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Unknotted Surfaces in 4-Space

By

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In this note we will discuss a concept of unknotted surfaces in the euclidean 4-space R^4 and study elementary topics related to it. Spaces and maps will be considered from a piecewise-linear point of view. We will denote by $R^3[t_o]$ the hyperplane whose fourth coordinate t is t_o in R^4, and for a subset A of $R^3[0]$, $A \times [a \leq t \leq b]$ means the subset $\{(x,t) \in R^4 \mid (x,0) \in A,\ a \leq t \leq b\}$ of R^4. The configurations of surfaces in R^4 will be described by adopting the <u>motion picture method</u>. (cf. R.H.Fox[1], F.Hosokawa[4] or A.Kawauchi-T.Shibuya[6].)

1. A Concept of Unknotted Surfaces

Consider a closed, connected and orineted surface F_n of genus n $(n \geq 0)$ in R^4. We will assume that F_n is <u>locally flat</u> in R^4. It is reasonable to note the following known basic fact before stating our definition of unknotted surfaces: <u>The surface F_n always bounds a compact, connected orientable 3-manifold in</u> R^4.

[For example, to see this, consider the regular neighborhood $N(F_n)$ of F_n in R^4. Since F_n is locally flat, we have $N(F_n)= F_n \times D^2$ for a 2-cell D^2. The projection $f: \partial N(F_n)(=F_n \times \partial D^2) \to \partial D^2$ is easily extendable to a piecewise-linear map $\bar{f}: cl(R^4 - N(F_n)) \to \partial D^2$ by an elementary obstruction theory. Then the transverse-regularity argument assures us to find a compact, connected orientable 3-manifold M in $cl(R^4 - N(F_n))$ with $\partial M = F_n \times x$ for some $x \in \partial D^2$. This M may be extended to a manifold $\bar{M}$ with $\partial \bar{M} = F_n$ in R^4. See H.Gluck[2] or A.Kawauchi-T.Shibuya[6,Chapter II] for other more constructive proofs.] We will define an unknotted surface as the boundary of a solid torus in R^4. Precisely,

1.1.Definition. F_n is said to be unknotted in R^4, if there exists a solid torus T_n of genus n in R^4 whose boundary ∂T_n is F_n. If such a T_n does not exist, then F_n is said to be knotted in R^4.

In the case of 2-spheres(i.e., surfaces of genera 0), Definition 1 is the usual definition of unknotted 2-spheres in R^4 and it is well-known that any unknotted 2-sphere is ambient isotopic[1)] to the boundary of a 3-cell in the hyperplane $R^3[0]$.

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1) An ambient isotopy of a space X is a family $\{h_t\}$ $(0 \leq t \leq 1)$ of auto-homeomorphisms of X with identity map h_0. For two subspaces X_1 and X_2 in a space X, X_1 is ambient isotopic to X_2, if there exists an ambient isotopy $\{h_t\}$ of X with $h_1(X_1)= X_2$. An auto-homeomorphism f of X is ambient isotopic to the identity, if there exists an ambient isotopy $\{h_t\}$ of X with $h_1 = f$.

The following theorem seems to justify Definition 1 for arbitrary unknotted surfaces.

1.2.<u>Theorem</u>. F_n <u>is unknotted if and only if</u> F_n <u>is ambient isotopic to the boundary of a regular neighborhood of an</u> n-<u>leafed rose</u> L_n <u>in</u> $R^3[0]$.

A 0-leafed rose L_0 in R^3 is understood as a point in R^3. For $n \geq 1$ an n-leafed rose L_n in R^3 is the union $\cup_{i=1}^{n} \partial\Delta_i$ of the boundaries $\partial\Delta_i$ of 2-simplices Δ_i in R^3 whose intersection $\cap_{i=1}^{n} \Delta_i$ is one vertex of each Δ_i and such that for each k,j, $k \neq j$, $\Delta_k \cap \Delta_j = \cap_{i=1}^{n} \Delta_i$. In Fig. 1 below, we illustrated L_n for the case $n = 6$.

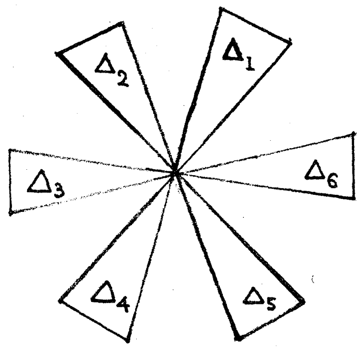


Fig. 1

1.3.<u>Example</u>. The surface of genus 1 in Fig. 2 is unknotted, since it bounds a solid torus of genus 1 that is shown in Fig.3.

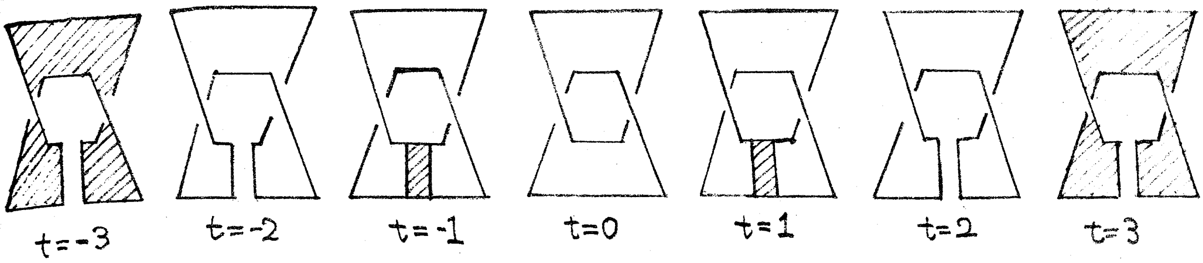


Fig. 2

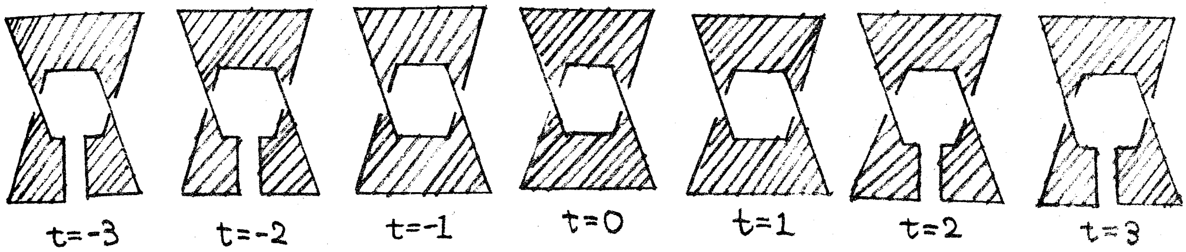


Fig. 3

Theorem 1.2 shows that this surface is ambient isotopic to the surface described in Fig. 4.

