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# On Isometric Structures of 3-Manifolds

By

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## 0. Introduction

As is well-known, J.W.Milnor[12] defined an isometric form of the knot exterior $E(k)$ of a classical tame knot $k \subset S^3$ associated with an element of $H^1(E(k);Z)$ that is specified by the orientations of the knot $k$ and the containing 3-sphere $S^3$. The isometric form, referred to as the quadratic form of the knot $k \subset S^3$, is necessarily non-singular and it was applied for the Fox-Milnor knot cobordism group[3]. The author defined analogously non-singular isometric forms for closed 3-manifolds having the integral homology group of an orientable handle $S^1 \times S^2$ and this isometric form was applied for the $\tilde{H}$-cobordism group $\Omega(S^1 \times S^2)$ of the homology orientable handles.(See [8].)

The main purpose of this note is to define isometric forms for arbitrary, compact,connected and oriented 3-manifolds with non-zero first Betti numbers and to deduce elementary properties of the forms.

In order to define the isometric form, we will need a version of the Blanchfield duality on the infinite cyclic covers of manifolds. (cf. R.C.Blanchfield[1].) Our isometric form will be, in many cases, singular and, as in the link theory(cf. K.Murasugi[13], F.Hosokawa[4].), we will define the Murasugi signature, the ( one variable)Alexander polynomial, the Hosokawa polynomial and the nullity.(However, it should be noted that the concepts of the Murasugi signature and the nullity of our form are strictly distinct from the original concepts of K.Murasugi[13].) The isometric forms and hence these invariants will be seen to be closely related to the various (restricted) cobordism problems of 3-manifolds.

The note will contain typical three applications. The first will concern a polynomial condition for a finitely presented group with the first Betti number one to be a 3-manifold group. For example, we shall show that the finitely presented group $G_{(p,q)}=( a, b: a^{-1}b^{p}a = b^{q} )$ is a 3-manifold group if and only if $pq = 0$ or $|p| = |q|$. This answers a question of William Jaco[5,Question 13]. The second will concern a codimension one piecewise-linear embedding of a 3-manifold into a 4-manifold. For example, we shall show that, for an orientable torus bundle $M$ over $S^1$, the bundle projection $p: M \to S^1 = S^1 \times * \subset S^1 \times S^3$ is not homotopic to any piecewise-linear embedding except for two possible cases. The third will concern a codimension two (possibly non-locally flat) piecewise-linear embedding of the disjoint union of 2-spheres in a 4-manifold. For example, we shall present a generalized version of an example shown by Y.Matsumoto[11] by using an invariant of R.A.Robertello[15]. That is, we shall show that there exist simply connected piecewise-linear 4-manifolds $W$ such that any basis of $H_2(W;Z)$ can not be represented by disjoint, piecewise-linearly embedded 2-spheres, though each two elements of $H_2(W;Z)$ has the intersection number zero and , for some basis of $H_2(W;Z)$, each generator can be represented by a locally flat 2-sphere.

In Section one we will state and prove Duality Theorems(I) and (II). In Section two we will define isometric forms for 3-manifolds and their related invariants, the Murasugi signature, the Alexander polynomial, the Hosokawa polynomial and the nullity. Section three will study the relationship between the invariants of our form and the cobordisms of 3-manifolds. Final Section is the applications on the three topics.

Throughout the note, spaces will be considered from a piecewise-linear point of view.

## 1. Duality Theorems

Let $F$ be a field and $<t>$ be the infinite cyclic ( multicative ) group generated by $t$. By $F[t]$ we denote the group algebra of $<t>$ over $F$. It is easy to see that $F[t]$ is a principal ideal domain.

Consider a compact, connected, piecewise-linear n-manifold $M^n$ with an epimorphism $\gamma: \pi_1(M^n) \to <t>$ and let $\tilde{M}^n$ be the infinite cyclic cover of $M^n$ associated with $\gamma$. It is easily checked that the homology F-modules $H_*(\tilde{M}^n;F)$ and $H_*(\tilde{M}^n,\partial\tilde{M}^n;F)$ form finitely generated $F[t]$-modules, since the original manifold $M^n$ is compact and $F[t]$ is a principal ideal domain. We denote by $\beta_*(M^n;F)$ and $\beta_*(M^n, M^n;F)$, respectively, the $F[t]$-ranks of the finitely generated homology $F[t]$-modules $H_*(\tilde{M}^n;F)$ and $H_*(\tilde{M}^n,\partial\tilde{M}^n;F)$, referred to as *the* $F[t]$-*Betti numbers of* $\tilde{M}^n$ *and* $(\tilde{M}^n, \partial\tilde{M}^n)$. By $T_*(\tilde{M}^n;F)$ and $T_*(\tilde{M}^n,\partial\tilde{M}^n;F)$, respectively, we denote the $F[t]$-torsion parts of $H_*(\tilde{M}^n;F)$ and $H_*(\tilde{M}^n,\partial\tilde{M}^n;F)$, referred to as *the homology* $F[t]$-*torsion modules of* $\tilde{M}^n$ *and* $(\tilde{M}^n,\partial\tilde{M}^n)$.

The cohomology $F$-modules $H^*(\tilde{M}^n;F)$ and $H^*(\tilde{M}^n,\partial\tilde{M}^n;F)$ form $F[t]$-modules that are not always finitely generated $F[t]$-modules. Define the finitely generated $F[t]$-torsion modules $T^*(\tilde{M}^n;F)$ and $T^*(\tilde{M}^n,\partial\tilde{M}^n;F)$ by the identities:

$$T^*(\tilde{M}^n;F) = \mathrm{Hom}_F[T_*(\tilde{M}^n;F),F]$$

$$T^*(\tilde{M}^n,\partial\tilde{M}^n;F) = \mathrm{Hom}_F[T_*(\tilde{M}^n,\partial\tilde{M}^n;F),F].$$

Further, we define $B_* = H_*/T_*$ and $B^* = \mathrm{Hom}_F[B_*,F]$. There are split short exact sequences of $F[t]$-modules:

$$0 \to T_* \longrightarrow H_* \longrightarrow B_* \longrightarrow 0$$

$$0 \to B^* \longrightarrow H^* \longrightarrow T^* \longrightarrow 0.$$

(Note that these sequences are canonical, but not *canonically split*.)

1.1. *Duality Theorems*. *Suppose* $\tilde{M}^n$ *is orientable over* $F$. *Then*,

(I) *For all* $i$, $\beta_i(\tilde{M}^n;F) = \beta_{n-i}(\tilde{M}^n,\partial\tilde{M}^n;F)$

(II) $T_{n-1}(\tilde{M}^n,\partial\tilde{M}^n;F) \approx F$, *and for a generator* $\mu$ *of* $T_{n-1}(\tilde{M}^n,\partial\tilde{M}^n;F)$ *and all* $i$, *the cup product with* $\mu$ $\cap\mu : H^i(\tilde{M}^n;F) \longrightarrow H_{n-i-1}(\tilde{M}^n,\partial\tilde{M}^n;F)$ *induces an isomorphism*

$$\cap\mu : T^i(\tilde{M}^n;F) \approx T_{n-i-1}(\tilde{M}^n,\partial\tilde{M}^n;F).$$

Duality Theorems (I) and (II) have been essentially known by R.C.Blanchfield[1]. However, the formulation of (II) is near to J.W.Milnor's[12] rather than the original R.C.Blanchfield's.

1.2. Remark. In Duality Theorems (I) and (II), the assumption of the triangulation of $M^n$ can be actually removed by using a method analogous to that of the author's paper[9].

1.3. Remark. In Duality Theorem (II), $t$ acts on $T_{n-1}(\tilde{M}^n, \partial\tilde{M}^n; F) \approx F$ as the identity map or the (-1)-multiple map according as the original manifold $M^n$ is orientable or non-orientable over $F$.

