数理解析研究所講究録
261巻 1976年 2-18

Causal calculus arising from Brownian Motion

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§0. Introduction

A new viewpoint of the analysis of functionals of Brownian motion is presented in this note. From this viewpoint a generalization of those functionals will naturally be introduced. Our approach, however, is still in line with the so-called causal analysis, where the propagation of time is taken into account in analysing functionals of Brownian motion.

Functionals of Brownian motion, call them Brownian functionals, with finite variance can be expressed in terms of white noise which is viewed as the time derivative of Brownian motion $B(t)$. Start with the probability measure μ of white noise introduced on the space $\mathcal{S}^*$ of tempered distributions. It is given by the characteristic functional

(1) $$C(\xi) = \exp[-\frac{1}{2}\|\xi\|^2], \qquad \xi \in \mathcal{S},$$

in such a way that

(2) $$C(\xi) = \int_{\mathcal{S}^*} \exp[i\langle x, \xi\rangle]d\mu(x).$$

With this measure μ each $x \in \mathcal{S}^*$ may be thought of as a sample function of $\dot{B}(t) = dB(t)/dt$. The Hilbert space $(L^2) = L^2(\mathcal{S}^*, \mu)$ is therefore the collection of all complex-valued Brownian functionals with finite variance.

Our main interest is to discuss differential and integral calculus on the space (L^2). Our approach involves the following three steps. i) The first thing which should be done is to visualize those members of (L^2) by using standard tools from analysis. For this purpose a transformation $\mathcal{T}$ is introduced :

$$(3) \qquad (\mathcal{T}\varphi)(\xi) = \int \exp[i\langle x, \xi\rangle]\varphi(x)\,d\mu(x), \qquad \varphi \in (L^2).$$

The collection $\mathcal{F} = \{\mathcal{T}\varphi \,;\, \varphi \in (L^2)\}$ becomes a reproducing kernel Hilbert space, so that $\mathcal{F}$ is isomorphic to (L^2) under $\mathcal{T}$. The Fock space expression for $\mathcal{F}$ follows immediately, and one has the integral representation of the multiple Wiener integrals. ii) One then must introduce a most suitable coordinate system in $\mathcal{S}^*$ with measure μ to carry on the causal analysis, that is the analysis where the propagation of time is always taken into account. Roughly speaking, the system $\{\dot{B}(t) \,;\, t \in R\}$ is taken to be a coordinate system. iii) With this system we shall be able to introduce a certain class of generalized Brownian functionals. Such kind of generalization can be done via the integral representation of (L^2)-functionals, and the idea comes from P. Lévy's approach to functional analysis (P. Lévy [1]).

The author should like to add a few words on the motivation of this work. He has been inspired by several problems, indeed actual problems, arising from Quantum Mechanics, in particular field theory or Feynman's path integral, Stochastic Control theory and stochastic evolution equations in Population Biology. Those problems require nonlinear, causal analysis of functionals of noise or fluctuation, a mathematical expression of which is to be the white noise $\{\dot{B}(t)\}$.

§1. Summary of known results

This section is devoted to a quick review of well-known results for Brownian functionals as well as their integral representations. For details we refer to [2], [3], [5].

Let $(\mathcal{S}^*, \mu)$ be the measure space of white noise, where μ is given by (1) and (2), and set $(L^2) = L^2(\mathcal{S}^*, \mu)$. The functional $C(\xi - \eta)$, $(\xi, \eta) \in \mathcal{S} \times \mathcal{S}$, $C(\xi)$ being given by (1), is positive definite, so that it defines a reproducing kernel Hilbert space which is denoted by $\mathcal{F}$. Let $\mathcal{T}$ be given by (3).

<u>Theorem 1.</u> <u>The reproducing kernel Hilbert space $\mathcal{F}$ with kernel C is isomorphic to the Hilbert space (L^2) under the transformation $\mathcal{T}$.</u>

Now observe the Taylor series expansion of the kernel C :

(4) $$C(\xi - \eta) = \sum_{n=0}^{\infty} C(\xi)\frac{(\xi, \eta)^n}{n!} C(\eta),$$

$(\ ,\ )$ the inner product in $L^2(R)$.