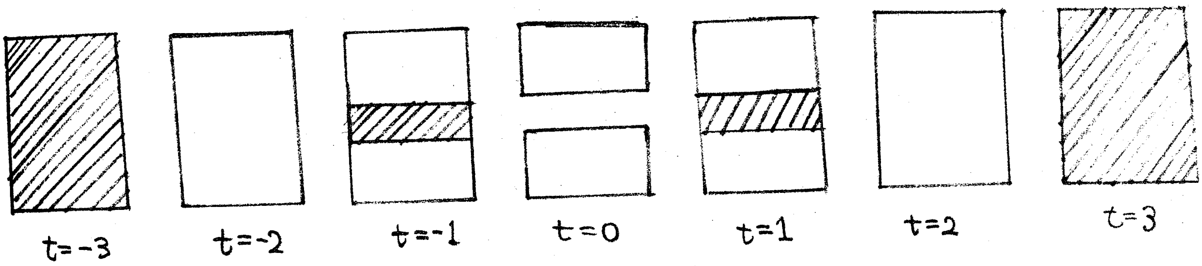


Fig. 4

1.4. Proof of Theorem 1.2. It suffices to prove Theorem 1.2 for the case $n \geq 1$. Assume F_n is unknotted. By definition, F_n bounds a solid torus T_n of genus n. Let a system $\{B_1,\dots,B_n\}$ be mutually disjoint n 3-cells in T_n, obtained by thickenning a system of meridian disks of T_n, such that $B = cl(T_n - B_1 \cup \dots \cup B_n)$ is a 3-cell. B is ambient isotopic to a 3-cell in $R^3[0]$; so we assume that B is contained in $R^3[0]$. Let L_n be a one-point-union of n 1-spheres at v in $Int(T_n)$ which is a spine of T_n, i.e., to which T_n collapses. Choose a sufficiently small , compact and connected neighborhood $U(v)$ of v in L_n so that $U(v)$ contains no vertices of L_n except for v. We may consider that $U(v) = L_n \cap B$ and $B \times [-1 \leq t \leq 1] \cap (L_n - U(v)) = \emptyset$. It is not hard to see that L_n is ambient isotopic to an n-leafed rose in $R^3[0]$ by an ambient isotopy of R^4 keeping $B \times [-1 \leq t \leq 1]$ fixed. So, we regard L_n as an n-leafed rose in $R^3[0]$. Let $R_0^4 = cl(R^4 - B \times [-1 \leq t \leq 1])$ and $cl(L_n - U(v)) = \ell_1 \cup \dots \cup \ell_n$, where ℓ_i are connected components. Note that $cl(T_n - B) = B_1 \cup \dots \cup B_n$. Now we shall show that there exist mutually disjoint regular neighborhoods H_i of ℓ_i in R_0^4 that meet the boundary ∂R_0^4 regularly and such that the pairs $(B_i \subset H_i)$ are proper, i.e., $\partial B_i = (\partial H_i) \cap B_i$. To show this, triangulate R_0^4 so that $B_1 \cup \dots \cup B_n$ is a subcomplex of R_0^4 and so that $\ell_1 \cup \dots \cup \ell_n$ is a subcomplex of $B_1 \cup \dots \cup B_n$. Let X and H' be the barycentric second derived neighborhoods of $\ell_1 \cup \dots \cup \ell_n$ in $B_1 \cup \dots \cup B_n$ and in R_0^4, respectively. It is easily seen that the pair $(X \subset H')$ is proper. Since $cl(B_1 \cup \dots \cup B_n - X)$ is homeomorphic to $cl(F_n - \partial B) \times [0,1]$, $B_1 \cup \dots \cup B_n$ is ambient isotopic to X by an ambient isotopy of R_0^4. Using this ambient isotopy, the desired pair $(B_1 \cup \dots \cup B_n \subset H_1 \cup \dots \cup H_n)$ is obtained.

Next, by using the uniqueness theorem of regular neighborhoods, we may assume that $H_i = N(\ell_i, R_0^3) \times [-1 \leq t \leq 1]$, $i = 1,2,\ldots,n$, where $R_0^3 = cl(R^3[0]-B)$ and $N(\ell_i, R_0^3)$ is a regular neighborhood of ℓ_i in R_0^3 meeting the boundary ∂R_0^3 regularly. More precisely, we can assume that $\partial R_0^3 \cap N(\ell_i, R_0^3) = (\partial B) \cap B_i$.

Now we need the following lemma:

1.5. <u>Lemma</u>. <u>Let a</u> 1-<u>sphere</u> S^1 <u>be contained in a</u> 2-<u>sphere</u> S^2 <u>and consider a proper surface</u> Y <u>in</u> $S^2 \times [0,1]$, (<u>abstructly</u>) <u>homeomorphic to</u> $S^1 \times [0,1]$. <u>If</u> $Y \cap S^2 \times 0 = S^1 \times 0$ <u>and</u> $Y \cap S^2 \times 1 = S^1 \times 1$, <u>then</u> Y <u>is ambient isotopic to</u> $S^1 \times [0,1]$ <u>by an ambient isotopy of</u> $S^2 \times [0,1]$ <u>keeping</u> $S^2 \times 0 \cup S^2 \times 1$ <u>fixed</u>.

