|_\infty < \delta \}
\]

\[
\geq \exp[-(a^{2/p}/2)(\| \phi \|^2_H + h)]
\]

for any \( h > 0 \), if \( a \) is sufficiently large. Thus, for any \( \phi \in H \cap D \),

\[
\liminf_{a \to \infty} (1/a^{2/p}) \log \mu\{ x \mid T(x) > a \} \geq -(1/2) \| \phi \|^2_H.
\]
and hence,

\[
\liminf_{\alpha \to \infty} (1/\alpha^{2/p}) \cdot \log \mu\{ x \mid T(x) > \alpha \} \\
\geq -(1/2) \cdot \inf \{ \|\phi\|_H^2 \mid T(\phi) > 1 \}.
\]

Since \( K_\cdot \) is compact in \( C \) (see, e.g. [6]) and \( T \) is continuous, there is a number \( r > 0 \) such that \( \sup\{ T(\phi) \mid \phi \in K_\cdot \} < 1 \), and for any such a number \( r \), there is a \( \delta > 0 \) such that \( d(K_\cdot, \overline{D}) > \delta \), where \( d(K_\cdot, \overline{D}) \) is the distance between \( K_\cdot \) and \( \overline{D} \). If \( T(x) > \alpha \), then \( x/\alpha^{1/p} \in D \), and by Theorem 2,

\[
\mu\{ x \mid T(x) > \alpha \} \leq \mu\{ x \mid d(x/\alpha^{1/p}, K_\cdot) > \delta \} \\
\leq \exp\{- (a^{2/p}/2) (r^2 - h)\}
\]

for any \( h > 0 \), if \( a \) is sufficiently large. Therefore,

\[
\limsup_{\alpha \to \infty} (1/\alpha^{2/p}) \cdot \log \mu\{ x \mid T(x) > \alpha \} \leq -(1/2) r^2,
\]

and hence

\[
\limsup_{\alpha \to \infty} (1/\alpha^{2/p}) \cdot \log \mu\{ x \mid T(x) > \alpha \} \\
\leq -(1/2) \cdot \sup\{ r^2 \mid \sup\{ T(\phi) \mid \phi \in K_\cdot \} > 1 \}.
\]

It is easy to see that \( \inf\{ \|\phi\|_H^2 \mid T(\phi) > 1 \} = \inf\{ \|\phi\|_H^2 \mid T(\phi) \geq 1 \} \) (in fact, \( = \inf\{ \|\phi\|_H^2 \mid T(\phi) = 1 \} \)) and \( r^2 < \inf\{ \|\phi\|_H^2 \mid T(\phi) \geq 1 \} \), then \( \sup\{ T(\phi) \mid \phi \in K_\cdot \} < 1 \), and so \( \sup\{ r^2 \mid \sup\{ T(\phi) \mid \phi \in K_\cdot \} < 1 \} = \inf\{ \|\phi\|_H^2 \mid T(\phi) > 1 \} \). This completes the proof.

**Remark.** Note that \( \sup\{ T(\phi) \mid \phi \in K_\cdot \} = 1 \). Since \( \sup\{ T(\phi) \mid \phi \in K_\cdot \} = B^p \cdot \sup\{ T(\phi) \mid \phi \in K_1 \} \), we have \( B^2 = (\sup\{ T(\phi) \mid \phi \in K_1 \})^{-2/p} \).
2. In what follows we consider several examples for which the values of $b^2$ can be explicitly given by evaluating $\sup\{T(\phi) \mid \phi \in K_1\}$.

(i) Let $X$ be a path continuous Gaussian process with mean zero and covariance function $R(s,t)$. Then

$$\lim_{\alpha \to \infty} (1/\alpha) \cdot \log P\{ \int_0^1 x^2(t) dt > \alpha \} = -1/(2\lambda_1),$$

where $\lambda_1$ is the largest eigenvalue of the covariance operator $R$ with kernel $R(s,t)$ on $L^2[0,1]$.

This is a known result, and so we just indicate briefly how it can be derived from Theorem 3. In this case $T(x) = \int_0^1 x^2(t) dt = ||x||_2^2$ and $p = 2$. Let $\{\lambda_1\}$ and $\{\psi_1\}$ be the eigenvalues and the corresponding normalized eigenfunctions of $R$. Then $\{\phi_1 = \lambda_1^{1/2} \psi_1\}$ is a complete orthonormal system in $H(R)$. It can be shown that $||\phi||_2^2 \leq \lambda_1 ||\phi||_H^2$ for any $\phi \in H(R)$. Hence $\sup\{T(\phi) \mid \phi \in K_1\} \leq \lambda_1$. Since $||\phi_1||_2^2 = \lambda_1$, we have $\sup\{T(\phi) \mid \phi \in K_1\} = \lambda_1$, and hence the result.