1.4. Proof of Duality Theorems. Take a triangulation of $M^n$ and choose a basis for the free $F[t]$-module $C_i(\tilde{M}^n;F)$ with one generator for each i-cell in $M^n$. Then it is easily done to set up the identification as $F[t]$-modules of the finite cochain complex $\mathrm{Hom}_c[C_*(\tilde{M}^n;F),F]$ with the cochain complex $\mathrm{Hom}_{F[t]}[C_*(\tilde{M}^n;F),F[t]]$. Hence we have an isomorphism $H^*_c(\tilde{M}^n;F) \approx H^*_{F[t]}(\tilde{M}^n;F[t])$ as $F[t]$-modules, where $H^*_{F[t]}(\tilde{M}^n;F[t])$ denotes the cohomology of the cochain complex $\mathrm{Hom}_{F[t]}[C_*(\tilde{M}^n;F),F[t]]$. Using the principal ideal domain $F[t]$, from the universal coefficient theorem, we obtain a short exact sequence

$$0 \to \mathrm{Ext}_{F[t]}[H_{i-1}(\tilde{M}^n;F),F[t]] \to H^*_{F[t]}(\tilde{M}^n;F[t]) \to \mathrm{Hom}_{F[t]}[H_i(\tilde{M}^n;F),F[t]] \to 0$$

for all i. In particular, we have

$$\begin{aligned}\beta_i(\tilde{M}^n;F) &= \mathrm{rank}_{F[t]}\mathrm{Hom}_{F[t]}[H_i(\tilde{M}^n;F),F[t]] \\ &= \mathrm{rank}_{F[t]}H^i_{F[t]}(\tilde{M}^n;F[t]) \\ &= \mathrm{rank}_{F[t]}H^i_c(\tilde{M}^n;F)\end{aligned}$$

and

$$\dim_F T_{i-1}(\widetilde{M}^n;F) = \dim_F \mathrm{Ext}_{F[t]}[H_{i-1}(\widetilde{M}^n;F),F[t]]$$
$$= \dim_F \mathrm{Tor}_{F[t]}(H^i_{F[t]}(\widetilde{M}^n;F[t]))$$
$$= \dim_F \mathrm{Tor}_{F[t]}(H^i_c(\widetilde{M}^n;F)).$$

On the other hand, we obtain the following commutative (up to sign) square of isomorphisms

$$\begin{array}{ccc} H^i_c(\widetilde{M}^n;F) & \xrightarrow[\approx]{\cap[\widetilde{M}^n]} & H_{n-i}(\widetilde{M}^n,\partial\widetilde{M}^n;F) \\ \approx\downarrow t & & t^{-1}\downarrow\approx \\ H^i_c(\widetilde{M}^n;F) & \xrightarrow[\approx]{\cap[\widetilde{M}^n]} & H_{n-i}(\widetilde{M}^n,\partial\widetilde{M}^n;F) \end{array}$$

, where $[\widetilde{M}^n]$ is a generator of the nth infinite homology F-module $H^{\inf}_n(\widetilde{M}^n,\partial\widetilde{M}^n;F) \approx F$. Therefore we have

$$\beta_i(\widetilde{M}^n;F) = \mathrm{rank}_{F[t]} H^i_c(\widetilde{M}^n;F)$$
$$= \mathrm{rank}_{F[t]} H_{n-i}(\widetilde{M}^n,\partial\widetilde{M}^n;F)$$
$$= \beta_{n-i}(\widetilde{M}^n,\partial\widetilde{M}^n;F).$$

This proves Duality Theorem (I).

Next, choose compact submanifolds $M_i$, $-\infty < i < \infty$, in $\widetilde{M}^n$ with $M_i \cap M_{i+1}$ a compact submanifold and such that $\widetilde{M}^n = \cup_{i=-\infty}^{\infty} M_i$ and $t(M_i) = M_{i+1}$. (cf. J.W.Milnor[12], A.Kawauchi[7],[9].) By [7,Theorem 1.1], we can assume that both $N_p = M_p \cup M_{p+1} \cup \ldots$ and $N'_q = M_{-q} \cup M_{-q-1} \cup \ldots$ are connected for all integers p and q. Let $j_p : \widetilde{M}^n \subset (\widetilde{M}^n, N_p)$ and $j'_q : \widetilde{M}^n \subset (\widetilde{M}^n, N'_q)$ be the inclusions. By considering the Mayer-Vietoris sequence of the triple $(\widetilde{M}^n; N_p, N'_q)$ and by taking the direct limit $p,\ q \to +\infty$, we obtain the following exact sequence

$$\longrightarrow H^i_c(\widetilde{M}^n;F) \longrightarrow \lim_{\to}\{H^i(\widetilde{M}^n,N_p;F)\} \oplus \{H^i(\widetilde{M}^n,N'_q;F)\} \xrightarrow{\{j^*_p\}+\{j'^*_q\}}$$
$$H^i(\widetilde{M}^n;F) \xrightarrow{\delta} H^{i+1}_c(\widetilde{M}^n;F) \longrightarrow \cdots\cdots .$$

Now we need the following lemma:

1.5. <u>Lemma</u>. $\operatorname{Im}\{j_p^*\} + \{j_q'^*\} \subset B^i(\tilde{M}^n;F)$ <u>for all</u> $i$.

The proof will be given later.

Lemma 1.5 tells us that the split exact sequence

$$0 \to B^i(\tilde{M}^n;F) \to H^i(\tilde{M}^n;F) \to T^i(\tilde{M}^n;F) \to 0$$

also induces a split exact sequence

$$0 \to B^i(\tilde{M}^n;F)/\operatorname{Im}\{j_p^*\}+\{j_q'^*\} \to H^i(\tilde{M}^n;F)/\operatorname{Im}\{j_p^*\}+\{j_q'^*\} \to T^i(\tilde{M}^n;F) \to 0.$$

Since the homomorphism $\delta: H^i(\tilde{M}^n;F) \to H_c^{i+1}(\tilde{M}^n;F)$ induces a monomorphism $\delta': H^i(\tilde{M}^n;F)/\operatorname{Im}\{j_p^*\} + \{j_q'^*\} \to H_c^{i+1}(\tilde{M}^n;F)$ and $\dim_F T^i(\tilde{M}^n;F) = \dim_F \operatorname{Tor}_{F[t]}[H_c^{i+1}(\tilde{M}^n;F)]$, from the above split exact sequence, we obtain a canonical isomorphism $\delta'': T^i(\tilde{M}^n;F) \to \operatorname{Tor}_{F[t]}[H_c^{i+1}(\tilde{M}^n;F)]$. Combined with the isomorphism $\cap[\tilde{M}^n]: \operatorname{Tor}_{F[t]}[H_c^{i+1}(\tilde{M}^n;F)] \to T_{n-i-1}(\tilde{M}^n,\partial\tilde{M}^n;F)$, we obtain the composite isomorphism $\cap[\tilde{M}^n]\circ\delta'': T^i(\tilde{M}^n;F) \to T_{n-i-1}(\tilde{M}^n,\partial\tilde{M}^n;F)$ for all $i$. For a unit $1 \in H^0(\tilde{M}^n;F)=T^0(\tilde{M}^n;F)$, we let $\mu=\delta''(1)\cap[\tilde{M}^n] \in T_{n-1}(\tilde{M}^n,\partial\tilde{M}^n;F)$. It is immediate to see that the isomorphism $\cap[\tilde{M}^n]\circ\delta'': T^i(\tilde{M}^n;F) \simeq T_{n-i-1}(\tilde{M}^n,\partial\tilde{M}^n;F)$ is induced from the cap product with $\mu$ $\cap\mu: H^i(\tilde{M}^n;F) \to H_{n-i-1}(\tilde{M}^n,\partial\tilde{M}^n;F)$. This proves Duality Theorem (II).

1.6. <u>Proof of Lemma 1.5</u>. Let $T_{i(p)}$ and $T'_{i(q)}$ be the images of the homomorphisms $j_{p*}: T_i(\tilde{M}^n;F) \to H_i(\tilde{M}^n,N_p;F)$ and $j'_{q*}: T_i(\tilde{M}^n;F) \to H_i(\tilde{M}^n,N'_q;F)$. For $p' \leq p$ and $q' \leq q$ let $\lambda_p^{p'}: H_i(\tilde{M}^n,N_p;F) \to H_i(\tilde{M}^n,N_{p'};F)$ and $\lambda_q'^{q'}: H_i(\tilde{M}^n,N'_q;F) \to H_i(\tilde{M}^n,N'_{q'};F)$ be the canonical homomorphisms. Since $tT_i(\tilde{M}^n;F) = T_i(\tilde{M}^n;F)$, we obtain $t^s T_{i(p)} = T_{i(p+s)}$ and $t^{-s}T'_{i(q)} = T'_{i(q+s)}$ for all $s$.