Set

(5) $$C_n(\xi, \eta) = C(\xi)\frac{(\xi, \eta)^n}{n!} C(\eta).$$

Since $C_n(\xi, \eta)$ is positive definite, it defines a reproducing kernel Hilbert space, call it $\mathcal{F}_n$, which is a subspace of $\mathcal{F}$. In addition, one can prove that

$$(C_n(\cdot, \eta_1), C_m(\cdot, \eta_2))_{\mathcal{F}} = 0, \quad \text{for } n \neq m, \ \eta_1, \eta_2 \in \mathcal{S},$$

which proves that the subspaces $\mathcal{F}_n$, $n \geq 0$, are mutually orthogonal. Thus a direct sum decomposition of $\mathcal{F}$ is obtained :

(6) $$\mathcal{F} = \sum_{n=0}^{\infty} \oplus \mathcal{F}_n \qquad \text{(Fock space)}$$

Now set

$$\mathcal{H}_n = \mathcal{T}^{-1}(\mathcal{F}_n).$$

The space $\mathcal{H}_n$ is called the multiple Wiener integral of degree n. Then the decomposition (6) of $\mathcal{F}$ gives us that of (L^2) :

$$(L^2) = \sum_{n=0}^{\infty} \oplus \mathcal{H}_n. \tag{7}$$

Theorem 2. i) For $\varphi(x) \in \mathcal{H}_n$ we have

$$(\mathcal{T}\varphi)(\xi) = i^n C(\xi) \int_{R^n} \cdots \int F(u_1, \cdots, u_n)\xi(u_1)\cdots\xi(u_n)du^n, \tag{8}$$

where $F \in \widehat{L^2(R^n)} \equiv$ {symmetric $L^2(R^n)$-functions}, and the mapping

$$\varphi \to F \in \widehat{L^2(R^n)}, \qquad \varphi \in \mathcal{H}_n,$$

is one-to-one.

ii) Under the relation in i) it holds that

$$\|\varphi\|_{(L^2)} = \sqrt{n!}\,\|F\|_{L^2(R^n)}. \tag{9}$$

Definition 1. The expression (8) for φ is called the integral representation of φ in $\mathcal{H}_n$, and F is called the kernel of the representation.

For general φ in (L^2) one uses the expansion

$$\varphi = \sum_{n=0}^{\infty} \varphi_n, \qquad \varphi_n \in \mathcal{H}_n$$

to have a series of integral representations and that of kernels $\{F_n\}$.

A special interest can be found in the case where φ is in $\mathcal{H}_2$. Associated with such φ is a symmetric $L^2(R^2)$-function $F(u, v)$. If, in addition, we assume that φ is real-valued, then F defines a Hermitian operator of Hilbert-Schmidt type acting on $L^2(R)$. One can therefore appeal to the Hilbert-Schmidt expansion theorem to have the eigenfunction expansion of F :

$$F(u,\ v) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \eta_n(u)\eta_n(v),$$

where $\{\lambda_n, \eta_n\ ;\ n \geq 1\}$ is the eigen system. This implies the expansion of $\mathcal{F}$ into a sum of independent random variables :

$$\mathcal{F}(x) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n} (<x,\ \eta_n>^2 - 1).$$

Having gotten this expansion, one can easily see the probability distribution of $\mathcal{F}$ by computing the characteristic function or semi-invariants.

§2. Coordinate systems in $(\mathcal{S}^*,\ \mu)$.

We discuss coordinate system in $\mathcal{S}^*$ and bases for the Hilbert space (L^2). Before we come to the main topics, let us remind some elementary concepts appearing in the analysis on $L^2([0,\ 1])$.