By using Lemma 1.5, $cl(\partial B_i - \partial B)$ is ambient isotopic to $cl(\partial N(\ell_i; R_0^3) - \partial B)$ by an ambient isotopy of $cl(\partial H_i - \partial B \times [-1 \leq t \leq 1])$ keeping the boundary fixed. Hence by using a collar neighborhood of $cl(\partial H_i - \partial B \times [-1 \leq t \leq 1])$ in R_0^4, we obtain that by an ambient isotopy of R_0^4 keeping ∂R_0^4 fixed. This implies that F_n is ambient isotopic to the boundary of a regular neighborhood of L_n in $R^3[0]$. Since the converse is obvious, the proof is completed.

1.6. <u>Proof of Lemma 1.5</u>. Let $D \subset S^2$ be a 2-cell with $\partial D = S^1$. The 2-sphere $Y \cup D \times 0 \cup D \times 1$ bounds the 3-cell C in $S^2 \times [0,1]$, since $S^2 \times [0,1] \subset S^3$. Let $p \in Int(D)$ and choose a proper simple arc α in C to which C collapses and such that $\alpha \cap S^2 \times 0 = p \times 0$ and $\alpha \cap S^2 \times 1 = p \times 1$. Since there is an ambient isotopy of $S^2 \times [0,1]$ keeping $S^2 \times 0 \cup S^2 \times 1$ fixed and carrying α to $p \times [0,1]$, it follows from the uniqueness theorem of regular neighborhoods that C is ambient isotopic to $D\times[0,1]$ by an ambient isotopy of $S^2\times[0,1]$ keeping $S^2\times 0\cup S^2\times 1$ fixed. This proves Lemma 1.5.

As one consequence of Theorem 1.2, we have the following corollary:

1.7. Corollary. For any unknotted surface F_n in R^4, the bounding solid torus T_n is unique up to ambient isotopies of R^4.

Proof. Let T_n be a solid torus in R^4 with $\partial T_n = F_n$. It sufficies to construct an ambient isotopy $\{h_t\}$ of R^4 such that $h_1(T_n)$ is a regular neighborhood of an n-leafed rose in $R^3[0]$. By Theorem 1.2, we can assume that F_n is the boundary of a regular neighborhood of an n-leafed rose in $R^3[0]$. Let $N(F_n)$ be a sufficiently thin regular neighborhood of F_n in $R^3[0]$. Then we may consider that the union of T_n and one component $C(F_n)$ of $N(F_n)-F_n$ is a solid torus T_n'. Since $C(F_n)$ is homeomorphic to $F_n\times(0,1]$, T_n' is ambient isotopic to T_n. Let T_n'' be a regular neighborhood of an n-leafed rose in $C(F_n)$ such that $cl(T_n'-T_n'')$ is homeomorphic to $F_n\times[0,1]$. Since T_n' is ambient isotopic to T_n'', we complete the proof.

1.8. Note. It should be noted that for $n \geq 1$ the bounding solid torus T_n is not unique up to ambient isotopies of R^4 keeping F_n setwise fixed. Consider, for example, an unknotted surface F_1 of genus 1 as in Fig.5.

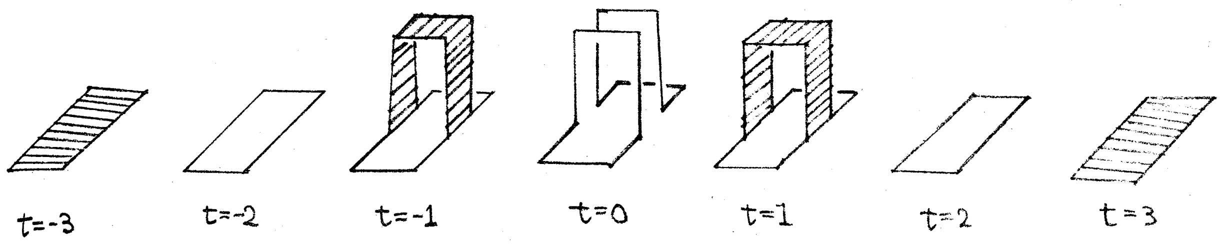


Fig. 5

This surface F_1 bounds two kinds of solid tori T_1, T_1' as shown in Fig. 6.

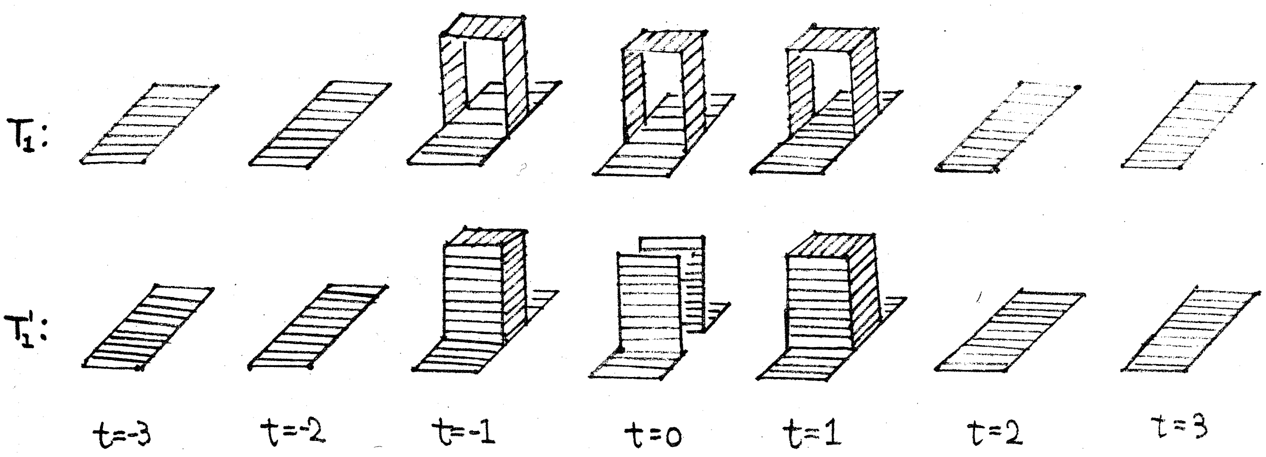


Fig. 6

Since the meridian curve of T_1 relating critical bands of F_1 is not a meridian curve of T_1', T_1 is not ambient isotopic to T_1' by an ambient isotopy of R^4 keeping F_1 setwise fixed.