Using that $T_{i(0)}$ and $T'_{i(0)}$ are finite-dimensional over F, there exists $s > 0$ such that $\lambda_0^{-s}(T_{i(0)})=0$ and $\lambda'^{-s}_0(T'_{i(0)})=0$ (cf. [7],[12].). Naturality , then, implies that for all p and q $\lambda^p_{p+s}(T_{i(p+s)}) = 0$ and $\lambda'^q_{q+s}(T'_{i(q+s)})=0$. Let $x^* \in \mathrm{Im}\{j^*_p\} + \{j'^*_q\}$ and write $x^* = \{j^*_p\}\{x^*_p\} + \{j'^*_q\}\{x'^*_q\} = j^*_p(x^*_p) + j'^*_q(x'^*_q)$. For all $y \in T_i(M;F)$, we have

$$\begin{aligned} x^*(y) &= x^*_p(j_{p*}(y)) + x'^*_q(j'_{q*}(y)) \\ &= x^*_{p-s}[\lambda^{p-s}_p(j_{p*}(y))] + x'^*_{q-s}[\lambda'^{q-s}_q(j'_{q*}(y))] \\ &= 0. \end{aligned}$$

Hence $x^* \in B^i(M^n;F)$. This completes the proof.

1.7. <u>Remark</u>. Consider a non-trivial homomorphism $\gamma: \pi_1(M^n) \to \langle t \rangle$ with order$[\langle t \rangle / \mathrm{Im}\,\gamma] = d$. The infinite cyclic cover $\widetilde{M}^n$ of $M^n$ associated with $\gamma$ has d components $\widetilde{M}_0, \widetilde{M}_1, \ldots, \widetilde{M}_{d-1}$ such that $t(\widetilde{M}_{j-1}) = \widetilde{M}_j$, j= 1, 2,..., d-1, and $t(\widetilde{M}_{d-1}) = \widetilde{M}_0$. Each component $\widetilde{M}_j$ is the infinite cyclic cover of $M^n$ associated with $\gamma': \pi_1(M^n) \to \mathrm{Im}\,\gamma$ . For simplicity, we assume $M^n$ is orientable over F. For each j, Duality Theorems (I) and (II) imply $\beta_i(\widetilde{M}^n_j;F) = \beta_{n-i}(\widetilde{M}^n_j, \partial\widetilde{M}^n_j;F)$ and $\cap\mu_j: T^i(\widetilde{M}^n_j;F) \approx T_{n-i-1}(\widetilde{M}_j, \partial\widetilde{M}_j;F)$, where we choose $\mu_0, \mu_1, \ldots, \mu_{d-1}$ so that $t\mu_{j-1} = \mu_j$, j = 1,2,...,d-1 and $t\mu_{d-1} = \mu_0$. Let $\mu = \mu_1 + \ldots + \mu_{d-1} \in T_{n-1}(\widetilde{M}^n, \partial\widetilde{M}^n;F)$. Duality Theorem in this case may be also formulated as follows:

(I) For all i, $\beta_i(\widetilde{M}^n;F) = \beta_{n-i}(\widetilde{M}^n, \partial\widetilde{M}^n;F)$

(II) For all i, $\cap\mu : T^i(\widetilde{M}^n;F) \approx T_{n-i-1}(\widetilde{M}^n, \partial\widetilde{M}^n;F)$.

## 2. Isometric Forms of 3-Manifolds

Consider a compact, connected and oriented 3-manifold $M$ with an epimorphism $\gamma : \pi_1(M) \to \langle t \rangle$. For the infinite cyclic cover $\tilde{M}$ of $M$ associated with $\gamma$, Duality Theorem (II) with rational coefficients $Q$ asserts a duality

$$\cap\mu : T^1(\tilde{M};Q) \approx T_1(\tilde{M},\partial\tilde{M};Q).$$

This implies that the cup product $\cup : T^1(\tilde{M};Q) \times T^1(\tilde{M},\partial\tilde{M};Q) \to T^2(\tilde{M},\partial\tilde{M};Q)$ $(\overset{\cap\mu}{\approx} T_0(\tilde{M};Q) = Q)$ is a dual pairing. From the exact sequence of the pair $(\tilde{M},\partial\tilde{M})$, we obtain the following commutative (up to sign) diagram of <u>semi-exact</u> sequences:

$$\begin{array}{ccccccccccc}
T^0(\tilde{M}) & \xrightarrow{i^*} & T^0(\partial\tilde{M}) & \xrightarrow{\delta} & T^1(\tilde{M},\partial\tilde{M}) & \xrightarrow{j^*} & T^1(\tilde{M}) & \xrightarrow{i^*} & T^1(\partial\tilde{M}) & \xrightarrow{\delta} & T^2(\tilde{M},\partial\tilde{M}) \\
\approx\downarrow\cap\mu & & \approx\downarrow\cap\mu' & & \approx\downarrow\cap\mu & & \approx\downarrow\cap\mu & & \approx\downarrow\cap\mu' & & \approx\downarrow\cap\mu \\
T_2(\tilde{M},\partial\tilde{M}) & \xrightarrow{\partial} & T_1(\partial\tilde{M}) & \xrightarrow{i_*} & T_1(\tilde{M}) & \xrightarrow{j_*} & T_1(\tilde{M},\partial\tilde{M}) & \xrightarrow{\partial} & T_0(\partial\tilde{M}) & \xrightarrow{i_*} & T_0(\tilde{M}).
\end{array}$$

We may have a ( generally singular) skew-symmetric cup product pairing $\cup : T^1(\tilde{M},\partial\tilde{M}) \times T^1(\tilde{M},\partial\tilde{M}) \to T^2(\tilde{M},\partial\tilde{M}) \approx Q$ such that the following triangle is commutative:

$$\begin{array}{ccc}
T^1(\tilde{M},\partial\tilde{M}) \times T^1(\tilde{M},\partial\tilde{M}) & \searrow & \\
\downarrow j^* \times \mathrm{id.} & & T^2(\tilde{M},\partial\tilde{M}) \approx Q. \\
T^1(\tilde{M}) \times T^1(\tilde{M},\partial\tilde{M}) & \nearrow &
\end{array}$$

2.1.<u>Definition</u>. The ideal order $A_\gamma(t)$ of $T_1(\tilde{M};Q)$ as a $Q[t]$-module is called the (<u>one-variable</u>) <u>Alexander polynomial</u> of $M$ with $\gamma$. Further, the ideal order $h_\gamma(t)$ of $T_1(\tilde{M};Q)/\mathrm{Im}\, i_*$ is called the <u>Hosokawa polynomial</u> of $M$ with $\gamma$. [Note that the Alexander polynomial is defined to be a non-zero polynomial. This attitude is near to that of R.C.Blanchfield[1]. If $M$ is without boundary, the Hosokawa polynomial is the Alexander polynomial.]

Now we let $H$ be the quotient $Q[t]$-module $T^1(\widetilde{M},\partial\widetilde{M})/\mathrm{Im}\,\delta$. It follows from the above diagram that the ideal order of $H$ is $h_\gamma(t^{-1})$. [Use the equality $(tu)\cap\mu = t^{-1}(u\cap\mu)$ and the duality $\cap\mu : H \approx T_1(\widetilde{M})/\mathrm{Im}\, i_*$.] The skew-symmetric pairing $U: T^1(\widetilde{M},\partial\widetilde{M}) \times T^1(\widetilde{M},\partial\widetilde{M}) \to Q$ induces a skew-symmetric pairing $U: H \times H \to Q$, since $j^*\delta = 0$.

We say that the pair $(M,\gamma)$ is <u>admissible</u> if the boundary $\partial M$ is empty or the union of tori of genera one, for each component $N$ of which the homomorphism $\gamma^*: \pi_1(N) \to \langle t \rangle$ induced from $\gamma$ is non-trivial.

