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<th>Title</th>
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On the disruption of Whitney's lemma for simply connected 4-manifolds (in piecewise-linear and homotopy versions)

By Yukio MATSUMOTO

Whitney's lemma[7] states that intersection points of smooth n-submanifolds of a simply connected 2n-manifold can be eliminated if the intersection number of the two submanifolds is equal to zero and 2n \geq 6. However, this lemma fails for 2n = 4. This was first pointed out by Kervaire and Milnor[2] who found 2-dimensional homology classes \( \xi_1, \xi_2 \) of a simply connected 4-manifold such that (i) \( \xi_1 \) and \( \xi_2 \) are represented by smoothly embedded 2-spheres, (ii) the intersection number \( \xi_1 \cdot \xi_2 = 0 \) but (iii) there are no smoothly embedded disjoint 2-spheres which represent \( \xi_1, \xi_2 \) respectively. However, one can easily verify that their classes \( \xi_1, \xi_2 \) can be represented by disjoint piecewise-linearly (PL) embedded 2-spheres (with locally knotted points).

In this paper we shall give an example (Example 1) which shows that it is not always possible to represent two homology classes \( \xi_1, \xi_2 \) with \( \xi_1 \cdot \xi_2 = 0 \) by disjoint PL embedded 2-spheres. We shall also give an example (Example 2) in which one cannot represent a homology class \( \xi \) with \( \xi[S_1] = 0 \) (\( S_1 \) being a finite set of embedded 2-spheres) by a continuous map of a
2-sphere whose image is disjoint of these 2-spheres \( \{ S_i \} \).

§1. The PL case.

EXAMPLE 1. There exists a compact 1-connected 4-manifold \( W^4 \) (with boundary) which satisfies the following conditions:

(i) There are two primitive homology classes \( \xi_1, \xi_2 \in H(W^4; \mathbb{Z}) \) with \( \xi_1 \cdot \xi_2 = 0 \), but (ii) one cannot represent \( \xi_1, \xi_2 \) by PL embedded 2-spheres with disjoint images.

We start with the following link:

![Diagram of a link](image)

Fig. 1.

Since each of the components \( C_1, C_2 \) is a trivial knot, it has a trivial framing in \( S^3: C_1 \times D^2, C_2 \times D^2 \). Attach 2-handles \( h_1, h_2 \) to \( D^4 \) along these trivially framed circles. Then we obtain the 1-connected 4-manifold \( \tilde{W}^4 \) with boundary. Clearly \( H_2(W; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \) of which each summand is generated by the respective 2-handles. Let \( \xi_1, \xi_2 \) be the two generators.

LEMMA 1. Suppose that \( \xi_1 \) (or \( \xi_2 \)) is represented by a PL embedded 2-sphere \( \Sigma^2 \) which has a singular point (i.e., a locally knotted point) of knot type \( k \) (cf. Fox and Milnor [1]).
Then \( \varphi(k) = 0 \), where \( \varphi(k) \) denotes the Robertello invariant of the knot \( k \). (See Robertello [3].)

**LEMMA 2.** Suppose that \( \xi_1 + \xi_2 \) is represented by a PL embedded 2-sphere \( \Sigma^2 \) with a singular point of knot type \( k \). Then \( \varphi(k) = 1 \).

These lemmas will be proved later. Since the linking number of our link is equal to zero, the intersection number \( k_1 \cdot k_2 = 0 \).

Now we shall show that \( k_1, k_2 \) cannot be represented by disjoint PL embedded spheres. Otherwise, we would have two 2-spheres \( \Sigma_1, \Sigma_2 \) \(( \subset W^4 \)) which represent \( k_1, k_2 \) respectively. By Lemma 1, the singularities \( k_1, k_2 \) of these 2-spheres have Robertello invariant zero. We take the connected sum of these two spheres and would obtain a PL embedded 2-sphere \( \Sigma_1 \# \Sigma_2 \) \(( \subset W^4 \)) which represents \( k_1 + k_2 \) and whose singularity has Robertello invariant \( \varphi(k_1) + \varphi(k_2) = 0 \).

This contradicts Lemma 2.

**Proof of Lemma 1.** We shall prove the lemma for \( k_1 \). The proof for \( k_2 \) is the same. Suppose \( k_1 \) is represented by a PL embedded disks 2-sphere \( \Sigma^2 \) with a singularity \( k \). Let \( D_1 \), \( D_2 \) be transverse, \( \wedge \) attached 2-handles \( h_1, h_2 \) \(( \text{i.e. cocores in the terminology of Rourke and Sanderson [4, p.74]} \)). We may assume that \( \Sigma^2 \) intersects \( D_1 \), \( D_2 \) transversely with algebraic intersection numbers 1, 0, respectively. Let \( U_1, U_2 \) be (sufficiently thin) tubular neighbourhoods of \( D_1, D_2 \) in \( W^4 \). Then \( V^4 = W^4 - (U_1 \cup U_2) \) is PL-homeomorphic with a 4-disk, and on the boundary of \( V^4 \) we have a link \( L = \Sigma^2 \cap (\partial U_1 \cup \partial U_2) \). observe
that one can obtain the link $l$ starting with the (trivial) knot $C_1$ (Fig.2) or with the link of Fig.3 by adding a finite number of $(0, L, K)$-pairs in Tristram's sense ([6], Def. 3.1), where $L$ is the knot $C_1$ or the link of Fig.3 and $K$ is any component of $L$.