1.9. <u>Note</u>. Let F_n be unknotted in R^4. Consider the homeotopy group $\mathcal{H}(R^4,F_n)$ of auto-homeomorphisms of the pair (R^4,F_n) modulo the homeomorphisms ambient isotopic to the identity. By Theorem 1.2, the homeotopy group $\mathcal{H}(R^4,F_n)$ is isomorphic to a homeotopy group $\mathcal{H}(R^4,\partial T_n)$, where ∂T_n is the boundary of a regular neighborhood T_n of an n-leafed rose in $R^3[0]$. So, we assume $F_n=\partial T_n$. Note 1.8 asserts that the group $\mathcal{H}(R^4,F_n)$ is non-trivial. Let $a_1,..,a_n$; $b_1,...,b_n$ be the standard meridian and longitude curves of $T_n \subset R^3[0]$. The homeotopy group $\mathcal{H}(R^4,F_n)$ contains the elements represented by the following auto-homeomorphisms; $h_{(i_1,...,i_n)}$ and $h^{(j)}$ such that

$$h_{(i_1,...,i_n)}(a_k) = a_{i_k}$$
$$h_{(i_1,...,i_n)}(b_k) = b_{i_k} ,$$

where $(i_1,...,i_n)$ is a permutation on $\{1,...,n\}$ defined by $(i_1,...,i_n)=\begin{pmatrix} 1 & \cdots & n \\ i_1 & \cdots & i_n \end{pmatrix}$, and

$$h^{(j)}(a_j) = b_j \qquad h^{(j)}(a_k) = a_k$$
$$h^{(j)}(b_j) = a_j \qquad h^{(j)}(b_k) = b_k \quad , k \neq j,$$

since T_n is contained in a 3-sphere in R^4. (Discussions on the orientation are now omitted.) Details of the homeotopy group $\mathcal{H}(R^4,F_n)$ remain as an open problem. For example, is $\mathcal{H}(R^4,F_n)$ isomorphic to the homeotopy group $\mathcal{H}(F_n)$ of the surface F_n ?

2. Hyperboloidal Transformations

Let F be a (possibly non-connected) closed and oriented surface in R^4. An oriented 3-cell B in R^4 is said to span F as a 1-handle, if $B \cap F = (\partial B) \cap F$ and this intersection is the union of disjoint two 2-cells, and if the surface $cl[F \cup \partial B - (\partial B) \cap F]$ can have an orientation compatible with both the orientations of $F - (\partial B) \cap F$ (induced from F) and $\partial B - (\partial B) \cap F$ (induced from B). Also, an oriented 3-cell B in R^4 spans F as a 2-handle, if $B \cap F = (\partial B) \cap F$ and this intersection is homeomorphic to the annulus $S^1 \times [0,1]$, and if the surface $cl[F \cup \partial B - (\partial B) \cap F]$ can have an orientation compatible with both the orientations of $F - (\partial B) \cap F$ and $\partial B - (\partial B) \cap F$.

2.1. Definition. If $B_1, \ldots, B_m$ are mutually disjoint oriented 3-cells in R^4 which span F as 1-handles, then the resulting oriented surface $h^1(F; B_1, \ldots, B_m) = cl[F \cup \partial B_1 \cup \ldots \cup \partial B_m - F \cap (\partial B_1 \cup \ldots \cup \partial B_m)]$ with orientation induced from $F - F \cap (B_1 \cup \ldots \cup B_m)$ is called the surface obtained from F by the hyperboloidal transformations along 1-handles $B_1, \ldots, B_m$. Likewise, if $B_1, \ldots, B_m$ span F as 2-handles, the resulting oriented surface $h^2(F; B_1, \ldots, B_m) = cl[F \cup \partial B_1 \cup \ldots \cup \partial B_m - F \cap (\partial B_1 \cup \ldots \cup \partial B_m)]$ is called the surface obtained from F by the hyperboloidal transformations along 2-handles $B_1, \ldots, B_m$.

We may have the following:

2.2. For arbitrary integers m and n with $1 \leq m \leq n$, if F_n is unknotted in R^4, then there exist mutually disjoint m 3-cells $B_1, \ldots, B_m$ in R^4 which span F_n as 2-handles and such that $h^2(F_n; B_1, \ldots, B_m)$ is an unknotted surface of genus $n-m$.

We shall show the following theorem which was partially suggested to the authors by T.Yajima:

2.3. Theorem. For arbitrary integers m and n with $1 \leq m \leq n$, if F_n is unknotted in R^4, then one can find mutually disjoint m 3-cells $B_1,\ldots,B_m$ in R^4 which span F_n as 2-handles and such that $h^2(F_n;B_1,\ldots,B_m)$ is a knotted surface of genus $n-m$. Further, every knotted surface in R^4 is ambient isotopic to a surface $h^2(F_n;B_1,\ldots,B_m)$ with an unknotted surface F_n and spanning 2-handles $B_1,\ldots,B_m$ for some m and n ($m \leq n$).

The proof will be given later.

Combined 2.2 with Theorem 2.3, we conclude that the knot type of the surface $h^2(F_n;B_1,\ldots,B_m)$ in R^4 depends on the choice of $B_1,\ldots,B_m$, even if F_n is unknotted. (In case F_n is knotted , the assertion has already known by T.Yajima[7].)

On the other hand, concerning 1-handles, we shall obtain the following:

2.4. Theorem. Given an unknotted surface F_n and mutually disjoint m 3-cells $B_1,\ldots,B_m$ in R^4 which span F_n as 1-handles, then the resulting surface $h^1(F_n;B_1,\ldots,B_m)$ of genus $n+m$ is necessarily unknotted.

2.5. Note. In case F_n is a knotted surface, then the knot type of the surface $h^1(F_n;B_1,\ldots,B_m)$ depends on the choice of $B_1,\ldots,B_m$. For example, let us consider the 2-sphere S illustrated in Fig. 7.