2.2. <u>Lemma</u>. <u>For any admissible pair</u> $(M,\gamma)$, <u>the row sequences of the following diagram</u>

$$\begin{array}{ccccccc} T^0(\partial\widetilde{M}) & \xrightarrow{\delta} & T^1(\widetilde{M},\partial\widetilde{M}) & \xrightarrow{j^*} & T^1(\widetilde{M}) & \xrightarrow{i^*} & T^1(\partial\widetilde{M}) \\ \approx\downarrow\cap\mu & & \approx\downarrow\cap\mu & & \approx\downarrow\cap\mu & & \approx\downarrow\cap\mu \\ T_1(\partial\widetilde{M}) & \xrightarrow{i_*} & T_1(\widetilde{M}) & \xrightarrow{j_*} & T_1(\widetilde{M},\partial\widetilde{M}) & \xrightarrow{\partial} & T_0(\widetilde{M}) \end{array}$$

<u>are exact</u>.

Proof. Since $T_1(\partial\widetilde{M}) = H_1(\partial\widetilde{M};Q)$, it follows from the exact sequence of the pair $(\widetilde{M},\partial\widetilde{M})$ that the lower row sequence is exact at $T_1(\widetilde{M})$ and $T_1(\widetilde{M},\partial\widetilde{M})$. Hence the upper row sequence is also exact at $T^1(\widetilde{M},\partial\widetilde{M})$ and $T^1(\widetilde{M})$. This completes the proof.

For simplicity, we will assume the pair $(\widetilde{M},\gamma)$ is admissible, throughout the section.

2.3. <u>Corollary</u>. <u>The skew-symmetric pairing</u> $H \times H \xrightarrow{U} Q$ <u>is non-singular</u>.

Proof. By Lemma 2.2, we have $\mathrm{Ker}\, j^* = \mathrm{Im}\, \delta$. Hence the homomorphism $H = T^1(\tilde{M},\partial\tilde{M})/\mathrm{Ker}\, j^* \longrightarrow T^1(\tilde{M})$ induced from $j^*$ is injective. Since the cup product $\cup: T^1(\tilde{M}) \times T^1(\tilde{M},\partial\tilde{M}) \to T^2(\tilde{M},\partial\tilde{M}) = Q$ is non-singular, the desired result follows.

2.4. Corollary. The Alexander polynomial $A_\gamma(t)$ is a reciprocal polynomial $A_\gamma(t) \doteq A_\gamma(t^{-1})$ and the Hosokawa polynomial $h_\gamma(t)$ is a reciprocal polynomial with even degree.

Proof. From Corollary 2.3, it follows that $\dim_Q H = \deg h_\gamma(t)$ is even and $h_\gamma(t^{-1}) \doteq h_\gamma(t)$. [Note that $tu \cup tv = u \cup v$ for all $u$, $v \in H$.] Further, the ideal order of $\mathrm{Im}\,\delta$ is reciprocal, since the ideal order of $T^0(\partial\tilde{M})$ is the product of a type $t^\lambda - 1$. So, the ideal order of $T^1(\tilde{M},\partial\tilde{M})$ is reciprocal, which implies $A_\gamma(t) \doteq A_\gamma(t^{-1})$. This completes the proof.

Define a bilinear form $\langle\ ,\ \rangle: T^1(\tilde{M},\partial\tilde{M}) \times T^1(\tilde{M},\partial\tilde{M}) \longrightarrow Q$ by the identity $\langle x,y\rangle = (x \cup ty + y \cup tx) \cap \mu$. (cf. J.W.Milnor[12], D.Erle[2], A.Kawauchi[8].) It is clear that $\langle x,y\rangle = \langle y,x\rangle$ and $\langle tx,ty\rangle = \langle x,y\rangle$ for all $x$, $y \in T^1(\tilde{M},\partial\tilde{M})$.

2.5. Definition. The pair $(\langle,\rangle,t)$ is called the isometric form of the oriented $M$ with the epimorphism $\gamma: \pi_1(M) \to \langle t\rangle$.

2.6. Definition. The signature of the form $\langle\ ,\ \rangle$ is called the Murasugi signature of the oriented $M$ with $\gamma$.

2.7. <u>Definition</u>. The <u>nullity</u> $n_\gamma(M)$ of $M$ with $\gamma$ is defined by the equality

$$n_\gamma(M) = \begin{cases} \dim_Q T^1(\tilde{M},\partial\tilde{M}) - \mathrm{rank}_Q\langle\ ,\ \rangle + 1 & \text{if } \partial M \neq \emptyset \\ \dim_Q T^1(\tilde{M}) - \mathrm{rank}_Q\langle\ ,\ \rangle & \text{if } \partial M = \emptyset . \end{cases}$$

Let $A$ be the $Q[t]$-submodule of $H$ consisting of elements $x$ with $(t-1)(t-1)x = 0$. Define $\hat{H} = H/A$. Then,

2.8. <u>Lemma</u>. <u>The isometric form</u> $(\langle\ ,\ \rangle, t)$ <u>induces a non-singular isometric form</u> $(\langle\ ,\ \hat{\rangle}, t): \hat{H} \times \hat{H} \longrightarrow Q$.

Proof. For all $y \in H$, $\langle x,y\rangle = 0$ if and only if $t(t-1)(t+1)x=0$ by Corollary 2.3 if and only if $x \in A$. This completes the proof.

Now consider an oriented tame link $\ell$ in the oriented 3-sphere $S^3$. Let $\ell$ have $\lambda$ components. By $E(\ell)$, we denote the exterior of the link (i.e. the closed link complement in $S^3$). The orientations of $\ell$ and $S^3$ specify a canonical basis of $H_1(E(\ell);Z)$. Choose an epimorphism $\gamma : \pi_1(E(\ell)) \twoheadrightarrow \langle t\rangle$ determined by sending each generator of the basis of $H_1(E(\ell);Z)$ to $t$. By $A_\ell(t)$, $h_\ell(t)$, $\sigma(\ell)$ and $n(\ell)$, respectively, we denote the Alexander polynomial, the Hosokawa polynomial, the Murasugi signature and the nullity of $E(\ell)$ with $\gamma$.

The following was first noticed by F.Hosokawa[4]:

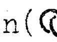
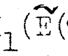
2.9. <u>Theorem</u>. <u>Suppose</u> $\beta_1(\tilde{E}(\ell);Q) = 0$. <u>Then</u> $A_\ell(t) = h_\ell(t)(t-1)^{\lambda-1}$.

Proof. Since the pair $(E(\ell),\gamma)$ is admissible and $\beta_1(\tilde{E}(\ell);Q)=0$, it follows from the exact sequence of the pair $(\tilde{E}(\ell),\partial\tilde{E}(\ell))$ that the sequence $0 \to T_2(\tilde{E}(\ell),\partial\tilde{E}(\ell)) \to T_1\partial\tilde{E}(\ell) \xrightarrow{i_*} T_1\tilde{E}(\ell) \xrightarrow{j_*} T_1(\tilde{E}(\ell),\partial\tilde{E}(\ell)) \longrightarrow$

$T_0(\partial\widetilde{E}(\ell)) \xrightarrow{i_*} T_0(\widetilde{E}(\ell)) \to 0$ is exact. [ Use the dual sequence.] Since $T_2(\widetilde{E}(\ell), \partial\widetilde{E}(\ell)) = Q[t]/t-1$ and $T_1(\partial\widetilde{E}(\ell)) = Q[t]/(t-1)^{\lambda}$ , the equality $A_\ell(t) = h_\ell(t)(t-1)^{\lambda-1}$ directly follows. This completes the proof.

2.10. Remark. $\beta_1(\widetilde{E}(\ell);Q) = 0$ if and only if the ideal order of $H_1(\widetilde{E}(\ell);Q)$ is not zero if and only if $A_\ell(t)$ is the ideal order of $H_1(\widetilde{E}(\ell);Q)$.