(ii) Let $\mu$ be the Wiener measure and let $T(x) = \int_0^1 |x(t)|^p dt$, $p \geq 1$. The RKHS $H(R)$ associated with the Wiener measure is the space of all absolutely continuous functions $\phi$ on $[0,1]$ such that $\phi(0) = 0$ and $d\phi/dt \in L^2[0,1]$, and $(\phi, \psi)_H = \int_0^1 (d\phi/dt)(d\psi/dt) dt$, where $(\cdot, \cdot)_H$ denotes the inner product of $H(R)$. V. Strassen ([8], p.220) proved that $\sup\{T(\phi) \mid \phi \in K_1\} = c(p)$, where

$$c(p) = 2(p+2)^{(p/2)-1/2}(\int_0^1 (1-t)^{p-1/2} dt)^{p/2}.$$

We thus obtain the result for Brownian motion stated at the beginning of this note. In particular, $C(1) = 3^{-1/2}$ and $c(2) = 4/\pi^2$. The case $p = 1$ has been previously obtained by Marlow [3] by a different method, and the
case \( p = 2 \) is of course a particular case of (i). If \( p \) is an integer, then the same formula holds for \( T(x) = \int_0^1 |x(t)|^p \, dt \).

(iii) Let \( \mu \) be the Wiener measure and let

\[
T(x) = \int_0^1 |x(t)|^2 \, dt / \int_0^1 |x(t)| \, dt.
\]

Then \( \sup\{T(\phi) \mid \phi \in \mathcal{K}_1 \} = 2q \), where \( 0 < q < 1 \) is the largest solution of

\[
(1-q)^{1/2} \sin((1-q)^{1/2}/q) + \cos((1-q)^{1/2}/q) = 0
\]

(see [8], p.222). Hence, if \( X \) is Brownian motion, then

\[
\lim_{a \to \infty} \left( 1/a^2 \right) \cdot \log P\left\{ \int_0^1 |X(t)|^2 \, dt / \int_0^1 |X(t)| \, dt > a \right\} = -1/(8q)^2.
\]

(iv) Let \( X \) be Brownian bridge. We shall show that

\[
\lim_{a \to \infty} \left( 1/a^{2/p} \right) \cdot \log P\left\{ \int_0^1 |X(t)|^{p} \, dt > a \right\} = -(c(p))^{-2/p}, \quad p \geq 1,
\]

where \( c(p) \) is the same as in (ii).

The covariance function of Brownian bridge is

\[
R(s,t) = \begin{cases} 
s(1-t), & \text{for } s \leq t, \\
t(1-s), & \text{for } s > t, 
\end{cases}
\]

\[
= \int_0^1 \Omega(u,s) \Omega(u,t) \, du,
\]

where

\[
\Omega(u,t) = \begin{cases} 
1-t, & \text{for } u \leq t, \\
-t, & \text{for } u > t.
\end{cases}
\]

Hence the RKHS \( H(R) \) with r.k. \( R \) is isometrically isomorphic to the closed subspace \( M \) of \( L^2[0,1] \), spanned by \( \{ \Omega(u,t), 0 \leq t \leq 1 \} \), and any function \( \phi \) in \( H(R) \) has a representation \( \phi(t) = \int_0^1 m(u) \Omega(u,t) \, du \) with \( m \in M \). Note that
\[ M \text{ is } 1, \text{i.e., } \int_0^1 m(u)du = 0 \text{ for all } m \in M, \text{ since } \int_0^1 q(u,t)du = 0 \text{ for all } t \in [0,1] \text{ and } \int_0^1 n(u)du = 0 \text{ and } \int_0^1 n(u)q(u,t)du = 0 \text{ for all } t \in [0,1], \text{ then } n = 0. \text{ Hence } \phi(t) = \int_0^1 m(u)q(u,t)du = \int_0^1 m(u)du, \text{ which shows that } \phi \text{ is absolutely continuous. Therefore, } \mathcal{H}(R) \text{ is the space of all absolutely continuous functions } \phi \text{ on } [0,1] \text{ such that } \phi(0) = \phi(1) = 0 \text{ and } \phi' = d\phi/dt \in L^2[0,1], \text{ and } (\phi, \psi)_H = \int_0^1 \phi' \psi' dt.

As in Strassen's proof [8] for Brownian motion case, we shall evaluate
\[ \sup\{T(\phi) \mid \phi \in K_1\} = \sup\{ \int_0^1 |\phi(t)|^2 dt \mid \phi(0) = \phi(1) = 0 \text{ and } \int_0^1 \phi'^2 dt \leq 1 \} \]
by classical methods of the calculus of variations. Since \( K_1 \) is compact and \( T \) is continuous, there is a maximizing point \( \phi \) with \( ||\phi||_H^2 = \int_0^1 \phi'^2 dt = 1 \).