Let $\{\xi_n\}$ be a complete orthonormal system (c.o.n.s) in $L^2([0,\ 1])$. Introduce Ψ_n and Φ_n by

$$\Psi_n(u,\ v) = \frac{1}{n} \sum_{i=1}^{n} \xi_n(u)\xi_n(v),$$

$$\Phi_n(u) = \Psi_n(u,\ u).$$

Then, one immediately sees

(10) $$\int_0^1\int_0^1 \Psi_n(u,\ v)^2 dudv = \frac{1}{n} \to 0,$$

namely Ψ_n converges to 0 in $L^2([0,\ 1]^2)$. For $\Phi_n(u)$,

$$\int_0^1 \Phi_n(u)du = 1, \qquad \text{for every } n$$

holds, however the convergence

(11) $$\Phi_n(u) \to 1 \qquad \text{in } L^1([0,\ 1])$$

is not always true. If (11) holds, then the c.o.n.s. $\{\xi_n\}$ is said to be equally dense.

Examples of an equally dense c.o.n.s.

i) $\{1, \sqrt{2}\sin 2k\pi t, \sqrt{2}\cos 2k\pi t ; k \ge 1\}$.

ii) Walsh functions.

Remark. One can think of an equally dense c.o.n.s. even in $L^2(R)$. The condition (11) could now be understood as the property that Φ_n should approximate the so-to-speak uniform probability measure on R. An example of such c.o.n.s. is the $\{\xi_n\}$ given by

$$\text{(12)} \qquad \xi_n(u) = \frac{1}{\sqrt{2^n n! \sqrt{\pi}}} H_n(u) e^{-\frac{1}{2}u^2}, \qquad n \ge 0,$$

where H_n is the Hermite polynomial of degree n.

We now come to a coordinate system in $\mathcal{S}^*$.

[I]. Let $\{\xi_n\}$ be a c.o.n.s. in $L^2(R)$. Each member x in $\mathcal{S}^*$ has the coordinate representation of the form

$$x \sim \{x_n ; n \ge 1\}, \qquad x_n = \langle x, \xi_n \rangle.$$

While, if the measure μ is introduced to $\mathcal{S}^*$, $\{\langle x, \xi_n\rangle\}$ forms a system of independent standard Gaussian random variables on the probability space $(\mathcal{S}^*, \mu)$. Hence we have

$$\frac{1}{N}\sum_{n=1}^{N} \langle x, \xi_n\rangle\langle y, \xi_n\rangle \to 0, \qquad \text{a.e. on } (\mathcal{S}^* \times \mathcal{S}^*, \mu \times \mu),$$

$$\frac{1}{N}\sum_{n=1}^{N} \langle x, \xi_n\rangle^2 \to 1, \qquad \text{a.e. on } (\mathcal{S}^*, \mu),$$

(the strong law of large numbers). Those observations show that the c.o.n.s. $\{\langle x, \xi_n\rangle\}$ enjoys the property "equally dense" in the Hilbert space $\mathcal{H}_1$.

One is naturally led to a c.o.n.s. in $\mathcal{H}_n$ defined by the Fourier-Hermite polynomials of the form

$$\prod_j (n_j!\,2^{n_j})^{-\frac{1}{2}} H_{n_j}\left(\frac{\langle x, \xi_j\rangle}{\sqrt{2}}\right), \qquad \sum n_j = n. \tag{13}$$

Unfortunately such a c.o.n.s. is not fitting for the causal calculus, because the propagation of time cannot be expressed explicitly.

[II]. Take the unit time interval $[0, 1]$ to fix the idea. Let Π_n, $n \geq 1$, be a sequence of partitions of $[0, 1]$ with $\Pi_n < \Pi_{n+1}$ (Π_{n+1} is finer than Π_n). For $\Pi = \{\Delta_i\}$ with $\Delta_i = [t_i, t_{i+1}]$, we introduce the quadratic variation of $f(t)$, $0 \leq t \leq 1$, by Π :

$$\Pi^2 f = \sum_{j=0}^{n-1} |f(t_{j+1}) - f(t_j)|^2.$$

Let $B(t, x)$ be a version of Brownian motion given by

$$B(t, x) = \langle x, \chi_{[0, t]}\rangle, \qquad 0 \leq t \leq 1.$$

Then we have

<u>Lemma</u> <u>If</u> $\delta\Pi_n = \mathrm{Max}|t_{j+1} - t_j| \to 0$, <u>then it holds that</u>

$$\lim_{n\to\infty} \Pi_n^2 B(\cdot, x) = 1, \qquad \text{a.e. } (\mu).$$

For proof we refer to [3, pp.60∿64].