2.11. Corollary. Suppose $\beta_1(\widetilde{E}(\ell);Q) = 0$. Then $n(\ell) = \dim_Q A + \lambda$.

2.12. Remark. From Theorem 2.9, we see that the Alexander polynomial and the Hosokawa polynomial are the generalizations of the usual concepts. However, the Murasugi signature and the nullity are different from the concepts of K.Murasugi[13]. For example, $\sigma$() = 0 and n() = 2, since $H_1(\widetilde{E}$()$;Q) = Q[t]/t-1$. On the other hand, (Classical Murasugi signature)() = $\pm 1$ and (Classical nullity)() = 1. [ It seems that our signature is related to the signature $\xi$ defined by K. Murasugi in [14].] One may note that if $\beta_1(E(\ell);Q) = 0$, then $n(\ell) \geq \lambda$ , though, for an arbitrary link, (Classical nullity) $\leq$ (the number of the components). (See [13, Lemma 6.1].) For a trivial link $O^\lambda$ with $\lambda$ components, we have $\sigma(O^\lambda)$ = (Classical Murasugi signature)$(O^\lambda) = 0$, $n(O^\lambda) = 1$ $(\leq \lambda)$ and (Classical nullity)$(O^\lambda) = \lambda$.

## 3. Cobordisms between 3-Manifolds

The following two theorems are basically important:

3.1. *Theorem. Let* $M$ *be a closed, connected, oriented 3-manifold with an epimorphism* $\gamma : \pi_1(M) \longrightarrow \langle t \rangle$.

*Suppose* $M$ *is the boundary of a compact, connected and oriented 4-manifold* $W$ *such that*

(1) *There is an epimorphism* $\bar{\gamma} : \pi_1(W) \rightarrow \langle t \rangle$ *such that the triangle*

$$\begin{array}{ccc} \pi_1(W) & \xrightarrow{\bar{\gamma}} & \\ \uparrow \text{inclusion} & & \langle t \rangle \\ \pi_1(M) & \xrightarrow{\gamma} & \end{array}$$

*is commutative,*

(2) *For the infinite cyclic cover* $\tilde{W}$ *of* $W$ *associated with* $\bar{\gamma}$, $H_2(\tilde{W},\tilde{M};Q)$ *is a torsion* $Q[t]$*-module.*

*Then we have* $\sigma_\gamma(M) = 0$ *and* $\hat{h}_\gamma(t) \doteq f(t)f(t^{-1})$, *where* $\hat{h}_\gamma(t)$ *is the ideal order of* $\hat{H}$.

3.2. *Theorem. Let* $K_i$, $i = 1,2$, *be connected finite complexes with* $\operatorname{rank} H_1(K_i;Z) \geq 1$.

*Suppose there exists a finite connected complex* $L$ *such that*

(1) *The disjoint union* $K_1 \cup K_2$ *is embedded in* $L$,

(2) $H_j(L,K_1;Z) = H_j(L,K_2;Z) = 0$, $j = 1,2$.

*Then for all compatible epimorphisms* $\gamma_i : \pi_1(K_i) \rightarrow \langle t \rangle$, $i=1,2$, $B_1$ *is isomorphic to* $B_2$ *as* $Q[t]$*-modules, where* $B_i$ *is the submodule of* $T_1(K_i;Q)$ *consisting of elements annihilated by some* multiple of $t-1$ or $t+1$.

3.3. Proof of Theorem 3.1. Consider the following commutative ( up to sign ) diagram:

$$\begin{array}{ccccc} T^1(\tilde W) & \xrightarrow{i^*} & T^1(\tilde M) & \xrightarrow{\delta} & H^2(\tilde W,\tilde M;Q) \\ \cong\big\downarrow \cap\tilde\mu & & \cong\big\downarrow\cap\mu & & \cong\big\downarrow\cap\tilde\mu \\ H_2(\tilde W,\tilde M;Q) & \xrightarrow{\partial} & T_1(\tilde M) & \xrightarrow{i_*} & T_1(\tilde W). \end{array}$$

It should be noted that the bottom sequence is exact at $T_1(\tilde M)$ and ,consequently, that the top sequence is exact at $T^1(\tilde M)$, because the vertical homomorphisms are isomorphisms by Duality Theorem (II). Denote by $\hat S$ the image of a subset $S$ of $H$ under the map $H \to \hat H = H/A$, where $H = T^1(\tilde M)$. Suppose, for all $\widehat{i^*(x)} \in \operatorname{Im}\widehat{i^*}$ $(\subset \hat H)$, $\langle\widehat{i^*(x)},\hat y\rangle = 0$. This situation is equivalent to $\delta(t-t^{-1})y = 0$ i.e. $(t-t^{-1})y \in \operatorname{Im} i^*$, because

$$\begin{aligned}\langle\widehat{i^*(x)},\hat y\rangle = \langle i^*(x),y\rangle &= (i^*(x)\cup(t-t^{-1})y)\cap\mu \\ &= (x\cup (t-t^{-1})y)\cap\tilde\mu \\ &= 0 \quad \text{for all} \quad x \in T^1(\tilde W).\end{aligned}$$

Hence we have $(t-t^{-1})\hat y \in \operatorname{Im}\widehat{i^*}$. However, $t-t^{-1} = t^{-1}(t-1)(t+1) : \hat H \to \hat H$ is a monomorphism and hence an isomorphism, because $\dim_Q H < +\infty$. Thus we have $\hat y \in \operatorname{Im}\widehat{i^*}$, since $(t-t^{-1})\operatorname{Im}\widehat{i^*} \subset \operatorname{Im}\widehat{i^*}$. Therefore, $\operatorname{Im}\widehat{i^*}$ is an orthogonal complement of $\operatorname{Im}\widehat{i^*}$ itself under the non-singular isometric form $\hat{\langle\ ,\ \rangle}$. This implies that $\sigma_\gamma(M) = 0$ and $\hat h_\gamma(t) = f(t)f(t^{-1})$. (cf. J.Levine[10].) This completes the proof.

3.4. Proof of Theorem 3.2. Consider an epimorphism $\gamma : \pi_1(L) \to \langle t\rangle$ such that the "restricted" homomorphism $\gamma|\pi_1(K_i) = \gamma_i : \pi_1(K_i) \to \langle t\rangle$ is also an epimorphism, $i = 1, 2$. It is always possible, since there is an inclusion isomorphism $H_1(M_i;Z) \cong H_1(L;Z)$, $i = 1, 2$.

Let $\tilde{L}$ be the infinite cyclic cover of $L$ associated with $\gamma$. Decompose $T_1(\tilde{K}_i;Q)$ into $B_i \oplus T_i'$, where $T_i'$ is the submodule that contains no elements annihilated by $t-1$ or $t+1$. By the Wang exact sequence (cf.[9].), we obtain an isomorphism $t-1: H_2(\tilde{L},\tilde{K}_i;Z) \approx H_2(\tilde{L},\tilde{K}_i;Z)$, since $H_2(L,K_i;Z) = 0$ and $H_2(\tilde{L},\tilde{K}_i;Z)$ is a finitely generated $Z[t]$-module. Let $\mathfrak{M}_i(t)$ be a presentation matrix of $H_2(\tilde{L},\tilde{K}_i;Z)$ as a $Z[t]$-module. By $\mathfrak{M}_i(t)_{Z_2}$ we denote the matrix $\mathfrak{M}_i(t)$ with coefficients reduced to $Z_2$. Let $\varepsilon_2:Z[t] \to Z_2$ be the augmentation reduced to $Z_2$. Since $\mathfrak{M}_i(t)$ is a presentation matrix of $H_2(\tilde{L},\tilde{K}_i;Z)$, $\mathfrak{M}_i(1)_{Z_2}$ is a presentation matrix of $H_2(\tilde{L},\tilde{K}_i;Z)\otimes_{\varepsilon_2} Z_2$. Hence if $A_i(t)$ is the first invariant factor of $\mathfrak{M}_i(t)$, then $A_i(1)$ mod 2 is the first invariant factor of $\mathfrak{M}_i(1)_{Z_2}$. However, $H_2(\tilde{L},\tilde{K}_i;Z)\otimes_{\varepsilon_2} Z_2 = 0$. Therefore, $A_i(1) \neq 0$ mod 2. So, $A_i(\pm 1) \neq 0$. $A_i(\pm 1) \neq 0$ insists that $T_2(\tilde{L},\tilde{K}_i;Q)=H_2(\tilde{L},\tilde{K}_i;Q)$ contains no elements annihilated by $t-1$ or $t+1$. Similarly, $T_1(\tilde{L},\tilde{K}_i;Q)=H_1(\tilde{L},\tilde{K}_i;Q)$ contains no elements annihilated by $t-1$ or $t+1$. The homology exact sequence of the pair $(\tilde{L},\tilde{K}_i)$ induces the following exact sequence:

$$T_2(\tilde{L},\tilde{K}_i) \to T_1(\tilde{K}_i) \to T_1(\tilde{L}) \longrightarrow T_1(\tilde{L},\tilde{K}_i).$$

This implies that $T_1(\tilde{L})$ is isomorphic to $B_i \oplus T_i''$, where $T_i''$ is a $Q[t]$-module that contains no elements annihilated by $t-1$ or $t+1$. Therefore $B_1$ is isomorphic to $B_2$, since $Q[t]$ is a principal ideal domain. This completes the proof.