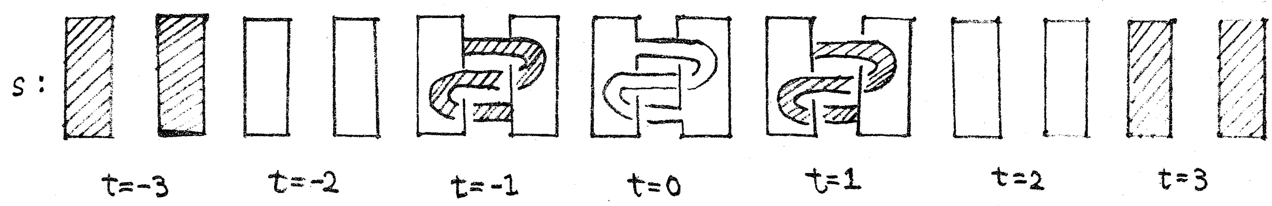


Fig. 7

This 2-sphere S is certainly knotted, since the fundamental group $\pi_1(R^4-S)$ has a presentation (a,b: aba=bab) whose Alexander polynomial is t^2-t+1.[In fact, this 2-sphere has the same knot type as the spun 2-knot of a trefoil.] Let B, B' be two 3-cells that span S as 1-handles, as shown in Fig.8.

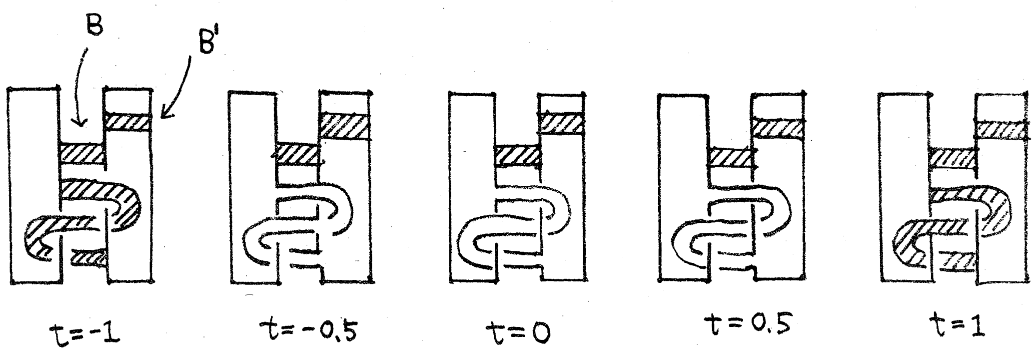


Fig. 8

The surfaces $F_1 = h^1(S;B)$ and $F_1' = h^1(S;B')$ of genera 1 related to the 3-cells B and B' are illustrated in Fig. 9.

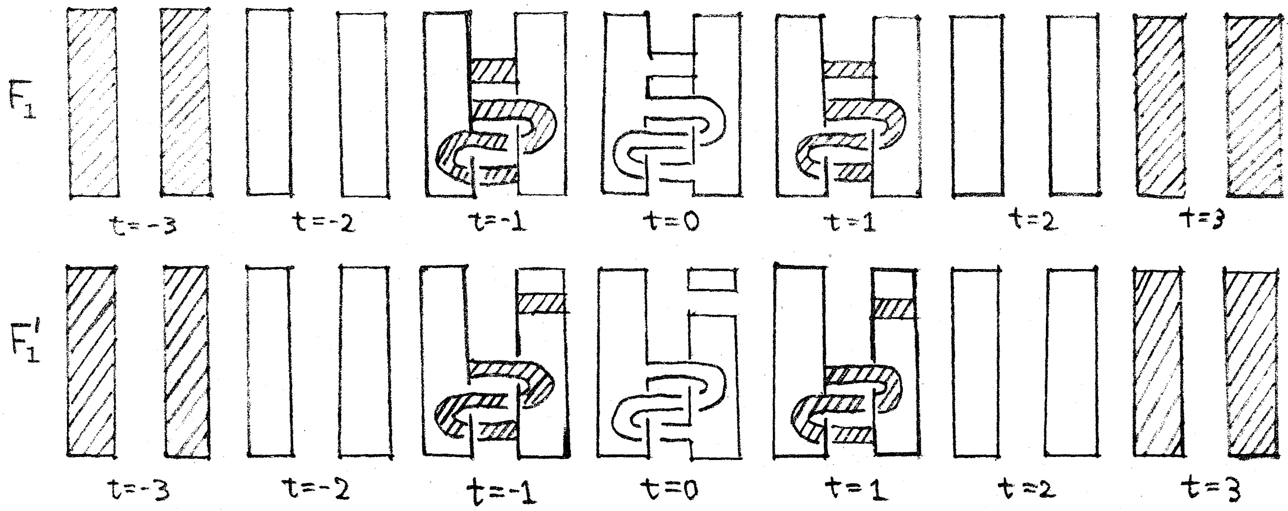


Fig. 9

It is easily seen that the fundamental group $\pi_1(R^4-F_1)$ is an infinite cyclic group [In 2.7 we shall show that this F_1 is actually unknotted.] and the fundamental group $\pi_1(R^4-F_1')$ is isomorphic to the fundamental group $\pi_1(R^4-S)$ that is non-abelian. Hence the knot types of F_1 and F_1' are distinct.

2.6. Proof of Theorem 2.4. We shall show the existence of a solid torus T_n of genus n in R^4 with $\partial T_n = F_n$ and $Int(T_n) \cap B_i = \emptyset$, $i = i,2,..,m$. Then the desired result follows , since $T_n \cup B_1 \cup .. \cup B_m$ is a solid torus of genus $n+m$ and since $h^1(F_n;B_1,..,B_m) = \partial(T_n \cup B_1 \cup .. \cup B_m)$. Choose for each i, $i = 1,2,..,m$, a simple proper arc α_i in B_i so that the union $F_n \cup \alpha_1 \cup ... \cup \alpha_m$ is a spine of the union

$F_n \cup B_1 \cup \ldots \cup B_m$. Since F_n is unknotted, we may consider F_n as the surface of genus n illustrated in Fig. 10.