If, in particular, Π_n is the uniform partition with $|\Delta_i| = 2^{-n}$, then

$$\sum_{k=1}^{2^n} (\Delta_k B)^2 = \frac{1}{2^n} \sum_{k=1}^{2^n} (\sqrt{2^n}\, \Delta_k B)^2 \to 1, \qquad \text{a.e. } (\mu).$$

Formally speaking, this says that $\left\{\frac{dB(t)}{\sqrt{dt}}\right\}$ is an equally dense c.o.n.s. in $\mathcal{H}_1$. In other words, the projective limit of

$$\left\{\frac{\Delta_k B}{\sqrt{\Delta_k}}\ ;\ \Delta_k \in \Pi_n\right\}, \qquad \Pi_n \text{ uniform,}$$

defines a c.o.n.s. which is equally dense.

One then proceed to a c.o.n.s. in $\mathcal{H}_2$. By using Fourier-Hermite polynomials all the members of the system are classified as follows :

$$\frac{dB(t)}{\sqrt{dt}} \cdot \frac{dB(s)}{\sqrt{ds}} \quad (t \neq s) \qquad \text{class (1)},$$

$$\frac{1}{\sqrt{2}}\left(\left(\frac{dB(t)}{\sqrt{dt}}\right)^2 - 1\right) \qquad \text{class (2)}.$$

If one uses the Hermite polynomials with parameter given by

$$H_n(x\ ;\ \sigma^2) = \frac{(-\sigma^2)^n}{n!} e^{\frac{x^2}{2\sigma^2}} \frac{d^n}{dx^n} e^{-\frac{x^2}{2\sigma^2}}, \qquad n \geq 0, \quad \sigma \geq 0,$$

then a c.o.n.s. in $\mathcal{H}_n$ for general n is easily expressed. In fact,

$$(14) \quad \begin{cases} \prod_{j=1}^{n} \{H_1(\dot{B}(t_j)\ ;\ \frac{1}{dt_j})\sqrt{dt_j}\}, & (t_j\text{'s different}) \quad \text{class (1)}, \\ \prod_j C_{n_j} H_{n_j}(\dot{B}(t_j)\ ;\ \frac{1}{dt_j})(\sqrt{dt_j})^{n_j} & \Sigma n_j = n, \qquad \text{class (2)}, \end{cases}$$

$$(t_j\text{'s are different}, \quad C_{n_j} = \left\{2^{n_j} n_j!\right\}^{-\frac{1}{2}}).$$

They are still formal expressions, but it is noted that they are consistent with the partitions Π_n through the addition formula for $H_n(x\ ;\ \sigma^2)$'s :

$$\sum_{k=0}^{n} H_{n-k}(x\ ;\ \sigma^2) H_k(y\ ;\ \tau^2) = H_n(x+y\ ;\ \sigma^2 + \tau^2).$$

§3. Generalized Brownian functionals

We start with an example which suggests us to define a class of generalized Brownian functionals. Set

$$B(t) = B(t, x) = \langle x, \chi_{[t\ 0,\ t\ 0]} \rangle.$$

Then $\{B(t)\}$ is a version of a Brownian motion. Being inspired by the class (2) base with $n = 2$ in (14), let us observe a functional of the form

$$\varphi_\Pi = \sum_j a_j \frac{1}{\sqrt{2}}\left\{\left(\frac{\Delta_j B}{\sqrt{\Delta_j}}\right)^2 - 1\right\}, \qquad \Pi = \{\Delta_j\} \quad \text{partition of } R,$$

which is in $\mathcal{H}_2$. Then the integral representation is

$$(\mathcal{T}\varphi_\Pi) = i^2 C(\xi)\frac{1}{\sqrt{2}}\iint \sum_j \frac{a_j}{\Delta_j}\ \chi_{\Delta_j^2}(u,\ v)\xi(u)\xi(v)dudv.$$