3.5. <u>Corollary</u>. <u>Let</u> $M$ <u>be a closed, connected and oriented 3-manifold with</u> $\text{rank}H_1(M;Z) \geq 1$. <u>If there exists a compact, connected</u>

*oriented* 4-*manifold* $W$ *with* $\partial W = M$ *and* $H_2(W;Z) = H^2(W;Z) = 0$, *then for any epimorphism* $\gamma: \pi_1(M) \to \langle t\rangle$ *we have* $\sigma_\gamma(M) = 0$, $h_\gamma(t) = f(t)f(t^{-1})$, $f(\pm 1) \neq 0$ , *and* $n_\gamma(M) = 0$.

Proof. $H_2(W;Z) = 0$ implies that the epimorphism $\gamma$ is extendable to an epimorphism $\tilde{\gamma}: \pi_1(W) \to \langle t\rangle$ and that $H_2(\tilde{W},\tilde{M};Q)$ is a torsion $Q[t]$-module. By Theorem 3.1, we have $\sigma_\gamma(M) = 0$ and $h_\gamma(t)= f(t)f(t^{-1})$. Further, $H^2(W;Z) = 0$ also says that $t-1: H_2(\tilde{W},\tilde{M};Z) \to H_2(\tilde{W},\tilde{M};Z)$ is an isomorphism, which assures that the ideal order $g(t)$ of $H_2(\tilde{W},\tilde{M};Q)$ satisfies $g(\pm 1) \neq 0$, as in 3.4. As was shown in 3.3, $f(t)$ is a factor of $g(t)$; so, $f(\pm 1) \neq 0$. This also implies that $n_\gamma(M) = 0$. This completes the proof.

3.6. *Corollary*. *Let* $M_i$, $i = 1, 2$, *be closed*, *connected and oriented 3-manifolds with rank* $H_1(M_i;Z) \geq 1$.

*Suppose there exists a finite connected complex* $K$ *such that*

(1) *The disjoint union* $M_1 \cup M_2$ *is embedded in* $K$,

(2) $H_j(K,M_1;Z) = H_j(K,M_2;Z) = 0$, $j = 1, 2$.

*Then we have* $n_{\gamma_1}(M_1)= n_{\gamma_2}(M_2)$ *for all compatible epimorphisms* $\gamma_i: \pi_1(M_i) \to \langle t\rangle$, $i= 1, 2$.

Proof. It follows immediately from Theorem 3.2.

For example, let $M = S^1 \times S^1 \times S^1$ and $\gamma: \pi_1(M) \to \pi_1(S^1)= \langle t\rangle$ be the epimorphism defined by the projection $M= S^1 \times S^1 \times S^1 \to S^1$ onto the first factor. Since $H_1(\tilde{M};Q) = Q[t]/t-1 \oplus Q[t]/t-1$, we have $\sigma_\gamma(M) = 0$ and $h_\gamma(t) = (t-1)^2$.[These are also justified, by Theorem 3.1, from the fact that $M$ bounds a 4-manifold $W$, say,

$S^1 \times S^1 \times B^2$ with a finitely generated $Q[t]$-module $H_2(\tilde{W},\tilde{M};Q)$.]
However, $n_\gamma(M) = 2$, which implies, by Corollary 3.5, that $M$ is not the boundary of a compact, connected, oriented 4-manifold $W$ with $H_2(W;Z) = H^2(W;Z) = 0$.

3.7. Corollary. Let $\ell_i \subset S^3$ be links with Alexander polynomials $A_i(t)$, $i = 0, 1$. Write $A_i(t) = (t-1)^{a_i}(t+1)^{b_i}A_i'(t)$, $A_i'(\pm 1) \neq 0$. If there exists a (possibly non-locally flat) piecewise-linear proper annulus $A(\cong S^1 \times [0,1])$ in $S^3 \times [0,1]$ with $\ell_0 = A \cap S^3 \times 0$ and $-\ell_1 = A \cap S^3 \times 1$, then we have $a_0 = a_1$ and $b_0 = b_1$.

Proof. It follows immediately from Theorem 3.2.

3.8. Corollary. If $\ell \subset S^3$ is a slice link in the strong sense, then we have $\sigma(\ell) = 0$, $A_\ell(t) = h_\ell(t) = f(t)f(t^{-1})$, $f(\pm 1) \neq 0$, and $n(\ell) = 1$. (cf. K. Murasugi[13, Theorem 8.4].)

Proof. It follows from Corollaries 3.5 and 3.7. [Note that, in the case, there is a canonical isomorphism $H_1(\tilde{E}(\ell)) \approx H_1(\tilde{M}(\ell))$ and $M(\ell)$ bounds a 4-manifold $W$ with $H^2(W;Z) = H_2(W;Z) = 0$, where $M(\ell)$ is a closed 3-manifold obtained from $S^3$ by surgery along $\ell$ exchanging the meridian curves on a tubular neighborhood with the (uniquely specified) longitude curves on the tubular neighborhood.]

## 4. Applications

Application 1. Consider a finitely presented group $G$ with rank $H_1(G;Z) \geq 1$. For an epimorphism $\gamma: G \longrightarrow \langle t \rangle$, $H_1(\mathrm{Ker}\gamma;Q)$ is a finitely generated $Q[t]$-module. By $T_\gamma(G;Q)$, we denote the $Q[t]$-torsion part . Let $A_\gamma(t)$ be the ideal order of $T_\gamma(G;Q)$. Also, define a polynomial $A_\gamma^{(2)}(t)$ by the identity $A_\gamma^{(2)}(t^2)= A_\gamma(t)A_\gamma(-t)$ (well-defined). We let $\gamma_2: G \longrightarrow \langle t \rangle \longrightarrow \langle t\rangle/\langle t^2\rangle = Z_2$.

Suppose there is a non-zero homomorphism $\sigma: G \rightarrow Z_2$ with $\sigma \neq \gamma_2$ for all epimorphisms $\gamma: G \longrightarrow \langle t\rangle$. [This situation is equivalent to saying that $H_1(G;Z)$ contains an element of order two.] Then denote $\mathrm{Ker}[G \xrightarrow{\sigma} Z_2]$ by $G^\sigma$ . Further, we let $\gamma^\sigma: G^\sigma \subset G \rightarrow \langle t\rangle$ be the composite epimorphism and $A_\gamma^\sigma(t)$ be the ideal order of $T_{\gamma^\sigma}(G^\sigma;Q)$.

We shall show the following:

Theorem A. Consider a finitely presented group $G$ without 2-torsion (i.e. $x^2= 1$ implies $x = 1$ in $G$) and with $H_1(G;Q)= Q$. Let $\gamma: G \longrightarrow \langle t\rangle$ be an epimorphism. If $G$ is a 3-manifold group , then the polynomial $A_\gamma^{(2)}(t)$ or $A_\gamma^\sigma(t)$ is reciprocal.

Remark. For an arbitrary finitely presented 3-manifold group $G$ with $H_1(G;Z) = Z$, there is a more explicite characterization of the polynomial. (cf. A.Kawauchi[6].)

A (finitely presented) 3-manifold group is a (finitely presented) group isomorphic to the fundamental group of a connected 3-manifold that need not have any other condition.