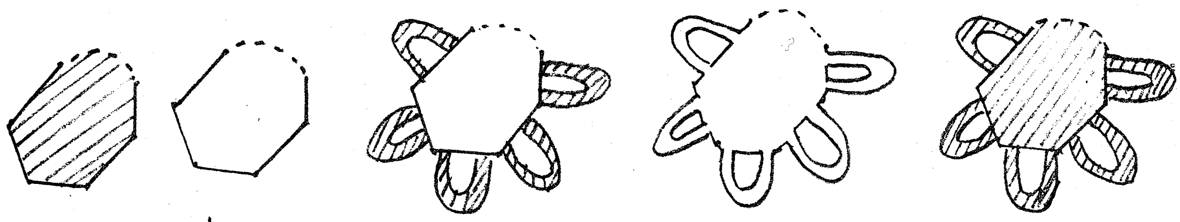


Fig. 10

By sliding $B_1,\ldots,B_m$ along F_n and by deforming $B_1,\ldots,B_m$ themselves, we can assume that $\alpha_1,\ldots,\alpha_m$ are attached to the circle in the level $t = 0$, i.e., $F_n \cap R^3[0]$ in well order and that for each i the two attaching points of α_i to $F_n \cap R^3[0]$ have compact and connected neighborhoods n_i^+ and n_i^- in α_i which are contained in the level $t = 0$. For $m = 3$ we illustrated the situation in Fig.11.

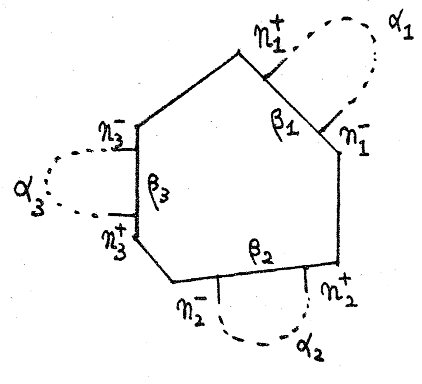


Fig. 11

For each i, let β_i be the part of $F_n \cap R^3[0]$ divided by α_i as in Fig. 12. (For $m = 1$ let β_1 be any one of the two components of $F_n \cap R^3[0]$ divided by α_1.) Further, for each i, let $\alpha_i' = cl(\alpha_i - n_i^+ \cup n_i^-)$. Now we join, for each i, the end points of α_i' with a simple arc γ_i such that the loop $\beta_i \cup n_i^+ \cup n_i^- \cup \gamma_i$ bounds a non-singular disk D_i in $R^3[0]$ with $Int(D_i) \cap (F_n \cup \alpha_1 \cup \ldots \cup \alpha_m) = \emptyset$, as in Fig. 12.

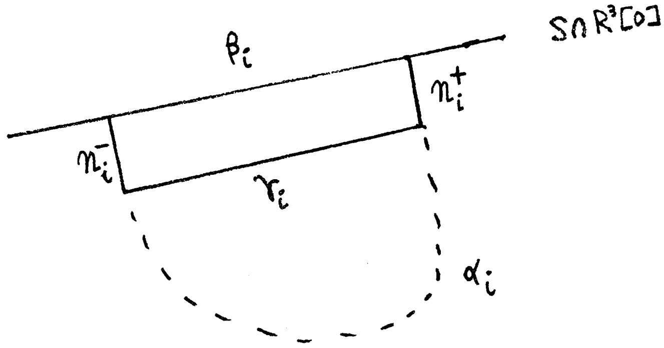


Fig. 12

The simple closed curve $\gamma_i \cup \alpha_i'$ is in general not homologous to 0 in $R^4 - F_n$. However, by twisting γ_i along the circle $F_n \cap R^3[0]$ (See for example Fig. 13.), we can assume that the simple closed curve $\gamma_i \cup \alpha_i'$ is homologous to 0 in $R^4 - F_n$.

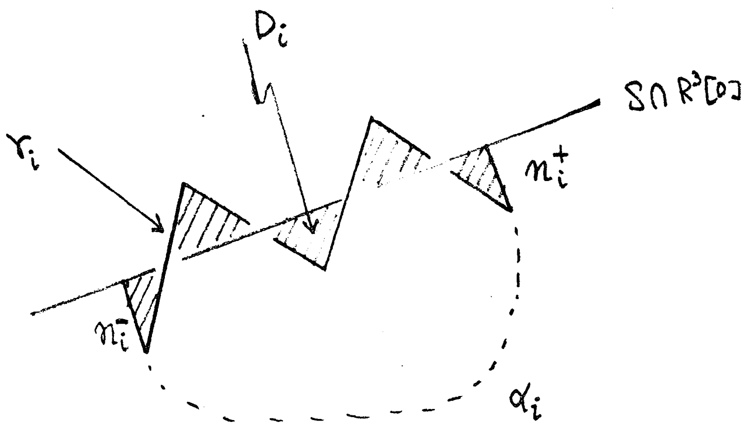
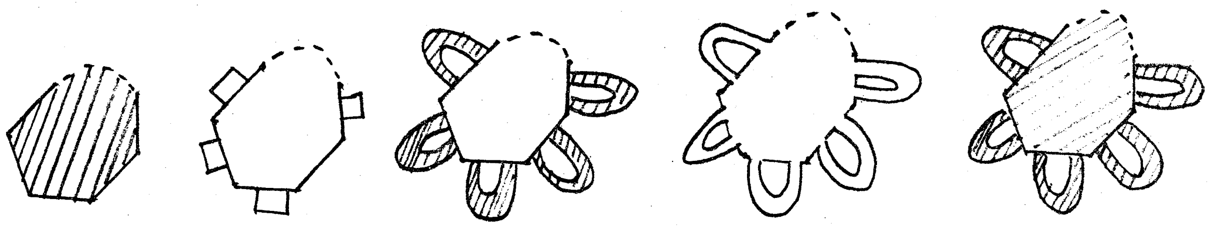


Fig. 13

Since F_n is unknotted, we have the Hurewicz isomorphism $\pi_1(R^4-F_n) \approx H_1(R^4-F_n;Z)$. Hence $\gamma_i \cup \alpha'_i$ is null-homotopic in R^4-F_n. By general position and by slight modifications, $\gamma_i \cup \alpha'_i$, $i = 1,2,..,m$, bound mutually disjoint non-singular disks d_i in R^4-F_n. Thus, $F_n \cup \alpha_1 \cup \ldots \cup \alpha_m$ is ambient isotopic to $F_n \cup (n_1^+ \cup \gamma_1 \cup n_1^-) \cup \ldots \cup (n_m^+ \cup \gamma_m \cup n_m^-)$. Hence $F_n \cup \alpha_1 \cup \ldots \cup \alpha_m$ is ambient isotopic to the standard surface of genus n with m attaching curves, as in Fig. 14.



t=-1 t=0 t=1 t=2 t=3

Fig. 14

Now by using the uniqueness theorem of regular neighborhoods, one can easily find a solid torus T_n of genus n in R^4 with $\partial T_n = F_n$ and $Int(T_n) \cap B_i = \emptyset$, $i = 1,2,\ldots,m$. This completes the proof.