If the partition $\Pi = \{\Delta_j\}$ becomes finer and finer with $\delta\Pi \to 0$, then the integral representation approaches

$$(15) \qquad i^2 C(\xi)\frac{1}{\sqrt{2}}\int f(u)\xi(u)^2 du,$$

provided $\Sigma a_j \chi_{\Delta_j}(u)$ approximates a function f. Such an expression can never be found in $\mathcal{F}_2$, however if it is written as

$$i^2 C(\xi)\frac{1}{\sqrt{2}}\iint f\left(\frac{u+v}{2}\right)\delta(u-v)\xi(u)\xi(v)dudv,$$

then one is led to think of a much wider class of functionals than $\mathcal{F}_2$. At the same time one considers a limit (in some sense, but not in (L^2)-sense) of the φ_Π when $\delta\Pi \to 0$. The limit, formally writing, would be expressed in the form

$$(16) \qquad \sqrt{2}\int f(u)H_2(\dot{B}(u)\ ;\ \frac{1}{du})du$$

and the integral representation would be given by (15).

We are now in the position to give a definition of generalized Brownian functionals. Let $H^m(R^n)$ be the Sobolev space of order m on R^n, and set $\widehat{H^m}(R^n) = H^m(R^n) \cap \widehat{L^2}(R^n)$. Define

$$\tilde{\mathcal{F}}_n^{(n)} = \{U(\xi) = \int\cdots\int F(u_1,\cdots,u_n)\xi(u_1)\cdots\xi(u_n)du^n\ ;\ F \in \widehat{H}^{\frac{n+1}{2}}(R^n)\}$$

and

$$\mathcal{F}_n^{(n)} = \{i^n C(\xi)U(\xi)\ ;\ U \in \tilde{\mathcal{F}}_n^{(n)}\}.$$

Introduce the $H^{\frac{n+1}{2}}(R^n)$-topology on $\mathcal{F}_n^{(n)}$; that is, if $i^n C(\xi)U(\xi)$

is in $\mathcal{F}_n^{(n)}$ with kernel F, then the norm of the functional is defined to be the $H^{\frac{n+1}{2}}(R^n)$-norm of F times $\sqrt{n!}$ (cf. Theorem 2, ii)). Set

$$\mathcal{H}_n^{(n)} = \mathcal{T}^{-1}(\mathcal{F}_n^{(n)})$$

and topologize $\mathcal{H}_n^{(n)}$ so that $\mathcal{T}$ is an isometry. Let $\mathcal{H}_n^{(-n)}$ and $\mathcal{F}_n^{(-n)}$ be the dual space for $\mathcal{H}_n^{(n)}$ and $\mathcal{F}_n^{(n)}$, respectively. Then one has the following diagram

$$\begin{array}{ccccc} \mathcal{F}_n^{(n)} & \hookrightarrow & \mathcal{F}_n & \hookrightarrow & \mathcal{F}_n^{(-n)} \\ \updownarrow & & \updownarrow & & \updownarrow \\ \mathcal{H}_n^{(n)} & \hookrightarrow & \mathcal{H}_n & \hookrightarrow & \mathcal{H}_n^{(-n)} \end{array}$$

The vertical bi-arrow means isomorphism under $\mathcal{T}$ (note that $\mathcal{T}$ can be extended naturally to $\mathcal{H}_n^{(-n)}$), and $\hookrightarrow$ means continuous injection.

Definition 2. A member of $\mathcal{H}_n^{(-n)}$ is called a generalized Brownian functional of degree n.

§4. Generalized random measures

As is easily seen,

$$(17) \qquad i^n C(\xi)\int\cdots\int f(u_1,\cdots,u_k)\xi(u_1)^{n_1}\cdots\xi(u_k)^{n_k}du^k, \qquad \sum_j n_j = n,$$

is a functional in $\mathcal{F}_n^{(-n)}$. One expects that the relationship between (15) and (16) will be extended to the case where (15) is replaced by (17). Start now with an $\mathcal{H}_n$-functional

$$\varphi = \sum_{i_1,\cdots,i_k} a_{i_1\cdots i_k} \prod_j \left\{ H_{n_j}\left(\frac{\Delta_{i_j}B}{\Delta_{i_j}} ; \frac{1}{\Delta_{i_j}}\right)\Delta_{i_j} \right\},$$

i_j's, j= 1, 2,··, k, different,

where $\Pi = \{\Delta_i\}$ is a partition of R. Then, if

$$C(n) \sum_{i_1,\cdots,i_k} a_{i_1\cdots i_k} \chi_{\Delta_{i_1}}(u_1)\cdots\chi_{\Delta_{i_k}}(u_k), \quad C(n) \text{ constant}$$

approximates $f(u_1,\cdots, u_k)$, one can prove that the integral representation of the $\mathcal{S}$ approaches the functional (17) in the space $\mathcal{F}_n^{(-n)}$.