Proof of Theorem A. Since $G$ is a finitely presented 3-manifold group, it follows from a result of D.E.Galewski-S.G.Hollingsworth-D.R.McMillan,Jr.: On the fundamental group and homotopy type of open 3-manifolds (preprint) that there exists a compact 3-manifold $M$ with $\pi_1(M) \approx G$. In that case, we can assume that $\partial M$ contains no copies of $S^2$. Also, we have that $\partial M$ contains no copies of $P^2$, since $G$ is without 2-torsion. First suppose $\partial M \neq \emptyset$. Using rank $H_1(M;Z) = 1$, we obtain that $\chi(M) \geq 0$; so, $\chi(\partial M) \geq 0$, for $\chi(\partial M) = 2\chi(M)$. Since neither $S^2$ nor $P^2$ is contained in $\partial M$, we have $\chi(\partial M) = \chi(M) = 0$. Accordingly, $H_3(M,\partial M;Z) \approx H_2(\partial M;Z)$ and hence it occurs either that $M$ is orientable and $\partial M = S^1 \times S^1$ or that $M$ is non-orientable and each component of $\partial M$ is a Klein bottle $S^1 \times_\tau S^1$. In case $M$ is non-orientable, we can further assume that , for each component $S^1 \times_\tau S^1$ of $\partial M$, the inclusion homomorphism $H_1(S^1 \times_\tau S^1;Z) \to H_1(M;Z)/(\text{torsion})$ $(\approx Z)$ is non-trivial. [ Otherwise, we will attach the solid Klein bottle $S^1 \times_\tau B^2$ to $M$ along this $S^1 \times_\tau S^1$: $M' = M \cup S^1 \times_\tau B^2$. Then we will have $H_1(M';Q) \approx H_1(M;Q)$ and $H_1(\tilde{M}';Q) \approx H_1(\tilde{M};Q)$.] Thus, the pair $(M_2, \gamma_2)$ is always admissible for any $M$, where $M_2$ is the orientation cover of $M$ and $\gamma_2 : \pi_1(M_2) \longrightarrow \langle t_2 \rangle$ is the epimorphism determined by the composite $\gamma_2 : \pi_1(M_2) < \pi_1(M) \longrightarrow \langle t \rangle$.

Case(1): $M$ is orientable. Since $\partial M = S^1\ S^1$ or $\emptyset$, the pair $(M,\gamma)$ is admissible. This implies that $A_\gamma(t) \doteq A_\gamma(t^{-1})$ by Corollary 2.4; hence $A_\gamma^{(2)}(t) \doteq A_\gamma^{(2)}(t^{-1})$.

Case(2): $M$ is non-orientable and $\gamma_2 : \pi_1(M) \to Z_2$ gives the first Stiefel-Whitney class of $M$. In this case, $M_2$ is the orbits space $\tilde{M}/\langle t^2 \rangle$ and a generator $t_2$ of the infinite cyclic group $\langle t_2 \rangle$ is is given by $t^2$. Let $A_{\gamma_2}(t_2)$ be the Alexander polynomial of $M_2$ with $\gamma_2$. Consider a rational matrix $\mathcal{A}$ representing the linear isomorphism $t\colon T_1(\tilde{M};Q) \to T_1(\tilde{M};Q)$ ( for a suitable basis ). Then $A_{\gamma_2}(t^2) \doteq \det(t^2E-\mathcal{A}^2) = \det(tE-\mathcal{A})\det(tE+\mathcal{A}) \doteq A_\gamma(t)A_\gamma(-t) = A^{(2)}_\gamma(t^2)$. Thus, $A_{\gamma_2}(t) \doteq A^{(2)}_\gamma(t)$. Since the pair $(M_2, \gamma_2)$ is admissible, by Corollary 2.4, we have $A_{\gamma_2}(t) \doteq A_{\gamma_2}(t^{-1})$; so, $A^{(2)}_\gamma(t) \doteq A^{(2)}_\gamma(t^{-1})$.

<u>Case(3)</u>: <u>M is non-orientable and the first Stiefel-Whitney class $\sigma\colon \pi_1(M) \to Z_2$ satisfies $\sigma \neq \gamma_2$</u>. In this case, $\gamma_2 = \gamma^\sigma$ and $t_2 = t$. Since the pair $(M_2, \gamma^\sigma)$ is admissible, from Corollary 2.4, the reciprocity $A^\sigma_\gamma(t) \doteq A^\sigma_\gamma(t^{-1})$ follows. This completes the proof.

<u>Example</u>. <u>The group</u> $G_{(p,q)} = (a, b;\ a^{-1}b^pa = b^q)$ <u>is a 3-manifold group if and only if</u> $|p| = |q|$ <u>or</u> $pq = 0$.

This answers a question due to William Jaco[5,Question 13].

Proof. It is not hard to construct a 3-manifold for the case $|p| = |q|$ or $pq = 0$. Hence it suffices to show that if $|p| \neq |q|$ and $pq \neq 0$, then $G_{(p,q)}$ is not a 3-manifold group. Suppose $G_{(p,q)}$ is a 3-manifold group. $G_{(p,q)}$ satisfies the assumption of Theorem A.

<u>$A^{(2)}_\gamma(t)$ is not reciprocal</u>: $A_\gamma(t) = qt - p$ ( or $pt - q$); hence $A^{(2)}_\gamma(t) = q^2t - p^2$ (or $p^2t - q^2$). Since $|p| \neq |q|$ and $pq \neq 0$, $A^{(2)}(t)$ is not reciprocal.

<u>$A^\sigma_\gamma(t)$ is not reciprocal</u>: If there exists a non-zero homomorphism $\sigma\colon G_{(p,q)} \to Z_2$ with $\sigma \neq \gamma_2$, then $|p-q|$ must be even, since $H_1(G_{(p,q)};Z) \approx Z \oplus Z_{p-q}$ . In case p and q are even, we have a presented group $G^\sigma_{(p,q)} = (a, b, c\colon a^{-1}b^{p/2}a = b^{q/2},\ c^{-1}b^{p/2}c = b^{q/2})$, where we have $\gamma^\sigma(a) = \gamma^\sigma(c) = t$.

In case $p$ and $q$ are odd, we have a presented group $G^{\sigma}_{(p,q)} = (a, b;\ a^{-1}b^{p}a = b^{q})$. [Apply the Reidemeister-Schreier method.] In either case, we also have $A^{\sigma}_{\gamma}(t) \doteq qt - p$ ( or $pt - q$). Since $|p| \neq |q|$ and $pq \neq 0$, $A(t)$ is not reciprocal. This completes the proof.

<u>Application</u> 2. Let $M$ be a closed, connected and orientable 3-manifold with rank $H_1(M;Z) = r \geq 1$ and $W$ be a closed, connected and orientable 4-manifold with rank $H_1(W;Z) = r-1$ or $r$, and $H_2(W;Z) = 0$.

<u>Theorem B</u>. <u>If there is a piecewise-linear embedding</u> $f: M \to W$ <u>such that the induced homomorphism</u> $f_*: H_1(M;Z) \to H_1(W;Z)$ <u>is an epimorphism</u>, <u>then for any epimorphism</u> $\gamma: \pi_1(M) \to \langle t \rangle$ <u>we have</u> $\sigma_{\gamma}(M) = 0$ <u>and</u> $h_{\gamma}(t) \doteq f(t)f(t^{-1})$. <u>Furthermore</u>, <u>if</u> $H_1(M;Z)$ <u>is free</u>, <u>then we also have</u> $n_{\gamma}(M) = 0$.

Proof. We identify $f(M)$ with $M$. $M$ separates $W$ into two submanifolds. [Notice the assumptions.] It is easily checked that one of the submanifolds, say, $W_1$ has $H_2(W_1;Z) = 0$. So, $H^2(W_1, M; Z)=0$ by Poincaré duality. This implies that any epimorphism $\gamma: \pi_1(M) \to \langle t \rangle$ is extendable to an epimorphism $\bar{\gamma}: \pi_1(W_1) \to \langle t \rangle$. Since $H_2(\tilde{W}_1, \tilde{M}; Q)$ is a finitely generated $Q[t]$-torsion module, from Theorem 3.1, we obtain that $\sigma_{\gamma}(M) = 0$ and $h_{\gamma}(t) = f(t)f(t^{-1})$. If $H_1(M;Z)$ is free, then we also have $H^2(W_1;Z) = 0$. By Corollary 3.5, $n_{\gamma}(M) = 0$. This completes the proof.