2.7. <u>Proof of Theorem 2.3</u>. We shall show that , for an unknotted surface F_1 of genus 1, there exists a 3-cell B_1 in R^4 which spans F_1 as a 2-handle and such that $h^2(F_1;B_1)$ is a knotted 2-sphere. Then it is easy to find mutually disjoint 3-cells $B_1,\ldots,B_m$ which span an unknotted surface F_n as 2-handles and such that $h^2(F_n;B_1,\dots,B_m)$ is a knotted surface of genus $n-m$ for arbitrary given $m \leq n$. We consider the surface F_1 in Fig. 9. This surface is actually unknotted. In fact, let $\bar{B}$ be the 3-cell which spans F_1 as a 2-handle, illustrated in Fig. 15.

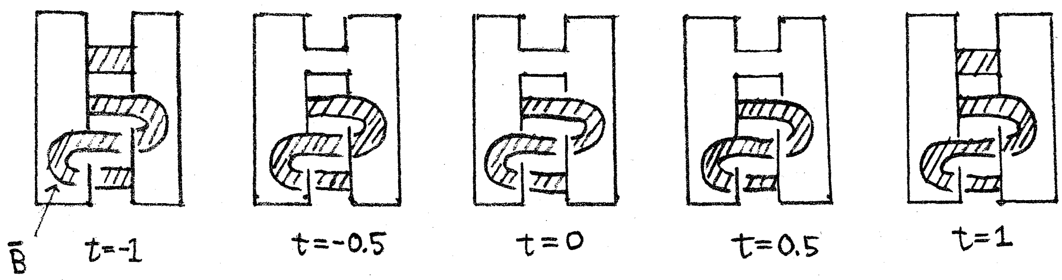


Fig. 15

The resulting 2-sphere $S_o = h^2(F_1;\bar{B})$ is clearly unknotted. Then Theorem 2.4 shows that the surface $F_1 = h^1(S_o;\bar{B})$ is unknotted. Consider the 3-cell B in Fig. 8 that spans F_1 as a 2-handle. The resulting 2-sphere $h^2(F_1;B)$ is knotted, because $h^2(F_1;B)$ is S in Fig. 7.

Secondly, we shall show that any knotted surface F in R^4 is ambient isotopic to a surface $h^2(F_n;B_1,\dots,B_m)$ with an unknotted surface F_n and spanning 2-handles $B_1,\dots,B_m$ for some m and n ($m \leq n$). Consider a compact, connected 3-manifold M in R^4 with $\partial M = F$. It is not difficult to find mutually disjoint 3-cells $B_1,\dots,B_m$ in M which span F as 1-handles and such that $T = cl(M-B_1 \cup \dots \cup B_m)$ is a solid torus. [In fact, take a 2-complex K that is a spine of M and let $K^{(1)}$ be the 1-skelton of K. Take the regular neighborhood $T' = N(K^{(1)};M)$ of $K^{(1)}$ in M. We may consider that cl(K-T') consists of m 2-cells $\Delta_1, \Delta_2, \ldots, \Delta_m$ for some m. For each i, let B_i' be a 3-cell thickenning Δ_i in cl(M-T'). The union $M' = T' \cup B_1' \cup \ldots \cup B_m'$ is a regular neighborhood of K in M. Using the uniqueness theorem of regular neighborhoods, we obtain that M' is homeomorphic to M. Divide M into a solid torus T and 3-cells $B_1, \ldots, B_m$ corresponding to T' and $B_1', \ldots, B_m'$, respectively, utilizing this homeomorphism. The result follows.] Let $F_n = \partial T$, where n is the genus of T. By definition, F_n is unknotted. From construction, we have $F = h^2(F_n; B_1, \ldots, B_m)$. This completes the proof.

A basic unsolved problem still remains that asks whether, given a knotted surface F_n of genus n, $n \geq 1$, one can always find mutually disjoint n 3-cells $B_1, \ldots, B_n$ in R^4 which spans F_n as 2-handles and such that $h^2(F_n; B_1, \ldots, B_n)$ is a 2-sphere. (The resulting 2-sphere will be necessarily knotted by Theorem 2.4.)

The following shows that there is a knotted surface from which one can never produce a 2-sphere by the hyperbolic transformation along 2-handle without changing the fundamental groups:

2.8. *Theorem*. *There exists a knotted surface* F_n *in* R^4 (*for each* $n \geq 1$) *such that*

(1) *One can find mutually disjoint* n 3-*cells* $B_1^o, \ldots, B_n^o$ *with* $h^2(F_n; B_1^o, \ldots, B_n^o)$ *a 2-sphere,*

(2) $\pi_1(R^4 - F_n)$ *is not isomorphic to* $\pi_1(R^4 - h^2(F_n; B_1, \ldots, B_n))$ *for any mutually disjoint* n 3-*cells* $B_1, \ldots, B_n$ *with* $h^2(F_n; B_1, \ldots, B_n)$ *a* 2-*sphere*.

Proof. It suffices to prove for the case $n = 1$. We shall show that the surface of genus 1 described in Fig. 16 is such a surface.