With this observation in mind, let us introduce notations. Set

$$M_n(dt) = H_n(\dot{B}(t) ; \frac{1}{dt})dt,$$

and set

$$\prod_{j=1}^{k} \circ M_{n_j}(dt_j) \quad \text{or} \quad M_{n_1}(dt_1) \circ \cdots \circ M_{n_k}(dt_k)$$

$$(18) \qquad = \begin{cases} \prod_{j=1}^{k} M_{n_j}(dt_j), & \text{if } t_j\text{'s are different,} \\ 0, & \text{otherwise.} \end{cases}$$

Of course, for a rigorous definition of the product $\Pi\circ$ one must do the same thing as in §2, [II] by using a sequences of partitions Π_n of R with $\delta\Pi_n \to 0$.

By using this notation the above discussion can be summarized as follows :

<u>Theorem</u> 3. <u>The product</u> $\prod_{j=1}^{k} \circ M_{n_j}(dt_j)$, $\Sigma n_j = n$, <u>is a random measure, and the integral with respect to it is in</u> $\mathcal{F}_n^{(-n)}$. <u>In addition,</u>

$$(19) \qquad \mathcal{T}\left(\prod_{j=1}^{k} \circ M_{n_j}(dt_j)\right) = C(n)\prod_j \delta_u^{n_j} du^{n_j},$$

<u>where</u> $\delta_u^n du^n$ <u>is a measure given by</u>

$$\int_{R^n}\cdots\int f(u_1,\cdots, u_n)\delta_u^n du^n = f(u,\cdots, u).$$

Definition 3. The measure defined by (18) is called a generalized random measure.

The integral, although it is symbolic, with respect to a generalized random measure, say (18), is written as

$$\int\cdots\int f(u_1,\cdots, u_k)M_{n_1}(du_1) \circ \cdots \circ M_{n_k}(du_k).$$

A multiplication of generalized random measures can be defined only for special cases. Namely, only multiplication by $M_1(dt)$ is possible.

$$M_n(dt)\cdot M_1(ds) = M_n(dt)\circ M_1(ds) + \delta_{t-s}M_{n-1}(dt)$$

$$(\prod_{j=1}^{k} \circ M_{n_j}(dt_j))\cdot M_1(ds) = \sum_{j=1}^{k} \{M_{n_j}(dt)\cdot M_1(ds) \prod_{i\neq j} \circ M_{n_j}(dt_i)\}.$$

For instance,

$$\int f(u)M_n(du)\cdot\int g(v)M_1(dv)$$

$$=\iint f \otimes g(u, v)M_n(du)\circ M_1(dv) + \int f(u)g(u)M_{n-1}(du).$$

Three remarks are now in order.

Remark 1. Integrals with respect to generalized random measure should not be thought of as definite integral, but they should be viewed as continuous analogues of polynomials. Such a consideration comes from the discussions in §2, [II].

Remark 2. The space $\mathcal{H}_n$ is in agreement with the collection of integrals with respect to $M_1(du_1) \circ \cdots \circ M_1(du_n)$.

Remark 3. The space $\mathcal{H}_n^{(-n)}$ can not be covered by the integrals with respect to generalized random measures of the form (18) with $\Sigma n_j = n$. An example of a member of $\mathcal{H}_n^{(-n)}$ which is not an integral will be presented in the next section. In terms of P. Lévy, a functional in $\tilde{\mathcal{F}}_n$ is said to be regular, and $U(\xi)$ such that $i^n C(\xi)U(\xi)$ is expressed in the form (17) is called a normal functional.

§5. Causal calculus

"Causal calculus" here means differential and integral calculi as well as related operations where the propagation of the time is involved or expressed explicitly. A most suitable base or c.o.n.s. for this calculus has been introduced in §2, and with this choice of a base we have discussed

[I] Integration

in the last section by introducing generalized random measures.

The next topic has to be

[II] Differentiation.

One is interested in a differentiation $\dfrac{d}{d\dot{B}(t)}$, the exact meaning is going to be illustrated in what follows.

By analogy with $\tilde{\mathcal{F}}_n$ a functional space $\tilde{\mathcal{F}}_n^{(n)}$ can be defined and is topologized in such a way that $\tilde{\mathcal{F}}_n^{(n)}$ is isomorphic to $\mathcal{F}_n^{(n)}$. The same for $\tilde{\mathcal{F}}_n^{(-n)} \cong \mathcal{F}_n^{(-n)}$. Let φ be in $\mathcal{H}_n^{(n)}$. Then one can find $U(\xi)$ in $\tilde{\mathcal{F}}_n^{(n)}$ such that $(\mathcal{T}\varphi)(\xi) = i^n C(\xi)U(\xi)$. Take the functional derivative (in the sense of Fréchet). Let it be denoted by $U'_\xi(\xi\ ;\ t)$. It always exists and belongs to $\tilde{\mathcal{F}}_{n-1}^{(n-1)}$ for every t. Applying the transformation $\mathcal{T}^{-1}$ one is given an $\mathcal{H}_{n-1}^{(n-1)}$-functional, call it $\varphi'(t, x)$:

$$\mathcal{T}(i^{n-1}C(\xi)U'_\xi(\xi\ ;\ t))(x) = \varphi'(t,\ x).$$

The mapping $\varphi(x) \to \varphi'(t,\ x)$ is denoted by

(20) $$\frac{d}{d\dot{B}(t)}\varphi(x) = \varphi'(t,\ x).$$

It is a derivative, indeed $\dot{B}(t)$-derivative of φ.

Higher order derivatives like $\dfrac{d^2}{d\dot{B}(t)^2}$, $\dfrac{d^2}{d\dot{B}(t)d\dot{B}(s)}$ can also be defined by using second order functional derivatives. They are simply illustrated by the following.

(21)
$$\begin{array}{ccc} \mathcal{H}_n^{(n)} & & \mathcal{F}_n^{(n)} \\ \dfrac{d^2}{d\dot{B}(t)^2} & \longleftrightarrow & U''_{\xi^2}(\xi\ ;\ t) \\ \dfrac{d^2}{d\dot{B}(t)d\dot{B}(s)} & \longleftrightarrow & U''_{\xi\xi'}(\xi\ ;\ t,\ s). \end{array}$$

<u>Note</u> i) variation δU when ξ varies by $\delta\xi$

a) linear in $\delta\xi$

b) $U(\xi + \delta\xi) - U(\xi) = \delta U + o(\delta\xi)$

functional derivative $U'_\xi(\xi;t)$: $\delta U = \int U'_\xi(\xi;t)\delta\xi(t)dt$

ii) second variation $\delta^2 U$

a) quadratic in $\delta\xi$

b) $U(\xi + \delta\xi) - U(\xi) = \delta U + \frac{1}{2}\delta^2 U + o(\delta\xi)^2$

functional derivatives $U''_{\xi^2}(\xi\ ;\ t)$ and $U''_{\xi\xi_1}(\xi;\ t,\ s)$:

$$\delta U'_\xi(t) = U''_{\xi^2}(\xi;t)\delta\xi(t) + \int U''_{\xi\xi_1}(\xi;t,s)\delta\xi(s)ds.$$

The infinite dimensional <u>Laplacian</u> Δ on $\mathcal{H}_n^{(-n)}$ can now be defined by

$$(22)\quad \begin{cases} \text{domain } \mathscr{D}(\Delta) = \mathcal{J}^{-1}\left\{ i^n C(\xi)U(\xi) \ ; \ U''_{\xi^2},\ U''_{\xi\xi_1} \text{ exist and } U''_{\xi^2}(\xi;t) \text{ is t-integrable} \right\}, \\ \Delta\varphi = \displaystyle\int \frac{d^2}{d\dot{B}(t)^2}\varphi dt. \end{cases}$$

It is straightforward to extend Δ on a certain subset of $\sum_n \mathcal{A}_n^{(-n)}$.

Examples.

i) $\varphi_t(x) = \exp[B(t) - \frac{1}{2}t]$.

$$\frac{d}{d\dot{B}(s)}\varphi_t = \begin{cases} \varphi_t, & 0 < s < t, \\ 0, & s > t. \end{cases}$$

ii) $\frac{d}{d\dot{B}(s)} H_n(B(t) \ ; \ t) = H_{n-1}(B(t) \ ; \ t), \quad s \le t.$

iii) $\varphi = \int f(u)M_2(du)$

$$\frac{d}{d\dot{B}(t)}\varphi = f(t)\dot{B}(t), \qquad \frac{d^2}{d\dot{B}(t)^2}\varphi = f(t)$$

$$\frac{d^2}{d\dot{B}(t)d\dot{B}(s)}\varphi = 0, \qquad \Delta\varphi = \int f(t)dt.$$

Definition 4. If φ is in $\mathscr{D}(\Delta)$, and if

$$\Delta\varphi = 0,$$

then φ is said to be harmonic.

Theorem 4. Every functional in $\mathcal{A}_n$, $n \ge 0$, is harmonic.

[III] Multiplication by $\dot{B}(t)$.

Since the system $\{e^{i\langle x, \eta\rangle}, \eta \in \mathscr{S}\}$ generates the entire (L^2), one start with the product $B(t)e^{i\langle x, \eta\rangle}$, which is approximated by

$$\langle x, \frac{\chi_\Delta}{\Delta}\rangle\, e^{i\langle x, \eta\rangle}.$$

Applying $\mathcal{T}$ to this approximation yields

$$iC(\xi)(\xi + \eta, \frac{\chi_\Delta}{\Delta})\exp[-\frac{1}{2}\|\eta\|^2 - (\xi, \eta)].$$

As $\Delta \to \{t\}$, this tends to

$$iC(\xi)(\xi(t) + \eta(t))\exp[-\frac{1}{2}\|\eta\|^2 - (\xi, \eta)].$$

The result enables us to prove

$$(23) \quad \mathcal{T}(\dot{B}(t)H_n(\langle x,\eta\rangle/\sqrt{2}))(\xi) = i^{n+1}2^{\frac{n}{2}}\{\xi(t)(\xi,\eta)^n + n\eta(t)(\xi, \eta)^{n-1}\}C(\xi),$$

$$\|\eta\| = 1.$$

This means that $\dot{B}(t)H_n(\langle x, \eta\rangle/\sqrt{2})$, $\|\eta\| = 1$, belongs to $\mathcal{H}_{n+1}^{(-n-1)} + \mathcal{H}_{n-1}$ and that the associated kernel of its integral representation is

$$2^{\frac{n}{2}}\{\widehat{\eta^{n\otimes} \otimes \delta_t} - n\eta(t)\eta^{(n-1)\otimes}\}, \quad \widehat{\quad} \text{ symmetrization.}$$

Thus one can prove the following theorem,

Theorem 5. One can multiply functionals in $\mathcal{H}_n^{(n)}$ by $\dot{B}(t)$ and the product is in $\mathcal{H}_{n+1}^{(-n-1)} + \mathcal{H}_{n-1}$.

[IV] Fourier transform.

So far there have been introduced several different kinds of Fourier transform, however none seems to be fitting for our causal calculus.

[References]

[1] P. Lévy, Problèmes concrets d'analyse fonctionelle. Gautier-Villars, 1951.

[2] T. Hida, Stationary stochastic processes. Princeton Univ.

Press, 1970.

[3] ———, Brownian motion (in Japanese). Iwanami Pub. Co. 1975.

[4] ———, Analysis of Brownian functionals. Carleton Univ. lecture notes. 1975.

[5] ———, N. Ikeda, Analysis on Hilbert space with reproducing kernel arising from multiple Wiener integral. Proc. 5th Berkeley Symp. on Math. Statist. and Probability. vol. 2, Part I, 117-143 (1967).