For the special case: $M$ = a homology orientable handle and $W$=

a homology 4-sphere, Theorem B has already obtained in [8].

Example. Consider an orientable torus bundle $M$ over $S^1$(i.e., $M$ is an orientable 3-manifold that is a fiber bundle over $S^1$ with fiber a torus of genus 1). Such a bundle $M$ is completely determined by the fundamental group presented as follows: $(t,u,w: uw = wu,\ tut^{-1}= u^a w^b,\ twt^{-1}= u^c w^d)$, where $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is an integral matrix with $\det \rho = 1$. We will fix the words $t$, $u$ and $w$ and set $M = M_\rho$. Let $p : M_\rho \longrightarrow S^1$ be the bundle projection that corresponds to an epimorphism $\gamma : \pi_1(M_\rho) \rightarrow \langle t \rangle$. Then it holds that the projection $p : M_\rho \longrightarrow S^1 = S^1 \times * \subset S^1 \times S^3$ is not homotopic to any piecewise-linear embedding $M_\rho \rightarrow S^1 \times S^3$ except for two possible cases: $\rho = E, -E$.

Proof. Let $\tilde{M}_\rho$ be the infinite cyclic cover associated with the above $\gamma$. In case $\rho \neq E, -E$, we have $1 \leq \operatorname{rank} H_1(M_\rho;Z) \leq 2$. $H_1(\tilde{M}_\rho;Q)$ has a presentation matrix $\begin{pmatrix} t-a & -b \\ -c & t-d \end{pmatrix}$. $h_\gamma(t) = t^2-(a+d)t+1$. If $a+d \neq \pm 2$, then $h_\gamma(t)$ is irreducible. If $a = d = \pm 1$ and $b^2+c^2 \neq 0$, or if $a+d= \pm 2$ and $a \neq d$, then $H_1(\tilde{M}_\rho;Q) \approx Q[t]/(t\mp 1)^2$. Hence $\sigma_\gamma(M_\rho) = \pm 1$. Thus, in case $\rho \neq E, -E$, from Theorem B, there is no embedding $M_\rho \rightarrow S^1 \times S^3$ homotopic to $p$. In case $\rho = E$ or $-E$, it is easy to construct the desired embedding. This completes the proof.

Remark. If $\rho = E$, then $M_\rho$ is clearly $S^1 \times S^1 \times S^1$. If $\rho = -E$, then $M_\rho$ can be visualized as the boundary of a regular neighborhood of the "standardly" embedded Klein bottle in $R^4$.

Application 3. Consider a compact, connected and orientable 4-manifold $W$ with connected boundary $\partial W$. Suppose $H_1(W;Z) = 0$ and

$H_2(W;Z) \approx H_1(\partial W;Z) \approx \oplus Z^m$ for some $m \geq 1$.

Theorem C. If there exists a basis for $H_2(W;Z)$ such that the generators of the basis are represented by mutually disjoint, piecewise-linearly embedded 2-spheres, then we have $n_\gamma(\partial W) = 0$ i.e. $h_\gamma(\pm 1) \neq 0$ for all epimorphisms $\gamma : \pi_1(\partial W) \longrightarrow \langle t\rangle$.

Proof. Let $\Sigma_1, \ldots, \Sigma_m \subset W$ be mutually disjoint piecewise-linearly embedded 2-spheres representing the basis of $H_2(W;Z)$ and $N_1, \ldots, N_m$ be the mutually disjoint regular neighborhoods of $\Sigma_1, \ldots, \Sigma_m$ in $W$, respectively. Since $W - \Sigma_1 \cup \ldots \cup \Sigma_m$ is connected, we can perform a disk sum of $N_1, \ldots, N_m$ in $W$. Let $N$ be the resulting 4-manifold $N_1 \natural \ldots \natural N_m$ in $W$. Since the inclusion $N \subset W$ induces an isomorphism $H_*(N;Z) \approx H_*(W;Z)$, we obtain that $H_*(W,N;Z) = H_*(\overline{W-N}, \partial N;Z) = 0$. By Poincaré duality, $H_*(\overline{W-N}, \partial W;Z) = 0$. Hence by Corollary 3.6 $n_{\gamma'}(\partial N) = n_\gamma(\partial W)$ for all compatible epimorphisms $\gamma'$ and $\gamma$. Note that $\partial N = N_1 \# \ldots \# N_m$ and $H_1(\partial N;Z) \approx H_1(\partial W;Z) \approx \oplus Z^m$. This implies that for all $i$ $H_*(\partial N_i;Z) \approx H_*(S^1 \times S^2;Z)$, since $N_i$ is a regular neighborhood of the 2-sphere $\Sigma_i$. Since, for any epimorphism $\gamma_i' : \pi_1(\partial N_i) \longrightarrow \langle t\rangle$, $h_{\gamma_i'}(\pm 1) \neq 0$ i.e. $n_{\gamma_i'}(\partial N_i) = 0$ (cf.[6].), it is not hard to see that $n_{\gamma'}(\partial N) = 0$ for all epimorphisms $\gamma' : \pi_1(\partial N) \rightarrow \langle t\rangle$. Therefore we have $n_\gamma(\partial W) = 0$ for all epimorphisms $\gamma$. This completes the proof.

Example (cf. Y.Matsumoto[11].) We consider the link $k_1 \cup k_2$ in $S^3$, illustrated in Fig. 1.

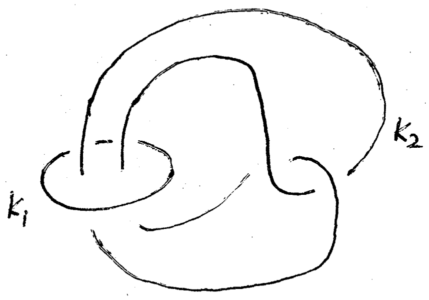



Fig. 1.

Since the linking number of $k_1$ and $k_2$ is 0, we can specify the longitude and meridian curves on a tubular neighborhood of each $k_i$ such that the longitude curve is homologous to 0 in $S^3-k_i$. Let W be the 4-manifold obtained from $D^4$ by attaching two 2-handles $D^2 \times D^2{}_1$, $D^2 \times D^2{}_2$ along the tubular neighborhoods with the specified longitude and meridian curves, so that the boundary $\partial W$ is a 3-manifold obtained from $S^3$ by surgery along the link $k_1 \cup k_2$ exchanging the meridian curves with the longitude curves. It is clear that W is a simply connected 4-manifold with connected boundary and that $H_2(W;Z) \approx H_1(\partial W;Z) \approx Z \oplus Z$. Note that the element $\xi_i$ of $H_2(W;Z)$ related to each $k_i$ is certainly represented by a locally flat 2-sphere and that $\{\xi_1, \xi_2\}$ forms a basis for $H_2(W;Z)$. It is immediate to see that the intersection numbers $\xi_1 \cdot \xi_1$, $\xi_2 \cdot \xi_2$, $\xi_1 \cdot \xi_2$ are all 0 and ,hence, that each two elements of $H_2(W;Z)$ has the intersection number 0. Y.Matsumoto[11] showed that $\xi_1$ and $\xi_2$ can not be represented by disjoint, piecewise-linearly embedded 2-spheres by using an invariant of R.A.Robertello[15]. We shall show that <u>any two elements of</u> $H_2(W;Z)$ <u>forming a basis can not be represented by disjoint, piecewise-linearly embedded</u>

<u>2-spheres</u>.

Proof. Let $t_1$ and $t_2$ be the elements of $H_1(\partial W;Z)$ related to $k_1$ and $k_2$, respectively. The set $\{t_1, t_2\}$ forms a basis for $H_1(\partial W;Z)$. Let $\gamma : \pi_1(\partial W) \rightarrow <t>$ be the epimorphism sending each $t_i$ to $t$. Since $T_1(\partial \widetilde{W};Q) \approx Q[t]/(t-1)^2$, we have $n_\gamma(\partial W) = 1$ ( or $h_\gamma(1) = 0$). Hence by Theorem C the desired assertion follows.

By considering the disk sums of copies of $W$, we can produce infinitely many simply connected 4-manifolds with similar properties.

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