(This construction of $l$ will be referred to as the explicit construction.) Thus $l$ is a proper link in the sense of Robertello [3, p. 546]. $l$ is clearly related (in Robertello's sense [3, p. 547]) to the singularity knot $k$. Since $l$ is a proper link, the Robertello invariant of a knot which is related to $l$ depends only on $l$. Therefore, we can compute $\varphi(k)$ by any knot which is related to $l$ ([3], Th. 2).

However, from the explicit construction of $l$ it is easily verified that $l$ is related to a trivial knot $C_1$. This implies that $\varphi(k) = 0$.

![Fig.2](image1.png) ![Fig.3](image2.png)

**Proof of Lemma 2.** Let $\Sigma^2$ be a PL embedded 2-sphere ($\subset W^4$) which represents $\xi_1 + \xi_2$. Then $\Sigma^2$ intersects $D_1, D_2$ with algebraic intersection numbers $1, 1$. 

-4-
Thus, by the same reasoning as the previous proof, the link $l = \Sigma^2 \cap (\Sigma^1 \cup \Sigma^2)$ is proper and is related to the link of Fig. 1. The link of Fig. 1 is related to a trefoil $z_1$ (See Fig. 4).

Since $\Phi(z_1) = 1$, we know that the singularity $k$ of $\Sigma^2$, which is also related to $l$, has Robertello invariant 1.

Q.E.D.

**Problem 1.** Find a closed example with the same property.

**Problem 2.** Determine whether $\xi_1, \xi_2$ are represented by topologically embedded 2-spheres with disjoint images.

§2. The homotopy case.

**Example 2.** There exists a closed 1-connected 4-manifold $M^4$ with the following properties: (i) There are smoothly embedded 16 2-spheres $S_1, \ldots, S_{16}$ with disjoint images, (ii) there is a continuous map $f:S^2 \to M^4$ of a 2-sphere to the manifold with $(f_*(S^2)) \cdot [S^2_i] = 0$ for $i = 1, \ldots, 16$, but (iii) $f$ cannot be homotopic to any map $g:S^2 \to M^4$.
with \( g(S^2) \cap \bigcup_{i=1}^{16} S^2_i = \emptyset \).

The manifold \( M^4 \) is, in fact, a Kummer manifold (Cf. Spanier[5]). Let us recall the construction. We take a 4-dimensional torus \( T^4 = S^1 \times S^1 \times S^1 \times S^1 \) and consider the involution \( \sigma \) defined by \( \sigma(z_1, z_2, z_3, z_4) = (\overline{z}_1, \overline{z}_2, \overline{z}_3, \overline{z}_4) \), where we are considering \( S^1 = \{ z \in \mathbb{C} ; |z| = 1 \} \). Then \( \sigma \) has 16 fixed points \( P_1, \ldots, P_{16} \). The quotient \( T^4/\sigma \) has thus 16 singular points each of which locally looks like a cone over a 3-dimensional (real) projective space. Blow up these singularities, in other words, delete small regular neighbourhoods of the singular points and glue copies of the total space \( E \) of \( \Lambda^2 \)-disk bundle over \( S^2 \) with Euler class \(-2\). Then we obtain a closed smooth 4-manifold \( M^4 \) which contains 16 smoothly embedded 2-spheres (as exceptional curves or zero-sections of \( E \)'s). Denote these spheres by \( S^2_1, \ldots, S^2_{16} \). Note that \( [S^2_i] \cdot [S^2_j] = -2 \delta_{ij} \) (Kronecker's delta). It is known that the second betti number \( b_2(M^4) = 22 \) (cf.[5]). Thus we have a non-zero homology class \( \xi \in H_2(M^4; \mathbb{Z}) \) such that \( \xi \cdot [S^2_i] = 0 \) (\( \forall i=1, \ldots, 16 \)). Since \( M^4 \) is 1-connected ([5]), \( H_2(M^4; \mathbb{Z}) = \pi_2(M^4) \). Hence \( \xi \) is represented by a continuous map \( f:S^2 \to M^4 \). Suppose \( f \) with \( g(S^2) \cap \bigcup_{i=1}^{16} S^2_i = \emptyset \). Then, since \( M^4 - \bigcup_{i=1}^{16} S^2_i = T^4/\sigma \) -(the 16 points), the map \( g \) would be lifted to \( \tilde{g} : S^2 \to \pi_2(T^4) \) -(the 16 points). However, \( \pi_2(T^4-16 \text{ points}) = \{0\} \). This
implies that \( g \neq 0 \), which contradicts \( \xi \neq 0 \).

Q.E.D.

PROBLEM 3. Find a similar example with a smaller number of spheres.
References


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