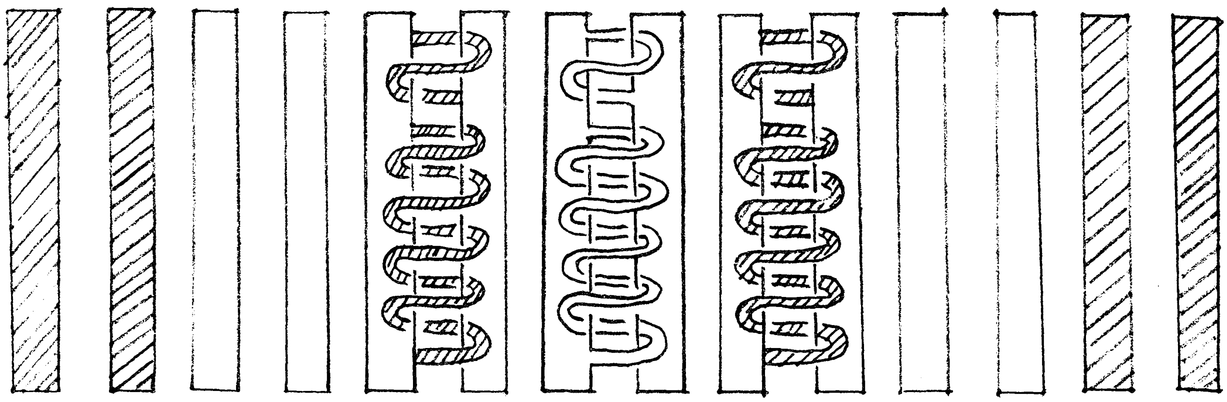


Fig. 16

This surface certainly satisfies (1). To see that it also satisfies (2), consider the fundamental group π of the complement of this surface in R^4. π has a presentation $(a,b \mid ab = ba^2,\ ba^5=a^5b)$.(See for example R.H.Fox[1]or T.Yajima[7] for a calculation.) Obviously, $H_1(\pi;Z)\approx Z$ and, by sending b of this presentation to t, a generator of an infinite cyclic group, the abelianized commutator subgroup π'/π'' of π is isomorphic to $Z[t]/(1-2t,5t-5)$ as $Z[t]$-modules, where $(1-2t,5t-5)$ denotes the ideal over $Z[t]$

generated by the polynomials $1-2t$ and $5t-5$. Using the identity $5 = 5(2t-1)+2(5-5t)$, π'/π'' is, consequently, isomorphic to $Z_5[t]/(2t-1)$ as $Z[t]$-modules. In particular, π'/π'' is isomorphic to Z_5 as abelian groups.

Now we need the following theorem that seems essentially the same as a result of M.A.Gutiérrez[3] (, although our approach is different from his.):

2.9.Theorem. Let G be a finitely presented group with $H_1(G;Z) = Z$ and such that G'/G'' is a finitely generated torsion group. If G is isomorphic to $\pi_1(R^4-S)$ for some 2-sphere S^2 in R^4, then for any finite field F the first polynomial invariant $a(t)$ of $(G'/G'')\otimes_Z F$ as $F[t]$-modules is reciprocal: $a(t) \doteq a(t^{-1})$ up to units of $F[t]$.(The first polynomial invariant $a(t)$ is defined to be the product $f_1(t)f_2(t)\ldots f_r(t)$ for a cyclic decomposition $(G'/G'')\otimes_Z F \approx F[t]/(f_1(t))\oplus F[t]/(f_2(t))\oplus\ldots\oplus F[t]/(f_r(t))$ as $F[t]$-modules.)

Note that $2t-1$ is the first polynomial invariant of $(\pi'/\pi'')\otimes Z_5$. Since $2t-1$ is not reciprocal in $Z_5[t]$, it follows from 2.9 that π is not the fundamental group of any 2-sphere in R^4. This is enough to show (2). This completes the proof.

2.10. Proof of Theorem 2.9. By assumption, G is isomorphic to the group $\pi_1(S^4-S)$ for some 2-sphere S in a 4-sphere S^4. Let $N(S)$ be the regular neighborhood of S in S^4 and $M = cl(S^4-N(S))$. Note that ∂M is homeomorphic to $S^1\times S^2$. Consider the infinite cyclic cover $\tilde{M}$ of M associated with the Hurewicz epimorphism $\pi_1(M) \longrightarrow H_1(M;Z)$. Since $H_1(\tilde{M};Z) \approx G'/G''$ is a finitely generated torsion group, it follows from A.Kawauchi[5,Theorem 2.3] that $H_*(\tilde{M};Z)$ is finitely generated as an abelian group and that there is a duality

$$\cap\mu: H^2(\tilde{M};Z) \approx H_1(\tilde{M},\partial\tilde{M};Z).$$

By the universal coefficient theorem, $H^1(\tilde{M};F)$ is canonically isomorphic to the torsion product $\mathrm{Tor}[H^2(\tilde{M};Z),F]$, for $H^1(\tilde{M};Z) = 0$. Since the inclusion map $\tilde{M} \subset (\tilde{M},\partial\tilde{M})$ induces an isomorphism $H_1(\tilde{M};Z) \approx H_1(\tilde{M},\partial\tilde{M};Z)$ as $Z[t]$-modules and $H_1(\tilde{M};Z)$ is a finitely generated torsion group and F is a finite field, we obtain the composite isomorphism

$$\begin{aligned}\Psi(\mu):\ H^1(\tilde{M};F) &\approx \mathrm{Tor}[H^2(\tilde{M};Z),F] \approx \mathrm{Tor}[H_1(\tilde{M},\partial\tilde{M};Z),F] \\ &\approx \mathrm{Tor}[H_1(\tilde{M};Z),F] \approx H_1(\tilde{M};F).\end{aligned}$$

The identity $(tu)\cap\mu = t^{-1}(u\cap\mu)$ for any $u \in H^2(\tilde{M};Z)$, then, induces the following commutative square of isomorphisms:

$$\begin{array}{ccc} H^1(\tilde{M};F) & \xrightarrow{\ \Psi(\mu)\ } & H_1(\tilde{M};F) \\ t \downarrow & & \downarrow t^{-1} \\ H^1(\tilde{M};F) & \xrightarrow{\ \Psi(\mu)\ } & H_1(\tilde{M};F). \end{array}$$

Since $H^1(\tilde{M};F)$ and $H_1(\tilde{M};F)$ are isomorphic as $F[t]$-modules, the first polynomial invariant $a(t)$ of $H_1(\tilde{M};F)$ must be reciprocal: $a(t) \doteq a(t^{-1})$. This completes the proof.

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