We may assume \( \lambda > 0 \), and \( \phi \) satisfies the equation
\[ \int_0^1 \phi' P^{-1} \psi dt = 2\lambda \int_0^1 \phi' \psi' dt, \text{ for any } \psi \in \mathcal{H}(R), \]
where \( \lambda > 0 \) is a Lagrange multiplier. Integrating by parts the left-hand side and noting that \( \psi' \perp 1 \), we obtain
\[ \int_0^1 (\int_0^s \phi' P^{-1} ds) - \int_0^1 \int_0^s \phi' P^{-1}(u)du)ds \psi'(t)dt = 2\lambda \int_0^1 \phi' \psi' dt \]
for all \( \psi' \in M \). Therefore,
\[ (1) \quad \int_0^1 \phi' P^{-1} (s) ds - \int_0^1 \int_0^s \phi' P^{-1}(u)du)ds = 2\lambda \phi'(t), \text{ for } 0 \leq t \leq 1. \]

Since \( \phi \geq 0 \) and \( \lambda > 0 \), (1) shows that \( \phi'(0) \geq 0 \), \( \phi'(1) \leq 0 \) and \( \phi' \) is differentiable and monotone decreasing. Hence there is a point \( t_0 \) such that \( \phi'(t_0) = 0 \) and \( \phi'(t) \geq 0 \) or \( \leq 0 \) according as \( 0 \leq t \leq t_0 \) or \( t_0 \leq t \leq 1 \).

Differentiating (1), multiplying with \( \phi' \) and integrating again, we have
\[ (2) \quad \phi^2(t) + \lambda \phi'^2(t) = \phi^2(1) + \lambda \phi'^2(1) = \lambda \phi'^2(1). \]
Hence $|\phi'(1)| > 0$ and

\[
\phi'(t) = \begin{cases} 
(\phi'^2(1) - (1/\lambda)\phi^P(t))^{1/2} & \text{for } 0 \leq t \leq t_0', \\
-(\phi'^2(1) - (1/\lambda)\phi^P(t))^{1/2} & \text{for } t_0' \leq t \leq 1.
\end{cases}
\]

Therefore, noting that $\phi(0) = \phi(1) = 0$, we get, for $0 \leq t \leq t_0$,

\[
(3) \quad t = \int_0^\phi(t) \frac{1}{|\phi'(1)|} (1 - u^p/(\lambda\phi'^2(1)))^{-1/2} du
\]

\[
= \lambda^{1/p} |\phi'(1)| (2/p - 1) \int_0^\phi(t) / (\lambda\phi'^2(1))^{1/p} (1 - v^p)^{-1/2} dv,
\]

and, for $t_0 \leq t \leq 1$,

\[
(4) \quad t - 1 = -\lambda^{1/p} |\phi'(1)| (2/p - 1) \int_0^1 (1 - v^p)^{-1/2} dv.
\]

Put $t = t_0$ in (3) and (4). Then $t_0 = 1 - t_0$, and so $t_0 = 1/2$. Put $t = 1/2$ in (2) and (3). Then $\phi^P(1/2) = \lambda\phi'^2(1)$ and

\[
(5) \quad 1/2 = \lambda^{1/p} |\phi'(1)| (2/p - 1) \int_0^1 (1 - v^p)^{-1/2} dv.
\]

Integrating (2) and noting $\int_0^1 \phi'^2 dt = 1$, we have

\[
(6) \quad \int_0^1 \phi^P(t) dt = \lambda (\phi'^2(1) - 1)
\]

Using (3) and (4), we obtain

\[
\int_0^{1/2} \phi^P(t) dt = \lambda^{1+1/p} |\phi'(1)|^{1+2/(2/p)} \int_0^1 (1 - v^p)^{-1/2} dv
\]

\[
= \int_{1/2}^1 \phi^P(t) dt.
\]

Hence

\[
(7) \quad \int_0^1 \phi^P(t) dt = 2\lambda^{1+1/(p)} |\phi'(1)|^{1+2/(2/p)} \int_0^1 (1 - v^p)^{-1/2} dv
\]

\[
= 4(p+2)^{-1} \lambda^{1+1/(p)} |\phi'(1)|^{1+2/(2/p)} \int_0^1 (1 - v^p)^{-1/2} dv.
\]
Eliminating $\lambda$ and $|\phi'(1)|$ from (5), (6) and (7), we obtain

$$\int_0^1 \phi^p(t) dt = 2^{1-p}(p+2)(p/2)-1/(\int_0^1 (1-t^p)^{-1/2} dt)^{p/2} = 2^{-p}c(p).$$

Thus $b^2 = 4(c(p))^{-2/p}$.

Remark. If $p$ is an integer, $T(x) = \int_0^1 |x(t)|^p dt$ can be replaced by $T(x) = \int_0^1 x^p(t) dt$. The above result $\sup\{T(\phi)\mid \phi \in K_1\} = 2^{-p}c(p)$ can be used to obtain an iterated logarithm result for the functional $T$ of empirical distributions (cf. H. Finkelstein [1]). Finkelstein discusses only the case $p = 2$, which can be obtained as a particular case of (i).

Acknowledgement. The author is grateful to Dr. N. A. Marlow for his comments on [7].

References:


